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Moments Of The Product F Distribution*

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Abstract

A new F distribution is introduced by taking the product of two F pdfs. Various particular cases and expressions for moments are derived.

The F distribution is the most familiar statistical distribution in finance, economics and related areas. The increasing applications in these areas have forced the need for more variations of the F distribution. In this note, we introduce a new F distribution with its pdf taken to be the product of two F densities, i.e.

$$f(x) = Cx^{\alpha+a-2}(1+cx)^{-(\alpha+\beta)}(1+dx)^{-(a+b)} \tag{1}$$

for $x > 0$, $a > 0$, $b > 0$, $c > 0$, $d > 0$, $\alpha > 0$ and $\beta > 0$, where C denotes the normalizing constant to be determined later. Like the F pdf, this pdf is unimodal with its mode given by the positive root of the quadratic equation:

$$-cd(b+\beta+2)x^2 + \left\{(c+d)(\alpha+a-2) - c(\alpha+\beta) - d(a+b)\right\}x + \alpha + a - 2 = 0.$$

The F pdf arises as the particular case of (1) for $c = d$. Figure 1 below illustrates possible shapes of (1) for selected values of a, b, α and β. Note that the y-axes are plotted on log scale. The effect of the parameters is evident.

The aim of this note is to provide detailed moment properties of (1). Other distributional properties including inference issues will be addressed in a subsequent paper. The calculations here involve several special functions, including the Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

the Legendre function of the first kind defined by

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right)$$

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and, the Legendre function of the second kind defined by

$$Q_\nu^\mu(x) = \frac{\sqrt{\pi}\exp(i\mu\pi)\Gamma(\mu+\nu+1)}{2^{\nu+1}\Gamma(\nu+3/2)} x^{-\mu-\nu-1}\left(x^2-1\right)^{\mu/2} \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \nu+\frac{3}{2}; \frac{1}{x^2}\right),$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. We also need the following important lemma.

LEMMA 1 (Equation (2.2.6.24), Prudnikov et al. [1], volume 1). For $0 < \alpha < \rho+\lambda$,

$$\int_0^\infty x^{\alpha-1}(x+y)^{-\rho}(x+z)^{-\lambda}dx = z^{-\lambda}y^{\alpha-\rho}B(\alpha, \rho+\lambda-\alpha)\ {}_2F_1\left(\alpha, \lambda; \rho+\lambda; 1-\frac{y}{z}\right).$$

Further properties of the above special functions can be found in Prudnikov *et al.* [1] and Gradshteyn and Ryzhik [2].

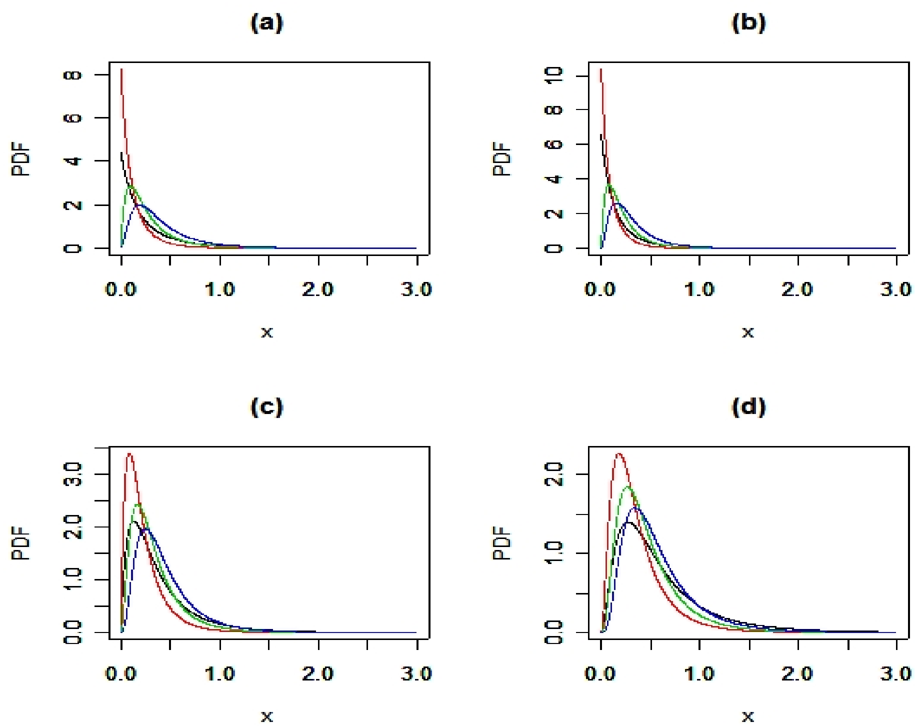

Figure 1. Plots of the pdf of (1) for (a): $(\alpha, \beta) = (1, 1)$; (b): $(\alpha, \beta) = (1, 3)$; (c): $(\alpha, \beta) = (2, 3)$; and, (d): $(\alpha, \beta) = (3, 3)$.
The four curves in each plot are: the black curve ($a = 1, b = 1$), the red curve ($a = 1, b = 3$), the green curve ($a = 2, b = 3$), and the blue curve ($a = 3, b = 3$).

THEOREM 1. If X is a random variable having the pdf (1) then

$$E\left(X^n\right) = Cc^{1-n-\alpha-a}B(n+\alpha+a-1,\beta+b-n+1) \times {}_2F_1\left(n+\alpha+a-1,a+b;\alpha+\beta+a+b;1-\frac{d}{c}\right) \quad (2)$$

for $n < 1+\beta+b$.

PROOF. One can write

$$E\left(X^n\right) = C\int_0^\infty x^{n+\alpha+a-2}(1+cx)^{-(\alpha+\beta)}(1+dx)^{-(a+b)}dx. \quad (3)$$

The result of the theorem follows by applying Lemma 1 to calculate the integral in (3).

Using special properties of the Gauss hypergeometric function, the following simpler forms for (2) can be obtained. Corollary 1 determines the normalizing constant C, Corollary 2 considers the case for $\alpha = a$ and $\beta = b$, and Corollary 3 considers the familiar F distribution for $c = d$.

COROLLARY 1. The normalizing constant C in (1) given by

$$\frac{1}{C} = c^{1-\alpha-a}B(\alpha+a-1,\beta+b+1)\ {}_2F_1\left(\alpha+a-1,a+b;\alpha+\beta+a+b;1-\frac{d}{c}\right).$$

COROLLARY 2. If $\alpha = a$ and $\beta = b$ then (2) can be reduced to one of the following equivalent forms

$$E\left(X^n\right) = \frac{C2^{2a+2b-1}\Gamma\left(a+b+1/2\right)\Gamma\left(n+2a-1\right)\Gamma\left(2b-n+1\right)}{c^{n+2a-5/4}d^{1/4}\Gamma\left(2a+2b\right)}\left(1-\frac{d}{c}\right)^{1/2-a-b} \times P_{n+a-b-3/2}^{1/2-a-b}\left(\frac{1+d/c}{2\sqrt{d/c}}\right),$$

$$E\left(X^n\right) = \frac{C4^{a+b}\Gamma\left(a+b+1/2\right)\Gamma\left(n+2a-1\right)}{\sqrt{\pi}c^{(3a+b+n-1)/2}d^{(n+a-b-1)/2}\Gamma\left(2a+2b\right)}\left(1-\frac{d}{c}\right)^{-(a+b)} \times \exp\left\{-i\pi(b-a-n+1)\right\}Q_{a+b-1}^{b-a-n+1}\left(\frac{1+d/c}{1-d/c}\right),$$

$$E\left(X^n\right) = \frac{C4^{a+b}\Gamma\left(a+b+1/2\right)\Gamma\left(2b-n+1\right)}{\sqrt{\pi}c^{(3a+b+n-1)/2}d^{(n+a-b-1)/2}\Gamma\left(2a+2b\right)}\left(\frac{d}{c}-1\right)^{-(a+b)} \times \exp\left\{i\pi(b-a-n+1)\right\}Q_{a+b-1}^{a+n-b-1}\left(-\frac{1+d/c}{1-d/c}\right)$$

for $n < 1+2b$.

COROLLARY 3. If $c = d$ then (2) can be reduced to the familiar form

$$E\left(X^n\right) = Cc^{1-n-\alpha-a}B(n+\alpha+a-1,\beta+b-n+1)$$

for $n < 1 + \beta + b$.

The proofs of the above corollaries are not difficult. Corollary 1 follows by setting $n = 0$ into (2). Corollary 2 follows by applying equations (7.3.1.70)–(7.3.1.72) in volume 3 of Prudnikov *et al.* [1] to reexpress the Gauss hypergeometric term in (2) in terms of the Legendre functions. Corollary 3 follows directly from (2) since ${}_2F_1(a, b; c; 0) = 1$.

We now derive particular forms of (2) for $(\alpha, a) = (\beta, b) = (1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 2)$, $(2, 3)$ and $(3, 3)$. The results are given in the forms of Corollaries 4 to 9 and note that all of the expressions are elementary. The results of the corollaries can be verified by using equations (7.3.1.10) and (7.3.1.128)–(7.3.1.130) in volume 3 of Prudnikov *et al.* [1]. These equations provide ways of reducing ${}_2F_1(a, b; c; x)$ to elementary forms when a, b and c take integer values.

COROLLARY 4. If $\alpha = \beta = 1$ and $a = b = 1$ then (2) can be reduced to

$$E(X) = C\Big\{ -2x + \ln(1-x)\,x - 2\ln(1-x) \Big\} \Big/ \Big\{ c^2 x^3 \Big\},$$

$$E\left(X^2\right) = C\Big\{ x^2 - 2\,x + 2\,\ln(1-x)\,x - 2\,\ln(1-x) \Big\} \Big/ \Big\{ c^3 x^3 (x-1) \Big\},$$

where $x = 1 - d/c$ and the normalizing constant C is given by

$$\frac{1}{C} = -\Big\{ x^2 - 2\,x + 2\,\ln(1-x)\,x - 2\,\ln(1-x) \Big\} \Big/ \Big\{ c x^3 \Big\}.$$

COROLLARY 5. If $\alpha = \beta = 1$ and $a = b = 2$ then (2) can be reduced to

$$\begin{aligned} E(X) &= C\Big\{ x^3 - 12x^2 + 12x + 6\ln(1-x)\,x^2 - 18\ln(1-x)\,x + 12\ln(1-x) \Big\} \\ &\quad \Big/ \Big\{ 3c^3 x^5 \Big\}, \end{aligned}$$

$$E\left(X^2\right) = C\Big\{ x^3 + 6x^2 - 24x + 18\ln(1-x)\,x - 24\ln(1-x) \Big\} \Big/ \Big\{ 6c^4 x^5 \Big\},$$

$$\begin{aligned} E\left(X^3\right) &= C\Big\{ 2x^3 + x^4 + 6x^2 - 12x + 12\ln(1-x)\,x - 12\ln(1-x) \Big\} \\ &\quad \Big/ \Big\{ 3c^5 x^5 (x-1) \Big\}, \end{aligned}$$

where $x = 1 - d/c$ and the normalizing constant C is given by

$$\begin{aligned} \frac{1}{C} &= \Big\{ -17x^3 + 42x^2 - 24x + 6\ln(1-x)\,x^3 - 36\ln(1-x)\,x^2 + 54\ln(1-x)\,x \\ &\quad -24\ln(1-x) \Big\} \Big/ \Big\{ 6c^2 x^5 \Big\}. \end{aligned}$$

COROLLARY 6. If $\alpha = \beta = 2$ and $a = b = 2$ then (2) can be reduced to

$$\begin{aligned} E(X) &= C\Big\{ -11x^3 + 60\,x^2 - 60x + 3\ln(1-x)\,x^3 - 36\ln(1-x)\,x^2 \\ &\quad +90\ln(1-x)\,x - 60\ln(1-x) \Big\} \Big/ \Big\{ 3c^4 x^7 \Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) &= C\Big\{x^4 - 32\,x^3 + 90\,x^2 - 60\,x + 12\,\ln(1-x)\,x^3 - 72\ln(1-x)\,x^2 \\ &\quad + 120\,\ln(1-x)\,x - 60\ln(1-x)\Big\} \Big/ \Big\{3c^5 x^7 (x-1)\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^3\right) &= C\Big\{x^5 + 10x^4 - 130x^3 + 240x^2 - 120x + 60\ln(1-x)\,x^3 \\ &\quad - 240\ln(1-x)\,x^2 + 300\ln(1-x)\,x - 120\ln(1-x)\Big\} \\ &\quad \Big/ \Big\{6c^6 x^7 (x-1)^2\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^4\right) &= C\Big\{x^6 + 3x^5 + 15x^4 - 110x^3 + 150x^2 - 60x + 60\,\ln(1-x)\,x^3 \\ &\quad - 180\ln(1-x)\,x^2 + 180\ln(1-x)\,x - 60\ln(1-x)\Big\} \\ &\quad \Big/ \Big\{3c^7 x^7 (x-1)^3\Big\}, \end{aligned}$$

where $x = 1 - d/c$ and the normalizing constant C is given by

$$\begin{aligned} \frac{1}{C} &= -\Big\{x^4 - 32x^3 + 90x^2 - 60x + 12\ln(1-x)\,x^3 - 72\ln(1-x)\,x^2 \\ &\quad + 120\ln(1-x)\,x - 60\ln(1-x)\Big\} \Big/ \Big\{3c^3 x^7\Big\}. \end{aligned}$$

COROLLARY 7. If $\alpha = \beta = 1$ and $a = b = 3$ then (2) can be reduced to

$$\begin{aligned} E(X) &= C\Big\{x^5 + 10x^4 - 130x^3 + 240x^2 - 120x + 60\ln(1-x)\,x^3 \\ &\quad - 240\ln(1-x)\,x^2 + 300\ln(1-x)\,x - 120\ln(1-x)\Big\} \Big/ \Big\{20c^4 x^7\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) &= C\Big\{x^5 + 5x^4 + 30x^3 - 210x^2 + 180x + 120\ln(1-x)\,x^2 \\ &\quad - 300\ln(1-x)\,x + 180\ln(1-x)\Big\} \Big/ \Big\{30c^5 x^7\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^3\right) &= C\Big\{3x^5 + 10x^4 + 30x^3 + 120x^2 - 360x + 300\ln(1-x)\,x \\ &\quad - 360\ln(1-x)\Big\} \Big/ \Big\{60c^6 x^7\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^4\right) &= C\Big\{3x^5 + 2x^6 + 5x^4 + 10x^3 + 30x^2 - 60x + 60\ln(1-x)\,x \\ &\quad - 60\ln(1-x)\Big\} \Big/ \Big\{10c^7 x^7 (x-1)\Big\}, \end{aligned}$$

where $x = 1 - d/c$ and the normalizing constant C is given by

$$\begin{aligned}\frac{1}{C} &= \Big\{6x^5 - 155x^4 + 480x^3 - 510x^2 + 180x + 60\ln(1-x)\,x^4 - 360\ln(1-x)\,x^3 \\ &\quad +720\ln(1-x)\,x^2 - 600\ln(1-x)\,x + 180\ln(1-x)\Big\}\Big/\Big\{30c^3x^7\Big\}.\end{aligned}$$

COROLLARY 8. If $\alpha = \beta = 2$ and $a = b = 3$ then (2) can be reduced to

$$\begin{aligned}E(X) &= C\Big\{3x^5 - 190x^4 + 1030x^3 - 1680x^2 + 840x + 60\ln(1-x)\,x^4 \\ &\quad -600\ln(1-x)\,x^3 + 1800\ln(1-x)\,x^2 - 2100\ln(1-x)\,x \\ &\quad +840\ln(1-x)\Big\}\Big/\Big\{15c^5x^9\Big\},\end{aligned}$$

$$\begin{aligned}E\left(X^2\right) &= C\Big\{3x^5 + 60x^4 - 1570x^3 + 4620x^2 - 3360x + 600\ln(1-x)\,x^3 \\ &\quad -3600\ln(1-x)\,x^2 + 6300\ln(1-x)\,x - 3360\ln(1-x)\Big\} \\ &\quad \Big/\Big\{60c^6x^9\Big\},\end{aligned}$$

$$\begin{aligned}E\left(X^3\right) &= C\Big\{9x^5 + x^6 + 90x^4 - 1370x^3 + 2940x^2 - 1680x + 600\ln(1-x)\,x^3 \\ &\quad -2700\ln(1-x)\,x^2 + 3780\ln(1-x)\,x - 1680\ln(1-x)\Big\} \\ &\quad \Big/\Big\{30c^7x^9(x-1)\Big\},\end{aligned}$$

$$\begin{aligned}E\left(X^4\right) &= C\Big\{63x^5 + 14x^6 + 3x^7 + 420x^4 - 4270x^3 + 7140x^2 - 3360x \\ &\quad +2100\ln(1-x)\,x^3 - 7560\ln(1-x)\,x^2 + 8820\ln(1-x)\,x \\ &\quad -3360\ln(1-x)\Big\}\Big/\Big\{60c^8x^9(x-1)^2\Big\},\end{aligned}$$

$$\begin{aligned}E\left(X^5\right) &= C\Big\{-840x + 2100x^2 + 42x^5 + 6x^7 + 210x^4 + 3x^8 + 14x^6 - 1540x^3 \\ &\quad +840\ln(1-x)\,x^3 - 2520\ln(1-x)\,x^2 + 2520\ln(1-x)\,x \\ &\quad -840\ln(1-x)\Big\}\Big/\Big\{15c^9x^9(x-1)^3\Big\},\end{aligned}$$

where $x = 1 - d/c$ and the normalizing constant C is given by

$$\begin{aligned}\frac{1}{C} &= \Big\{-247x^5 + 2660x^4 - 7870x^3 + 8820x^2 - 3360x + 60\ln(1-x)\,x^5 \\ &\quad -1200\ln(1-x)\,x^4 + 6000\ln(1-x)\,x^3 - 12000\ln(1-x)\,x^2 \\ &\quad +10500\ln(1-x)\,x - 3360\ln(1-x)\Big\}\Big/\Big\{60c^4x^9\Big\}.\end{aligned}$$

COROLLARY 9. If $\alpha = \beta = 3$ and $a = b = 3$ then (2) can be reduced to

$$\begin{aligned} E(X) &= C\Big\{15120x^2 - 137x^5 + 2310x^4 - 9870x^3 - 7560x + 30\ln(1-x)x^5 \\ &\quad -900\ln(1-x)x^4 + 6300\ln(1-x)x^3 - 16800\ln(1-x)x^2 \\ &\quad +18900\ln(1-x)x - 7560\ln(1-x)\Big\}\Big/\Big\{30c^6x^{11}\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) &= C\Big\{3150x^2 - 107x^5 + 945x^4 + x^6 - 2730x^3 - 1260x + 30\ln(1-x)x^5 \\ &\quad -450\ln(1-x)x^4 + 2100\ln(1-x)x^3 - 4200\ln(1-x)x^2 \\ &\quad +3780\ln(1-x)x - 1260\ln(1-x)\Big\}\Big/\Big\{5c^7x^{11}(x-1)\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^3\right) &= C\Big\{15120\,x^2 - 1288x^5 + x^7 + 7560x^4 + 28x^6 - 16380x^3 - 5040x \\ &\quad +420\ln(1-x)x^5 - 4200\ln(1-x)x^4 + 14700\ln(1-x)x^3 \\ &\quad -23520\ln(1-x)x^2 + 17640\ln(1-x)x - 5040\ln(1-x)\Big\} \\ &\quad \Big/\Big\{20c^8x^{11}(x-1)^2\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^4\right) &= C\Big\{26460x^2 - 4592x^5 + 12x^7 + 19950x^4 + x^8 + 168x^6 - 34440x^3 \\ &\quad -7560x + 1680\ln(1-x)x^5 - 12600\ln(1-x)x^4 + 35280\ln(1-x)x^3 \\ &\quad -47040\ln(1-x)x^2 + 30240\ln(1-x)x - 7560\ln(1-x)\Big\} \\ &\quad \Big/\Big\{30c^9x^{11}(x-1)^3\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^5\right) &= C\Big\{20160x^2 - 6258x^5 + 36x^7 + 21420x^4 + 6x^8 + 336x^6 + x^9 \\ &\quad -30660x^3 - 5040x + 2520\ln(1-x)x^5 - 15120\ln(1-x)x^4 \\ &\quad +35280\ln(1-x)x^3 - 40320\ln(1-x)x^2 + 22680\ln(1-x)x \\ &\quad -5040\ln(1-x)\Big\}\Big/\Big\{20c^{10}x^{11}(x-1)^4\Big\}, \end{aligned}$$

$$\begin{aligned} E\left(X^6\right) &= C\Big\{-2520x + 11340x^2 - 5754x^5 + 60x^7 + 16170x^4 + 15x^8 + 420x^6 \\ &\quad +5x^9 + 2x^{10} - 19740x^3 + 2520\ln(1-x)x^5 - 12600\ln(1-x)x^4 \\ &\quad +25200\ln(1-x)x^3 - 25200\ln(1-x)x^2 + 12600\ln(1-x)x \\ &\quad -2520\ln(1-x)\Big\}\Big/\Big\{10c^{11}x^{11}(x-1)^5\Big\}, \end{aligned}$$

where $x = 1 - d/c$ and the normalizing constant C is given by

$$\begin{aligned}\frac{1}{C} \quad = \quad & -\Big\{3150x^2 - 107x^5 + 945x^4 + x^6 - 2730x^3 - 1260x + 30\ln(1-x)\,x^5 \\ & -450\ln(1-x)\,x^4 + 2100\ln(1-x)\,x^3 - 4200\ln(1-x)\,x^2 \\ & +3780\ln(1-x)\,x - 1260\ln(1-x)\Big\}\Big/\Big\{5c^5x^{11}\Big\}.\end{aligned}$$

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[2] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (sixth edition), Academic Press, 2000.