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Asymptotic Position And Shape Of The Limit Cycle In A Cardiac Rheodynamic Model*

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Abstract

In this paper, we estimate the relative asymptotic position and shape of the limit cycle in a rheodynamic model of cardiac pressure pulsations and prove that when the bifurcation parameters are sufficiently large, the limit cycle will be asymptotically close to a rectangle.

1 Introduction

In [2], based on rheodynamic principles, Petrov and Nikolov presented a mathematical model for cardiac pressure pulsations in the form of a two-dimensional autonomous system. In [1], Feng and Liu proved the existence, uniqueness of the limit cycle in this system and determined its stability, and also obtained the global parameter bifurcation diagram (see also [3] and [4]). In this paper, we estimate the relative asymptotic position and shape of the limit cycle in the system presented in [2], and prove that when the bifurcation parameters are sufficiently large, the limit cycle will be asymptotically close to a rectangle. The parameters in the system have biological interpretations. Although the system may lose its biological meaning for large parameters, the limit cycle represents its asymptotic behavior, and hence our discussion is meaningful in certain extreme circumstances.

The model proposed in [2] can be expressed in the form

$$\begin{cases} \frac{d\zeta}{dt} = \xi - e, \\ \frac{d\xi}{dt} = [a - b(\xi - c - e)^2](\xi - c - e) - g\zeta, \end{cases}$$

where a, b, c, e, g are positive parameters, ζ is the dimensionless blood volume momentum, ξ is the dimensionless left ventricular pressure, and t is the time. Let $x = \zeta - \zeta_s$, $y = \xi - \xi_s$ and $\lambda = a - 3bc^2$ where $\zeta_s = c(bc^2 - a)/g$ and $\xi_s = e$. Then we have

$$\begin{cases} \dot{x} = y = P(x, y), \\ \dot{y} = -gx + \lambda y + 3bcy^2 - by^3 = Q(x, y). \end{cases} \tag{1}$$

By [1], we know that if $\lambda > 0$, then system (1) has a unique limit cycle Γ which is asymptotically stable; and if $\lambda \leq 0$, then system (1) does not have any limit cycles.

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2 A Parallelogram With Vertices On The Horizontal Isocline

Let A and B be functions of λ in a neighborhood of $+\infty$. We write $A \sim B$ if $A/B \to 1$ as $\lambda \to +\infty$. The horizontal isocline of system (1) is

$$H : x = h(y) = \frac{\lambda y + 3bcy^2 - by^3}{g}. \tag{2}$$

H has two knee points $L(x_-, y_-)$ and $R(x_+, y_+)$ (i.e. the curve H has vertical tangents at these two points), where $x_\pm = h(y_\pm)$ and $y_\pm$ defined by

$$y_\pm = \frac{6bc \pm \sqrt{36b^2c^2 + 12b\lambda}}{6b} \sim \pm\sqrt{\frac{\lambda}{3b}} \tag{3}$$

are the negative and positive roots respectively of the equation

$$x'_y = 3by^2 - 6bcy - \lambda = 0. \tag{4}$$

Since

$$3by_\pm^2 - 6bcy_\pm - \lambda = 0,\ y_\pm^2 = 2cy_\pm + \frac{\lambda}{3b},\ by_\pm^3 = 2bcy_\pm^2 + \frac{\lambda}{3}y_\pm, \tag{5}$$

we have

$$\lambda y_\pm + 3bcy_\pm^2 - by_\pm^3 = \frac{1}{3}c\lambda + 2bc^2y_\pm + \frac{2}{3}\lambda y_\pm, \tag{6}$$

so that

$$x_\pm = \frac{1}{g}(\frac{1}{3}c\lambda + 2bc^2y_\pm + \frac{2}{3}\lambda y_\pm) \sim \pm\frac{2\lambda\sqrt{\lambda}}{3g\sqrt{3b}}. \tag{7}$$

Draw a straight line through L parallel to the y-axis, intersecting the curve H at $\overline{L}(x_-, \overline{y})$. Through R draw a straight line parallel to the y-axis, intersecting H at another point $\underline{R}(x_+, \underline{y})$ (see Figure 1). Now, we compute $\overline{y}$ and $\underline{y}$. Since L and $\overline{L}$ are all on the curve H, we see that $h(y) = h(y_-)$ or

$$by^3 - 3bcy^2 - \lambda y - (by_-^3 - 3bcy_-^2 - \lambda y_-) = 0, \tag{8}$$

furthermore, y_- also satisfies the equation $h'(y) = 0$. Thus y_- is the double root of $h(y) - h(y_-) = 0$. We may compute

$$h(y) - h(y_-) = b(y - y_-)^2(y + 2y_- - 3c).$$

Then the other root of the above equation is $y = 3c - 2y_-$ and therefore

$$\overline{y} = 3c - 2y_- \sim 2\sqrt{\frac{\lambda}{3b}} > 3c. \tag{9}$$

Similarly, we may obtain

$$\underline{y} = 3c - 2y_+ \sim -2\sqrt{\frac{\lambda}{3b}}. \tag{10}$$

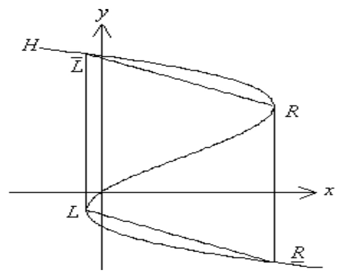

Figure 1.

LEMMA 1. The quadrilateral $\overline{L}R\underline{R}L$ is a parallelogram.

PROOF. By the definition of $\overline{L}$ and $\underline{R}$, $\overline{L}L$ and $R\underline{R}$ are parallel to the y-axis, so $y_+ + y_- = 2c$. In view of (9), we have

$$\overline{L}L = \bar{y} - y_- = 3c - 2y_- - y_- = 3c - 3y_- = 3y_+ - 3c = y_+ - (3c - 2y_+) = y_+ - \underline{y} = R\underline{R},$$

which shows that the quadrilateral $\overline{L}R\underline{R}L$ is a parallelogram.

LEMMA 2. Let k denote the slope of $\overline{L}R$. Then $k < 0$ and $k \to 0$ as $\lambda \to \infty$.

PROOF. Since

$$k = \frac{y_+ - \bar{y}}{x_+ - x_-} = -\frac{3c - 2y_- - y_+}{x_+ - x_-},$$

by (7),

$$x_+ - x_- = \frac{2(\lambda + 3bc^2)(y_+ - y_-)}{3g}, \tag{11}$$

and

$$k = -\frac{3g}{2(\lambda + 3bc^2)} \cdot \frac{3c - 2y_- - y_+}{y_+ - y_-} = -\frac{3g}{4(\lambda + 3bc^2)} < 0, \tag{12}$$

where we have used (3) in obtaining the second equality. Therefore $k \to 0$ as $\lambda \to +\infty$.

LEMMA 3. $\frac{\overline{L}R}{\overline{L}L} \sim \frac{4(\lambda+3bc^2)}{9g}$ as $\lambda \to +\infty$.

PROOF. By (11) and (12), we know $x_+ - x_- = \frac{1}{2|k|}(y_+ - y_-)$, and by (3) and (10), $\frac{y_+ - y_-}{y_+ - \underline{y}} = \frac{2}{3}$. Thus

$$\begin{aligned}
\frac{\overline{L}R}{\overline{\underline{L}}L} &= \frac{\sqrt{(x_+ - x_-)^2) + (y_+ - \bar{y})^2)}}{y_+ - \underline{y}} = \frac{\sqrt{(x_+ - x_-)^2) + k^2(x_+ - x_-)^2)}}{y_+ - \underline{y}} \\
&= \frac{\sqrt{1+k^2}(x_+ - x_-)}{y_+ - \underline{y}} = \frac{\sqrt{1+k^2}}{2|k|} \cdot \frac{y_+ - y_-}{y_+ - \underline{y}} = \frac{2}{3} \cdot \frac{\sqrt{1+k^2}}{2|k|} \\
&= \frac{\sqrt{1+k^2}}{3|k|} = \frac{\sqrt{16(\lambda + 3bc^2)^2 + 9g^2}}{9g} \sim \frac{4(\lambda + 3bc^2)}{9g}.
\end{aligned}$$

3 Internal And External Boundary Lines

We will find the internal and external boundary lines for system (1).

LEMMA 4. When λ is sufficiently large, the orbits of system (1) pass through the line segment $y = \sqrt{\lambda}(x - x_+)$, where $\underline{y}^* \le y \le 0$, from its left-top to its right-bottom, and pass through $y = \sqrt{\lambda}(x - x_-)$, where $0 \le y \le \overline{y}^*$, from its right-bottom to its left-top. Here

$$\underline{y}^* = \frac{gx_+\underline{y}}{gx_+ - (\sqrt{\lambda} + \frac{g}{\lambda})\underline{y}} > \underline{y},\ \overline{y}^* = \frac{-gx_-\overline{y}}{-gx_- + (\sqrt{\lambda} + \frac{g}{\lambda})\overline{y}} < \overline{y},$$

and, $\underline{y}^* \to \underline{y}$, $\overline{y}^* \to \overline{y}$ as $\lambda \to +\infty$.

PROOF. The normal vector of the line $y = \sqrt{\lambda}(x - x_+)$ from its left-top to right-bottom is $n = (\sqrt{\lambda}, -1)$. The inner product of (P, Q) and n is

$$F = (P, Q) \bullet n = \sqrt{\lambda}y + gx - \lambda y - 3bcy^2 + by^3 = gx_+ + (\sqrt{\lambda} + \frac{g}{\sqrt{\lambda}}) + by^3 - 3bcy^2 - \lambda y.$$

It is easy to show that the function $by^3 - 3bcy^2$ is monotone increasing and convex in $[\underline{y}, 0]$, so the line segment $(b\underline{y}^2 - 3bc\underline{y})y$ is under the curve $by^3 - 3bcy^2$. Therefore we have

$$by^3 - 3bcy^2 \ge (b\underline{y}^2 - 3bc\underline{y})y,\ \ \underline{y} \le y \le 0,$$

and,

$$F \ge (b\underline{y}^2 - 3bc\underline{y} + \sqrt{\lambda} + \frac{g}{\sqrt{\lambda}} - \lambda)y + gx_+.$$

When

$$\begin{aligned}
y &\ge -\frac{gx_+}{b\underline{y}^2 - 3bc\underline{y} + \sqrt{\lambda} + \frac{g}{\sqrt{\lambda}} - \lambda} = -\frac{gx_+\underline{y}}{b\underline{y}^3 - 3bc\underline{y}^2 - \lambda\underline{y} + (\sqrt{\lambda} + \frac{g}{\sqrt{\lambda}})\underline{y}} \\
&= \frac{gx_+\underline{y}}{gx_+ - (\sqrt{\lambda} + \frac{g}{\sqrt{\lambda}})\underline{y}} = \underline{y}^*,
\end{aligned}$$

$F \geq 0$. Obvious $\underline{y}^* > \underline{y}$ and $\underline{y}^* \to \underline{y}$ as $\lambda \to +\infty$. This proves the first part of Lemma 4. Similarly ,we can prove the other part of Lemma 4.

Now we construct a closed curve $\underline{\Sigma} : \underline{A_1}\underline{A_2}\underline{A_3}L\underline{A_4}\underline{A_5}\underline{A_6}R$ as demonstrated in Figure 2. Draw a straight line through R parallel to the y-axis intersecting the positive x-axis at point $\underline{A_1}(x_+, 0)$, and through A_1 to the left-bottom draw a straight line $y = \sqrt{g}(x - x_+)$ intersecting the line $y = \underline{y}^*$ at $\underline{A_2}(x_2, \underline{y}^*)$, where $x_2 = \underline{y}^*/\sqrt{g} + x_+$, through A_2 to left draw a straight line $y = \underline{y}^*$ intersecting the horizontal isocline H at the point $\underline{A_3}(\underline{x_3}, \underline{y}^*)$. Along H from $\underline{A_3}$ to $\underline{L}$ draw a curve, and though L draw a straight line parallel to the y-axis intersecting the x-axis at $\underline{A_4}(x_-, 0)$, through A_4 to the right-top draw a straight line $y = \sqrt{g}(x - x_-)$ intersecting the line $y = \overline{y}^*$ at $\underline{A_5}(x_5, \overline{y}^*)$, where $x_5 = \overline{y}^*/\sqrt{g} + x_-$, through A_5 to the right draw a straight line $y = \overline{y}^*$ intersecting the horizontal isocline H at the point $\underline{A_6}(\underline{x_6}, \overline{y}^*)$. Along H from $\underline{A_6}$ to R draw a curve.

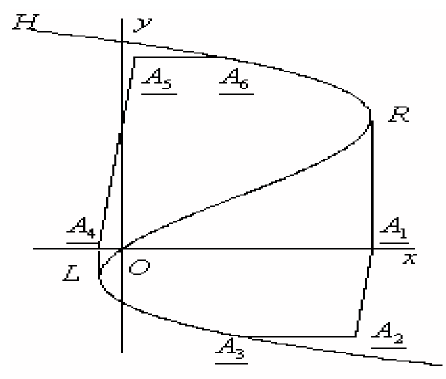

Figure 2.

LEMMA 5. If λ is sufficiently large, then the orbits of system (1) will cross over the closed curve $\underline{\Sigma}$ from its interior to the exterior, so the closed curve $\underline{\Sigma}$ forms the internal boundary line of system (1).

PROOF. Since the point $\underline{A_2}$ on the line segment $\underline{A_1 A_2}$ is above $\underline{R}$, so $\underline{y} \leq y \leq 0$. By Lemma 3, we know that on $\underline{A_1 A_2}$ the orbits of system (1) pass through $\underline{\Sigma}$ from its interior to the exterior. On the line segment $\underline{A_2 A_3}$, the outward normal vector of $\underline{\Sigma}$ is n = (0, -1), and $\dot{y} \leq 0$ (because the line segment $\underline{A_2 A_3}$ is above the horizontal isocline), so the inner product between (P,Q) and n is $F = (P, Q) \bullet n = -\dot{y} \geq 0$. This means that the orbits of (1) pass through $\underline{\Sigma}$ from its interior to its exterior on the line segment $\underline{A_2 A_3}$. The curved segment $\underline{A_3}L$ is a horizontal isocline, in other words, the vector field defined by system (1) points from the right to the left horizontally, so the trajectories of system (1) pass through $\underline{\Sigma}$ from its interior to its exterior. On the line segment $L\underline{A_4}$, the outward normal vector of $\underline{\Sigma}$ is $n = (-1, 0)$, and $y \leq 0$, so the inner product between (P, Q) and n is $F = (P, Q) \bullet n = -y \geq 0$. This means that the orbits of (1) pass through $\underline{\Sigma}$ from its interior to its exterior on the line segment $L\underline{A_4}$. Similarly, on $\underline{A_4}\underline{A_5}$, $\underline{A_5}\underline{A_6}$,$\underline{A_6}R$, and $R\underline{A_1}$, the trajectories of system (1) pass through $\underline{\Sigma}$ from its interior to its exterior. This completes the proof.

Let $\epsilon > 0$. Consider two curves parallel to the horizontal isocline $H : x = h(y)$ as follows: $H^+(\epsilon) : x = h(y) + \epsilon$ and $H^-(\epsilon) : x = h(y) - \epsilon$. Let

$$y^{\pm}(\epsilon) = \frac{6bc \pm \frac{1}{\epsilon} \pm \sqrt{(6bc + \frac{1}{\epsilon})^2 + 12b\lambda}}{6b}. \tag{13}$$

Note that $y^+(\epsilon) > y_+$ and $y^-(\epsilon) < y_-$.

LEMMA 6. When $y \geq y^+(\epsilon)$, the orbits of system (1) pass through $H^+(\epsilon)$ from its left-top to its right-bottom; when $y \leq y^-(\epsilon)$, the orbits of system (1) pass through $H^-(\epsilon)$ from its right-bottom to its left-top.

PROOF. The curves $H^-(\epsilon)$, H, $H^+(\epsilon)$ have the same slope k at the point $(x, y-\epsilon)$, (x, y) and $(x, y+\epsilon)$ respectively, where (x, y) belongs to H, and

$$k = y'_x = \frac{1}{x'_y} = \frac{1}{h'(y)} = \frac{g}{\lambda + 6bcy - 3by^2}. \tag{14}$$

The normal vector of $H^+(\epsilon)$ from the left-top to right-bottom is $n^+_\epsilon = (k, -1)$. The inner product between the vector field defined by system (1) and n^+_ϵ is

$$\begin{aligned} F^+_\epsilon &= (P, Q) \bullet n^+_\epsilon = kP - Q = \frac{gy}{\lambda + 6bcy - 3by^2} + gx - \lambda y - 3bcy^2 + by^3 \\ &= g\left(\epsilon - \frac{y}{3by^2 - 6bcy - \lambda}\right). \end{aligned}$$

If $\epsilon \geq y/(3by^2 - 6bcy - \lambda)$, that is, $3by^2 - (6bc + 1/\epsilon)y - \lambda \geq 0$, then $F^+_\epsilon > 0$. Since $y \geq y^+(\epsilon)$ is a solution of the above inequality, the first part of our lemma is true. The other part is similarly proved.

LEMMA 7. Let z_+ and z_- be respectively the positive and negative roots of the equation

$$3by^2 - 6bcy - (\lambda + 2\sqrt{g}) = 0$$

and let

$$M_\pm = \pm\frac{1}{g}\left[\frac{c(\lambda + 2\sqrt{g})}{3} + 2bc^2 z_\pm + \frac{2}{3}(\lambda + 2\sqrt{g})z_\pm\right].$$

If $M \geq M_+$, the orbits of system (1) will pass through the semi-straight line $x + \frac{y}{\sqrt{g}} = M$, where $y \geq 0$, from its left-top to its right-bottom. If $M \geq M_-$, the orbits of system (1) will pass through the semi-straight line $x + \frac{y}{\sqrt{g}} = -M$, where $y \leq 0$, from its right-bottom to its left-top.

PROOF. The normal vector from the left top to right bottom of $x + y/\sqrt{g} = M$ is $n = (-\sqrt{g}, -1)$. The inner product of (P, Q) and n is

$$\begin{aligned} F &= (P, Q) \bullet (-\sqrt{g}, -1) = -\sqrt{g}y + gx - \lambda y - 3bcy^2 + by^3 \\ &= -\sqrt{g}y + g(M - \frac{y}{\sqrt{g}}) - \lambda y - 3bcy^2 + by^3 = by^3 - 3bcy^2 - (\lambda + 2\sqrt{g})y + gM, \end{aligned}$$

and

$$F'_y = 3by^2 - 6bcy - (\lambda + 2\sqrt{g}).$$

Thus the minimal value of $F(y \geq 0)$ is obtained at $y = z_+$. By the definition of z_+, we know

$$3bz_+^2 - 6bcz_+ - (\lambda + 2\sqrt{g}) = 0,\ z_+^2 = 2cz_+ + \frac{\lambda + 2\sqrt{g}}{3b},\ bz_+^3 = 2bcz_+^2 + \frac{\lambda + 2\sqrt{g}}{3} z_+.$$

Thus

$$\begin{aligned} F &\geq bz_+^3 - 3bcz_+^2 - (\lambda + 2\sqrt{g})z_+ + gM_+ \\ &= -\frac{c(\lambda + 2\sqrt{g})}{3} - 2bc^2 z_+ - \frac{2}{3}(\lambda + 2\sqrt{g})z_+ + gM_+ \geq 0. \end{aligned}$$

This proves the first part of our lemma. The other part is similarly proved.

Now, we construct a closed curve $\overline{\Sigma} : \overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5\overline{A}_6\overline{A}_7\overline{A}_8$ as demonstrated in Figure 3. From $\overline{A_1}(M_+, 0)$ draw to the left-top a straight line $x + y/\sqrt{g} = M_+$ intersecting the line $y = y^+(M_+)$ (the value of $y^+(\varepsilon)$ at $\varepsilon = M_+$, see (13) and Lemma 7) at the point $\overline{A}_2(\overline{x}_2, y^+(M_+))$; from $\overline{A}_2$ draw to the left a straight line $y = y^+(M_+)$ intersecting the curve $H^+(M_+)$ (the curve $H^+(\varepsilon)$ when $\varepsilon = M_+$) at the point $\overline{A}_3(\overline{x}_3, y^+(M_+))$; from $\overline{A}_3$ draw to the left-top the curve $H^+(M_+)$ intersecting the line $x = -M_-$ at the point $\overline{A}_4(\overline{x}_4, \overline{y}_4)$; from $\overline{A}_4$ draw to the bottom a straight line $x = -M_-$ intersecting the x-axis at the point $\overline{A}_5(-M_-, 0)$; from $\overline{A}_5$ draw to the right-bottom a straight line $x + y/\sqrt{g} = -M_-$ intersecting the straight line $y = y^-(M_-)$ (the value of $y^-(\varepsilon)$ at $\varepsilon = M_-$ see (13) and Lemma7) at the point $\overline{A}_6(\overline{x}_6, y^-(M_-))$; from $\overline{A}_6$ draw to the right a straight line $y = y^-(M_-)$ intersecting the curve $H^-(M_-)$ (the curve $H^-(\varepsilon)$ when $\varepsilon = M_-$) at the point $\overline{A}_7(\bar{x}_7, y^-(M^-))$; from $\overline{A}_7$ draw down the curve $H^-(M_-)$ intersecting the straight line $x = M_+$ at the point $\overline{A}_8(M_+, \overline{y}_8)$; finally from $\overline{A}_8$ draw a straight line $x = M_+$ to the starting point $\overline{A}_1$.

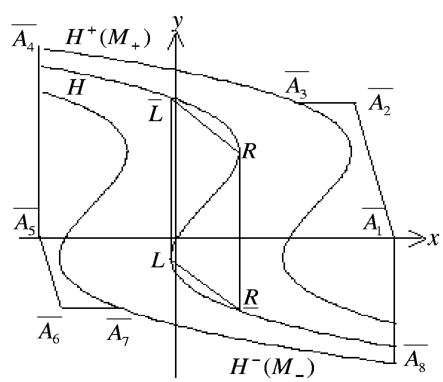

Figure 3.

LEMMA 8. The trajectories of system (1) pass through $\overline{\Sigma}$ from its exterior to its interior, so $\overline{\Sigma}$ forms the external boundary line of system (1).

PROOF. On $\overline{A}_1\overline{A}_2$, by Lemma 7, the orbits of system (1) pass though $\overline{\Sigma}$ from the left-top to the right-bottom, equivalently, from its exterior to its interior. On $\overline{A}_2\overline{A}_3$, the inward normal vector of $\overline{\Sigma}$ is $n = (0, -1)$; the inner product of (P, Q) and n is $F = (P, Q) \bullet n = -Q$. Since the line segment $\overline{A}_2\overline{A}_3$ is above the horizontal isocline, so on $\overline{A}_2\overline{A}_3$ we have $Q < 0$, and therefore $F > 0$. This means that on $\overline{A}_2\overline{A}_3$, the trajectories of system (1) pass through $\overline{\Sigma}$ from its exterior to its interior. On $\overline{A}_3\overline{A}_4$, $y \geq y^+(M_+)$. By Lemma 6, the trajectories pass through $\overline{\Sigma}$ from its exterior to its interior. On $\overline{A}_4\overline{A}_5$, the inward normal vector of Σ is $n = (1, 0)$. The inner product of (P, Q) and n is $F = (P, Q) \bullet n = y$; and on $\overline{A}_4\overline{A}_5$, $y \geq 0$, so $F \geq 0$. This shows that the trajectories of system (1) pass through $\overline{\Sigma}$ from its exterior to its interior. Similarly, we can prove that on $\bar{A}_5\bar{A}_6$, $\overline{A}_6\overline{A}_7$, $\overline{A}_7\overline{A}_8$ and $\overline{A}_8\overline{A}_1$, the orbits of system (1) pass through $\overline{\Sigma}$ from its exterior to its interior. So $\overline{\Sigma}$ forms the internal boundary line of system (1).

4 Asymptotic Behavior Of The Limit Cycle

We are now ready for our main results.

THEOREM 1. As $\lambda \to +\infty$, the limit cycle Γ of system (1) asymptotically approaches a parallelogram $\overline{L}R\underline{R}L$, and the relative error between Γ and $\overline{L}R\underline{R}L$ is of order $1/\sqrt{\lambda}$.

PROOF. We first define the distance from Σ to Γ as follows: For a fixed $M_2 \in \Gamma$, $d(\Sigma, M_2) = \inf\{M_1M_2 | M_1 \in \Sigma\}$ and $\underline{d} = \sup\{d(\Sigma, M_2) | M_2 \in \Gamma\}$. By the construction of the internal boundary line $\underline{\Sigma}$, the slopes of the line segments $\underline{A}_1\underline{A}_2$ and $\underline{A}_3\underline{A}_4$ tend to $+\infty$, so $\underline{A}_2 \to \underline{R}$ and $\underline{A}_4 \to \bar{L}$. Therefore $\underline{d} \to 0$. Now consider the error d between $\overline{\Sigma}$ and Γ. Clearly d is not greater than the error $\overline{d}$ between $\overline{\Sigma}$ and $\underline{\Sigma}$. It is easy to prove that $\overline{d}$ is attained on $\underline{A}_1\overline{A}_1$ or $\overline{A}_5\underline{A}_3$. From the coordinates of $\overline{A}_1$ and $\overline{A}_5$, we know $\overline{A}_5\underline{A}_3 < \underline{A}_1\overline{A}_1$. Through direct computation, we have

$$
\begin{aligned}
g(M_+ - x_+) &= \frac{c(\lambda + 2\sqrt{g})}{3} + 2bc^2 z_+ + \frac{2}{3}(\lambda + 2\sqrt{g})z_+ - \frac{1}{3}c\lambda - 2bc^2 y_+ - \frac{2}{3}\lambda y_+ \\
&= 2c\sqrt{g} + \frac{8bc^2}{\sqrt{36b^2c^2 + 12b(\lambda + 2\sqrt{g})} + \sqrt{36b^2c^2 + 12b\lambda}} \\
&\quad + \frac{24\sqrt{g}\lambda^2 + 48(bc^2\sqrt{g} + g)\lambda + 48bc^2 g + 32g\sqrt{g}}{3(\lambda + 2\sqrt{g})\sqrt{36b^2c^2 + 12b(\lambda + 2\sqrt{g})} + \lambda\sqrt{36b^2c^2 + 12b\lambda}} \\
&\sim 2\sqrt{\frac{g\lambda}{3b}}.
\end{aligned}
$$

Thus $\bar{d} = \underline{A}_1\bar{A}_1 = M_+ - x_+ \sim 2\sqrt{\frac{\lambda}{3bg}}$ which implies $\frac{\bar{d}}{\lambda} \sim \frac{2}{3bg\lambda} \to 0$ as $\lambda \to \infty$. This completes the proof.

We will prove that the order of the relative error of approximation in Theorem 1 is the best possible.

THEOREM 2. The order of relative error of approximation between Γ and the parallelogram $\bar{L}R\underline{R}L$ cannot be higher than $1/\sqrt{\lambda}$.

PROOF. Let $A_1(0, w)$ be the intersecting point of the horizontal isocline $x = h(y)$ and the positive y-axis. Then w satisfies the equation $by^2 - 3bcy - \lambda = 0$. Thus

$$w = \frac{3bc + \sqrt{9b^2c^2 + 4b\lambda}}{2b} \sim \sqrt{\frac{\lambda}{b}}.$$

Consider the straight line segment $A_1A_2 : y - w = -\sqrt{g}x, \quad 0 \le x \le w/(2\sqrt{g})$. Its normal vector from left bottom to right top is $n = (\sqrt{g}, 1)$; the inner product of (P, Q) and n is

$$F = (P, Q) \bullet n = \sqrt{g}y - gx + \lambda y + 3bcy^2 - by^3.$$

It is easy to verify that when $0 \le y \le w$, $\lambda y + 3bcy^2 - by^3 \ge 0$. Therefore, on the line segment, we have

$$F \ge \sqrt{g}y - gx \ge \sqrt{g}(w - \sqrt{g}x) - gx \ge \sqrt{g}w - 2gx \ge 0.$$

This means that the orbits of system (1) pass through from the underside of the line segment A_1A_2 to its upside. Through A_2 construct a straight line parallel to the y-axis intersecting the x-axis at A_3. It is clear that the orbits cross the line segment A_2A_3 from its left to its right. So the orbit γ of system (1) starting from A_1 must intersect the broken line $A_1A_2A_3$ above the x-axis in the first quadrant; however, γ is in the inner part of the limit cycle Γ (because A_1L is in the inner part of the internal boundary line). Therefore, the greatest distance between γ and the parallelogram $\overline{L}R\underline{R}L$, d, is not less than the greatest distance between γ and $\underline{\Sigma}$ (since a part of Σ belongs to the parallelogram $\overline{L}R\underline{R}L$); furthermore, it is not less than the greatest distance between γ and $\underline{\Sigma}$. So

$$d \ge \underline{d} \ge OA_3 = \frac{w}{2\sqrt{g}} \sim \sqrt{\frac{\lambda}{b}} \text{ and } \frac{d}{\lambda} \ge \frac{\overline{d}}{\lambda} \ge \frac{OA_3}{\lambda} \sim \frac{1}{\sqrt{6\lambda}}.$$

This completes the proof of Theorem 2.

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