Applied Mathematics E-Notes, 6(2006), 159-166 © ISSN 1607-2510
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Minimizing Energy Use For A Road Expansion In A Transportation System Using Optimal Control Theory*

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Received 9 March 2005

Abstract

The objective of the present research is to make use of optimal control theory to solve a serious environment and transportation problem.

1 Introduction

Optimal control theory is a branch of mathematics developed to find optimal ways to control a dynamic system. It has been successfully applied to many areas of operations research such as: finance [5, 7, 19], economics [3, 8, 12], inventory [10, 11, 17], marketing [9, 18], maintenance [15, 16], and the consumption of natural resources [2, 6].

In the present research we intend to apply optimal control theory to solve a general and important problem related to both transportation and environment sectors; for a real-life application see [14].

An inadequate road infrastructure creates traffic jams, which have undoubtedly harmful consequences and negative effects on the environment. Therefore, there is an agreement between transportation policy and environment policy since they both tend to avoid and diminish as much as possible of the traffic jams.

In this study we quantify the relative energy effects of intense traffic and the energy costs of construction and maintenance of roads, and then compare these energies: Higher energy costs of construction and maintenance of roads imply lower energy effects of intense traffic, while lower energy costs of construction and maintenance of roads imply higher energy effects of intense traffic. The balance is typically not easy to find.

Our goal is to minimize the life cycle energy consumption of the construction and maintenance of the roads by controlling the capacity of the transportation system. So, the problem can be seen as an optimal control problem which will be solved using Pontryagin maximum principle.

*Mathematics Subject Classifications: 49J15, 90B30.

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2 Problem Formulation and Modelling

Consider a road section of capacity $x(t)$ (veh·km/h) in a transportation system. Let the function $y(t)$ (veh·km/h) forecast the transport demand of the road. The system can be improved by a production rate $p(t)$ (veh·km/(yr·h)). The system also looses quality at a deterioration rate $\theta(t)$ (yr^{-1}). This deterioration can be neutralized by conducting maintenance at rate $\mu(t)$ (yr^{-1}). It follows that the system capacity evolves according to the following dynamics

$$\dot{x} = p(t) - [\theta(t) - \mu(t)]\, x(t). \tag{1}$$

Let us introduce the control variable $u(t) = \mu(t)x(t)$ representing the maintenance activity. The above equation becomes

$$\dot{x} = p(t) + u(t) - \theta(t)x(t). \tag{2}$$

Given a constant discount rate $\rho \geq 0$ and a fixed time horizon $T > 0$, we assume in the sequel, that the deterioration and discount rates satisfy $\theta(t) \geq \rho$. This is the case if for example the value of money is not discounted, i.e., $\rho = 0$. The model we are considering here is represented as an optimal control problem with one state variable (the system capacity x) and two control variables (the production rate and maintenance activity p and u, respectively). It aims at minimizing the following objective function representing the time discounted life cycle energy use of the system:

$$J = \int_0^T e^{-\rho t}\, [E(p) + K(x,u) + G(x,y)]\, dt, \tag{3}$$

subject to the state equation (2). The three utility functions E, K, and G represent the energy use (measured in MJ) of respectively the production phase of the infrastructure, the maintenance and operation phase of the infrastructure, and the operation phase of the vehicles. These functions are all nonnegative.

- The function $E(p)$ represents the energy use for the construction of a new capacity and has the form
$$E(p) = \alpha p, \quad \alpha > 0.$$
- The function $K(x,u)$ represents the energy use for the maintenance of an existing capacity and has the form
$$K(x,u) = \beta u, \quad \beta > 0.$$
The linear form of E and K is common in the literature. It is also customary to take $\alpha = \beta$ when considering a traditional road construction, see for example [1, 4, 14].
- The function $G(x,y)$ represents the energy use of the traffic related to the vehicular fuel consumption and has the form
$$G(x,y) = g(v(x,y))y,$$
where g is the relative energy use of the traffic and v (km/h) is the velocity, see [14, 21].

Velocity estimation: An important parameter in determining the velocity v is the actual flux $\Phi(t)$ (veh/h) on the road section. It represents the actual number of vehicles that are passing one point in a certain amount of time and is given by, see [14]

$$\Phi(t) = \frac{y(t)}{l}, \tag{4}$$

where l is the length of the road. It is assumed that all vehicles are queued and so the vehicles have to maintain a safe distance between them to avoid collisions. The safe distance is of the form $D = c_1 + c_2 v$. This is used to give an explicit form of the flux in terms of the velocity as follows

$$\Phi(v) = \frac{vw}{c_1 + c_2 v}, \quad 0 \leq v \leq \hat{v}. \tag{5}$$

Here w represents the number of lanes in the road and $\hat{v}$ is the optimum velocity, (see [13, 14]). The constant c_1 is the minimum distance between vehicles, and the constant c_2 is the reaction time of an individual driver, thus $c_2 v$ is the approximate safe distance to avoid collisions. According to the literature, see [13, 14], the flux attains its maximum $\hat{\Phi}$ at the optimum velocity $\hat{v}$, i.e.,

$$\max \Phi(v) = \Phi(\hat{v}). \tag{6}$$

The optimum velocity depends on the architecture of the road. In [13], it lies between 50 km/h and 75 km/h and in [14] the author takes $\hat{v} = 60$ km/h. Using Equation (6) to compute the value of c_2, we have

$$c_2 = \frac{w}{\hat{\Phi}} - \frac{c_1}{\hat{v}}. \tag{7}$$

Now from (5) and (7), we have

$$v(\Phi) = \frac{\hat{v}\hat{\Phi}c_1\Phi}{w\hat{v}\hat{\Phi} + \Phi\left(c_1\hat{\Phi} - w\hat{v}\right)}. \tag{8}$$

The dependence of v on y follows easily from (4) and (8), while its dependence on x depends on $\hat{\Phi}$ which may or not depend on the capacity x of the road (see for instance [14]).

3 Problem Solution

The main tool in the study of the optimal control problem

$$(\mathcal{P}) \qquad \begin{cases} \displaystyle\min_{p,u} \int_0^T e^{-\rho t}\left[E(p) + K(x,u) + G(x,y)\right] dt, \\ \dot{x} = p(t) + u(t) - \theta(t)x(t), \quad x(0) = x_0, \\ 0 \leq p(t) \leq p_{\max}, 0 \leq u(t) \leq u_{\max}, \end{cases}$$

is to determine the necessary optimality conditions using the Pontryagin maximum principle [20] as follows: if the triplet (p^*, u^*, x^*) is an optimal solution for problem $(\mathcal{P})$ then there exists a continuous and piecewise continuously differentiable function λ such that

$$H(t, x^*(t), p^*(t), u^*(t), \lambda(t)) \geq H(t, x^*(t), p(t), u^*(t), \lambda(t)), \text{ for } 0 \leq p(t) \leq p_{\max} \quad (9)$$

$$H(t, x^*(t), p^*(t), u^*(t), \lambda(t)) \geq H(t, x^*(t), p^*(t), u(t), \lambda(t)), \text{ for } 0 \leq u(t) \leq u_{\max} \quad (10)$$

$$-\dot{\lambda}(t) = \frac{\partial}{\partial x} H(t, x^*(t), p^*(t), u^*(t), \lambda(t)), \quad (11)$$

$$x^*(0) = x_0, \qquad \lambda(T) = 0, \quad (12)$$

where

$$\begin{aligned} H(t,x,p,u,\lambda) &= -e^{-\rho t}\left[E(p) + K(x,u) + G(x,y)\right] + \lambda(t)\left\{p(t) + u(t) - \theta(t)x(t)\right\} \\ &= \left[\lambda(t) - \alpha e^{-\rho t}\right]\left[p(t) + u(t)\right] - \gamma e^{-\rho t} v(x(t), y(t))y(t) - \lambda(t)\theta(t)x(t) \end{aligned}$$

Because the Hamiltonian is linear in p and u, the form of the optimal control (p^*, u^*), i.e., the one that would satisfy (9) and (10), is

$$(p^*(t), u^*(t)) \in \begin{cases} \{(p_{\max}, u_{\max})\}, & \text{if } \lambda(t) > \alpha e^{-\rho t}, \\ [0, p_{\max}] \times [0, u_{\max}], & \text{if } \lambda(t) = \alpha e^{-\rho t}, \\ \{(0,0)\}, & \text{if } \lambda(t) < \alpha e^{-\rho t}. \end{cases} \quad (13)$$

To find λ, we make use of Equation (11) to get

$$\dot{\lambda}(t) = -\gamma e^{-\rho t} y(t) \frac{\partial}{\partial x} v(x(t), y(t)) - \lambda(t)\theta(t). \quad (14)$$

It is well known that the concavity of the Hamiltonian H with the respect to the state variable x implies that the necessary optimality conditions become sufficient. Therefore, when the velocity v is convex in x, the necessary optimality conditions (9)-(12) become sufficient for (p^*, u^*, x^*) to be an optimal solution for $(\mathcal{P})$.

In order to further develop the solution, we need at this point, an explicit form for the velocity v in terms of x and y.

3.1 State Independent Velocity

For simplicity, we first assume that the maximum flow $\hat{\Phi}$ is constant which makes the velocity v independent of the capacity x. In this case, Equation (14) can be rewritten as

$$\dot{\lambda}(t) + \theta(t)\lambda(t) = 0, \quad \lambda(T) = 0. \quad (15)$$

The solution of this differential equation is $\lambda(t) = 0$, for all $t \in [0, T]$. Hence by (13), the optimal control is $(p^*, u^*) = (0, 0)$.

3.2 State Dependent Velocity

Now assume that the maximum flow $\hat{\Phi}$ depends on the capacity x as follows, see [14]

$$\hat{\Phi} = \frac{x}{l}. \tag{16}$$

The velocity v becomes dependent on x and Equations (4), (8), and (16) yield

$$v(x(t), y(t)) = \frac{\hat{v}c_1 x(t)y(t)}{lw\hat{v}x(t) + y(t)\left[c_1 x(t) - lw\hat{v}\right]}. \tag{17}$$

In this case, Equation (14) can be rewritten as

$$\dot{\lambda}(t) + \theta(t)\lambda(t) = -c_1 lw \left(\frac{\hat{v}y(t)}{\left[lw\hat{v} + c_1 y(t)\right] x(t) - lw\hat{v}y(t)} \right)^2, \quad \lambda(T) = 0. \tag{18}$$

This equation involves x so that we cannot solve it directly as in the previous case. Because the state equation (2) involves p and u, which depend on λ (see Equation (13)), we also cannot integrate it independently without knowing λ. So, we propose that (p, u) takes its lowest value, i.e., $(0, 0)$ in the interval $[0, T]$ so that from the state equation we have

$$x(t) = x_0 e^{-\int_0^t \theta(s)ds}.$$

Substituting this into Equation (18) gives

$$\dot{\lambda}(t) + \theta(t)\lambda(t) = -c_1 lw \left(\frac{\hat{v}y(t)}{\left[lw\hat{v} + c_1 y(t)\right] x_0 e^{-\int_0^t \theta(s)ds} - lw\hat{v}y(t)} \right)^2, \quad \lambda(T) = 0. \tag{19}$$

Equation (19) can now be solved to obtain

$$\lambda(t) = \int_t^T \varphi(s) e^{\eta(s) - \eta(t)} ds,$$

where

$$\varphi(t) = c_1 lw \left(\frac{\hat{v}y(t)}{\left[lw\hat{v} + c_1 y(t)\right] x(t) - lw\hat{v}y(t)} \right)^2 \quad \text{and} \quad \eta(t) = \int_0^t \theta(s)ds.$$

To compare $\lambda(t)$ with $\alpha e^{-\rho t}$, we put $Q(t) = \lambda(t)/\alpha e^{-\rho t}$. Then

$$\dot{Q}(t) = \frac{e^{\rho t}}{\alpha}\left[-\varphi(t) + (\rho - \theta)\lambda(t)\right].$$

Since $\varphi(t)$ and $\lambda(t)$ are nonnegative functions and by our assumption $\rho \leq \theta$, it is clear that the function $Q(t)$ is always decreasing towards 0. Therefore, if $Q(0)$ is smaller than 1, then the optimal control throughout the interval $[0, T]$ is $(p^*, u^*) = (0, 0)$. This is the case when T is relatively small. For larger values of T, it may happen than $Q(0)$ is larger than 1, in which case there exists some instant $t_0 \in [0, T]$ such that $Q(t) \leq 1$ for all $t_0 \leq t \leq T$. Thus,

$$(p^*(t), u^*(t)) = \begin{cases} \{(p_{\max}, u_{\max})\}, & \text{if } 0 \leq t < t_0, \\ \{(0, 0)\}, & \text{if } t_0 \leq t \leq T. \end{cases} \tag{20}$$

Numerical Example. We illustrate the solution methodology in the case of state dependent velocity. Take for example $\theta(t) \equiv 0.05$ yr^{-1}, $y \equiv 2000$ veh·km/h, $\rho = 0.001$, $\alpha = 45$ MJ·h·veh^{-1}·km^{-1}, $w = 4$ lanes, $l = 1$ km, $x_0 = 10$ veh·km/h, $\hat{v} = 120$ km/h, and $c_1 = 7.5$ m.

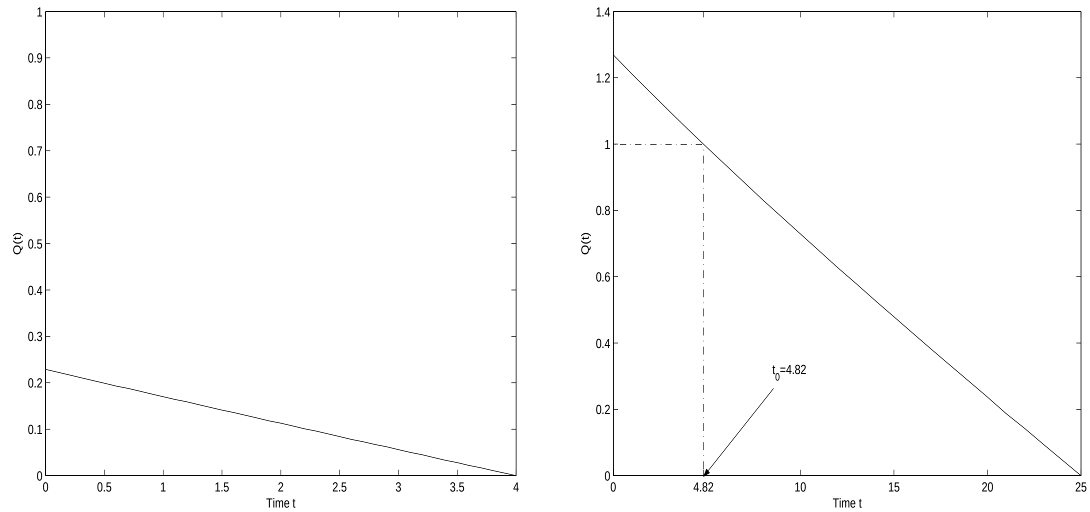

Figure 1: Variations of $Q(t)$ for $T = 4$ (left) and $T = 25$ (right).

For $T = 4$, we take $(p, u) = (0, 0)$ and compute $\lambda(t)$. The graph of the quotient $Q(t)$ in Figure 1 (left) shows that the adjoint function $\lambda(t)$ is smaller than $\alpha e^{-\rho t}$ throughout the interval $[0, T]$, which means that our assumption is satisfied and so by Equation (13), the optimal control is $(p^*, u^*) = (0, 0)$.

For $T = 25$, we take $(p, u) = (0, 0)$ and compute $\lambda(t)$. The graph of the quotient $Q(t)$ in Figure 1 (right) shows that there exists some instant $t_0 = 4.82$ for which we have the adjoint function $\lambda(t)$ is larger than $\alpha e^{-\rho t}$ throughout the interval $[0, t_0]$, and smaller than $\alpha e^{-\rho t}$ throughout the interval $[t_0, T]$, which ensures by Equation (13) that the optimal control is given by (20).

4 Conclusion

To summarize, we used in this paper optimal control theory to determine the optimal way to expand a road in a transportation system subject to deterioration and maintenance. There are many ways to expand this work. For example, the results could be extended to the case of an infinite planning horizon ($T = \infty$). Also, quadratic, instead of linear functions could be used to represent the various energies involved in the objective function. Another possible line of investigation is to represent these energies by non-convex functions. Finally, this model could be dealt with in a stochastic environment.

Acknowledgement. This work was supported by the Research Center at KSU, project number (Stat/24-25/11). The authors would also like to thank the referee for his careful and thorough reading of the paper. His valuable suggestions and critical remarks made numerous improvements throughout.

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