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Uniqueness Of A Solution Of The Cauchy Problem For One-Dimensional Compressible Viscous Micropolar Fluid Model*

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Abstract

The Cauchy problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid is considered. It is assumed that the fluid is thermodynamically perfect and polytropic. This problem has a strong solution on $\mathbf{R}\times]0,T[$ for each $T>0$. We prove that the solution of the Cauchy problem is unique.

1 Statement of the problem and the main result

In this paper we consider nonstationary 1-D flow of a compressible and heat-conducting micropolar fluid. The equations of motion for this fluid are derived from the integral form of conservation laws for polar fluids, under a number of supplementary assumtions such as politropy, Fourier's law, Boyle's law and selection of constitutive equations (see [7] and [6]). The corresponding Cauchy problem has a strong solution on $\Pi=\mathbf{R}\times]0,T[$, for each $T>0$ ([8]). We prove the uniqueness of the solution of this problem.

Let ρ, v, ω and θ denote, respectively, the mass density, velocity, microrotation velocity and temperature of the fluid in the Lagrangean description. Then the problem which we consider has the formulation as follows ([7]):

$$\frac{\partial\rho}{\partial t}+\rho^2\frac{\partial v}{\partial x}=0\ , \tag{1}$$

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left(\rho\frac{\partial v}{\partial x}\right)-K\frac{\partial}{\partial x}\left(\rho\theta\right)\ , \tag{2}$$

$$\rho\frac{\partial\omega}{\partial t}=A\left[\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial\omega}{\partial x}\right)-\omega\right]\ , \tag{3}$$

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$$\rho\frac{\partial\theta}{\partial t} = -K\rho^2\theta\frac{\partial v}{\partial x} + \rho^2\left(\frac{\partial v}{\partial x}\right)^2 + \rho^2\left(\frac{\partial\omega}{\partial x}\right)^2 + \omega^2 + D\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial\theta}{\partial x}\right) \tag{4}$$

in $\mathbf{R}\times\mathbf{R}^+$, where K, A and D are positive constants. The equations (1)-(4) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment and energy. We take the following non-homogeneous initial conditions :

$$\rho(x,0) = \rho_0(x)\ , \tag{5}$$

$$v(x,0) = v_0(x)\ , \tag{6}$$

$$\omega(x,0) = \omega_0(x)\ , \tag{7}$$

$$\theta(x,0) = \theta_0(x) \tag{8}$$

for $x\in\ \mathbf{R}$. We assume that there exist $m, M\in\mathbf{R}^+$, such that

$$0 < m \le \rho_0(x) \le M, \quad m \le \theta_0(x) \le M, \quad x\in\mathbf{R}\ . \tag{9}$$

If the initial functions satisfy conditions (9) and

$$\rho_0 - 1, v_0, \omega_0, \theta_0 - 1 \in H^1(R) \tag{10}$$

then for each $T\in\mathbf{R}^+$ there exists a state function

$$S(x,t) = (\rho, v, \omega, \theta)(x,t) \quad (x,t)\in\Pi = \mathbf{R}\times]0,T[\ , \tag{11}$$

with the properties

$$\rho - 1 \in L^\infty(0,T;H^1(R))\cap H^1(\Pi)\ , \quad \inf_{\Pi}\rho \ge c > 0, \ \ c\in\mathbf{R}^+, \tag{12}$$

$$v, \omega, \theta - 1 \in L^\infty(0,T;H^1(R))\cap H^1(\Pi)\cap L^2(0,T;H^2(R)) \tag{13}$$

which satisfies equations (1)-(4) in the sense of distributions in Π and conditions (5)-(8) in the sense of traces (see Theorem 1.1 in [8]).

We denote by $B^k(R), k\in\ \mathbf{N_0}$, the Banach space

$$B^k(R) = \{u\in C^k(R) : \lim_{|x|\to\infty}|D^n u(x)| = 0,\ 0\le n\le k\}, \tag{14}$$

where D^n is n-th derivative; the norm is defined by

$$\|u\|_{B^k(R)} = \sup_{n\le k}\{\sup_{x\in R}|D^n u(x)|\}. \tag{15}$$

REMARK 1. From Sobolev's embedding theorem ([3, 5, 2]) and theory of vector-valued distributions ([4]) one can conclude that from (12) and (13) it follows:

$$\rho - 1 \in L^\infty(0,T;B^0(R)) \cap C([0,T];L^2(R)), \tag{16}$$

$$v,\omega,\theta - 1 \in L^2(0,T;B^1(R)) \cap C([0,T];H^1(R)) \cap L^\infty(0,T;B^0(R)) \tag{17}$$

and hence

$$v,\omega \in C([0,T];B^0(R)), \quad \rho,\theta \in L^\infty(\Pi). \tag{18}$$

The state function S and its distributional derivatives that occur in (1)-(4) are locally integrable functions in Π and the system (1)-(4) is satisfied a. e. in Π. In other words, state function (11) is a strong solution of the system (1)-(4).
The aim of this paper is to prove the following result.

THEOREM 1. For each $T > 0$, the problem (1)-(10) has in Π at most one solution $S = (\rho, v, \omega, \theta)$ with properties (12)-(13).

2 Proof of Theorem 1.

The proof is very similar to that of Theorem 2.1 in [7] and in [1, Chapter 2].

Let $S_1 = (\rho_1, v_1, \omega_1, \theta_1)$ and $S_2 = (\rho_2, v_2, \omega_2, \theta_2)$ be two distinct solutions of (1)-(10) with properties (12)-(13). We can easily show that for all $\varphi \in L^2(R)$ the function $(\rho, v, \omega, \theta) = (\rho_1 - \rho_2, v_1 - v_2, \omega_1 - \omega_2, \theta_1 - \theta_2)$ satisfies the following system:

$$\int_R [\frac{\partial \rho}{\partial t} + \rho_1^2 \frac{\partial v}{\partial x} + \rho(\rho_1 + \rho_2)\frac{\partial v_2}{\partial x}]\varphi dx = 0 \ , \tag{19}$$

$$\int_R [\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}(\rho_1 \frac{\partial v}{\partial x} + \rho \frac{\partial v_2}{\partial x}) + K \frac{\partial}{\partial x}(\rho_1 \theta + \rho \theta_2)]\varphi dx = 0 \ , \tag{20}$$

$$\int_R [\frac{\partial \omega}{\partial t} - A(\frac{\partial}{\partial x}(\rho_1 \frac{\partial \omega}{\partial x} + \rho \frac{\partial \omega_2}{\partial x}) - \frac{\omega}{\rho_1} + \omega_2 \frac{\rho}{\rho_1 \rho_2})]\varphi dx = 0 \ , \tag{21}$$

$$\begin{aligned} \int_R [\frac{\partial \theta}{\partial t} - D\frac{\partial}{\partial x}(\rho_1 \frac{\partial \theta}{\partial x} + \rho \frac{\partial \theta_2}{\partial x}) + K(\rho_1 \theta \frac{\partial v_1}{\partial x} + \theta_2 \rho \frac{\partial v_1}{\partial x} + \rho_2 \theta_2 \frac{\partial v}{\partial x}) \\ -\rho_1 \frac{\partial v}{\partial x}(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x}) - \rho(\frac{\partial v_2}{\partial x})^2 - \frac{\omega}{\rho_1}(\omega_1 + \omega_2) + \omega_2^2 \frac{\rho}{\rho_1 \rho_2} \\ -\rho_1 \frac{\partial \omega}{\partial x}(\frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial x}) - \rho(\frac{\partial \omega_2}{\partial x})^2]\varphi dx = 0 \end{aligned} \tag{22}$$

in $]0,T[$. From (12)-(13) it follows

$$\rho \in L^\infty(0,T;H^1(R)) \cap H^1(\Pi) \ , \tag{23}$$

$$v,\omega,\theta \in L^\infty(0,T;H^1(R))\cap H^1(\Pi)\cap L^2(0,T;H^2(R)). \tag{24}$$

Evidently, we have

$$\rho(x,0)=v(x,0)=\omega(x,0)=\theta(x,0)=0,\quad x\in\mathbf{R}. \tag{25}$$

From (16) and (17) we conclude that

$$\lim_{|x|\to\infty}\rho(x,t)=\lim_{|x|\to\infty}v(x,t)=\lim_{|x|\to\infty}\omega(x,t)=\lim_{|x|\to\infty}\theta(x,t)=0, \tag{26}$$

$$\lim_{|x|\to\infty}\frac{\partial v}{\partial x}(x,t)=\lim_{|x|\to\infty}\frac{\partial\omega}{\partial x}(x,t)=\lim_{|x|\to\infty}\frac{\partial\theta}{\partial x}(x,t)=0,\quad t\in]0,T[. \tag{27}$$

By $C>0$ we denote a generic constant, having possibly different value at different places. We also use the notation

$$\|f\|=\|f\|_{L^2(R)}.$$

First we replace φ by $\rho(t)$ in (19). Taking into account (18) and applying Young's inequality we get

$$\frac{d}{dt}\|\rho(t)\|^2\leq C[1+\sup_R|\frac{\partial v_2}{\partial x}|^2(\tau)]\|\rho(t)\|^2+\|\frac{\partial v}{\partial x}(t)\|^2. \tag{28}$$

Integrating over $]0,t[$ $(t\in]0,T[)$, using (25) and (17), after application of the Gronwall inequality we obtain

$$\|\rho(t)\|^2\leq C\int_0^t\|\frac{\partial v}{\partial x}(\tau)\|^2d\tau. \tag{29}$$

For $\varphi=v(t)$ from (20) it follows

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2+\int_R\rho_1(\frac{\partial v}{\partial x})^2dx=-\int_R\rho\frac{\partial v_1}{\partial x}\frac{\partial v}{\partial x}dx+K\int_R(\rho_1\theta+\rho\theta_2)\frac{\partial v}{\partial x}dx. \tag{30}$$

Integrating again over $]0,t[$ and using (12), (18), (25) and Hölder's inequality, we have

$$\|v(t)\|^2+\int_R\|\frac{\partial v}{\partial x}(\tau)\|^2d\tau\leq C\int_0^t[(1+\sup_R|\frac{\partial v_2}{\partial x}|(\tau))\|\rho(\tau)\|\|\frac{\partial v}{\partial x}(\tau)\|$$
$$+\|\theta(\tau)\|\|\frac{\partial v}{\partial x}(\tau)\|]d\tau \tag{31}$$

or applying (29) and Young's inequality with a parameter $\varepsilon_i>0(i=1,2)$, we find that

$$\|v(t)\|^2+\int_0^t\|\frac{\partial v}{\partial x}(\tau)\|^2d\tau\leq(\varepsilon_1+\varepsilon_2)\int_0^t\|\frac{\partial v}{\partial x}(\tau)\|^2d\tau$$
$$+C\int_0^t[(1+\sup_R|\frac{\partial v_1}{\partial x}|(\tau))^2(\|v(\tau)\|^2+\int_0^\tau\|\frac{\partial v}{\partial x}(s)\|^2ds)+\|\theta(\tau)\|^2]d\tau. \tag{32}$$

Taking into account $\varepsilon_i (i = 1,2)$ sufficiently small and using Gronwall's inequality, from (32) we obtain

$$\|v(t)\|^2 + \int_0^t \|\frac{\partial v}{\partial x}(\tau)\|^2 d\tau \leq C \int_0^t \|\theta(\tau)\|^2 d\tau \tag{33}$$

Now we replace φ in (21) and (22), respectively, by $\omega(t)$ and $\theta(t)$ and integrate by parts over $\mathbf{R}$. With the help of (12), (18), (25) and the Young inequality we get

$$\|\omega(t)\|^2 + \int_0^t \|\frac{\partial \omega}{\partial x}(\tau)\|^2 d\tau \leq C \int_0^t [\|\omega(\tau)\|^2 + \int_0^\tau \|\frac{\partial \omega}{\partial x}(s)\|^2 ds + (1 + (\sup_R |\frac{\partial \omega_2}{\partial x}|)^2(\tau))\|\rho(\tau)\|^2] d\tau + \varepsilon_1 \int_0^t \|\frac{\partial \omega}{\partial x}(\tau)\|^2 d\tau \tag{34}$$

and

$$\|\theta(t)\|^2 + \int_0^t \|\frac{\partial \theta}{\partial x}(\tau)\|^2 d\tau \leq \int_0^t A(\tau)[\|\theta(\tau)\|^2 + \int_0^\tau \|\frac{\partial \theta}{\partial x}(s)\|^2 ds] d\tau + \int_0^t B(\tau)\|\rho(\tau)\|^2 d\tau + \varepsilon_2 \int_0^t \|\frac{\partial \theta}{\partial x}(\tau)\|^2 d\tau + C \int_0^t [\|\frac{\partial v}{\partial x}(\tau)\|^2 + \|\omega(\tau)\|^2 + \|\frac{\partial \omega}{\partial x}(\tau)\|^2] d\tau \tag{35}$$

where

$$A(\tau) = C[1 + (\sup_R |\frac{\partial v_1}{\partial x}|)^2 + (\sup_R |\frac{\partial v_2}{\partial x}|)^2 + (\sup_R |\frac{\partial \omega_2}{\partial x}|)^2 + (\sup_R |\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x}|)^2 + (\sup_R |\frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial x}|)^2], \tag{36}$$

$$B(\tau) = C[1 + (\sup_R |\frac{\partial \theta_2}{\partial x}|)^2 + (\sup_R |\frac{\partial v_2}{\partial x}|)^2 + (\sup_R |\frac{\partial v_1}{\partial x}|)^2 + (\sup_R |\frac{\partial \omega_2}{\partial x}|)^2]. \tag{37}$$

Taking into account (17), (29), (33) and $\varepsilon_i (i = 1,2)$ sufficiently small, with the help of the Gronwall inequality from (34) and (35) we obtain

$$\|\omega(t)\|^2 + \int_0^t \|\frac{\partial \omega}{\partial x}(\tau)\|^2 d\tau \leq C \int_0^t \|\theta(\tau)\|^2 d\tau, \tag{38}$$

$$\|\theta(t)\|^2 + \int_0^t \|\frac{\partial \theta}{\partial x}(\tau)\|^2 d\tau \leq C \int_0^t [\|\frac{\partial v}{\partial x}(\tau)\|^2 d\tau + \|\omega(\tau)\|^2 + \|\frac{\partial \omega}{\partial x}(\tau)\|^2] d\tau \leq C \int_0^t [\|\theta(\tau)\|^2 + \int_0^\tau \|\frac{\partial \theta}{\partial x}(s)\|^2 ds] d\tau. \tag{39}$$

Using again Gronwall's inequality from (39) we find that $\theta = 0$ and from (38), (33) and (29) we get immediately

$$\rho = v = \omega = 0. \tag{40}$$

References

[1] S. N. Antontsev, A. V. Kazhikhov and V. N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, North-Holland, 1990.

[2] H. Brezis, Analyse fonctionnelle, Masson, Paris, 1983.

[3] R. Dautray and J. L. Lions, Mathematical Analysis and Numerical Methods for Science and Techonology, Vol.2, Springer-Verlag, Berlin, 1988.

[4] R. Dautray and J. L. Lions, Mathematical Analysis and Numerical Methods for Science and Techonology, Vol.5, Springer-Verlag, Berlin, 1992.

[5] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol.1, Springer-Verlag, Berlin, 1972.

[6] G. Lukaszewicz, Micropolar Fluids: Theory and Applications, Birkh äuser, Boston, 1999.

[7] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: a local existence theorem, Glasnik Matematički 33(53) (1998),71–91.

[8] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: The Cauchy problem, Mathematical Communications 10(2005), 1–14.