

Solving A System Of Second Order Obstacle Problems By A Modified Decomposition Method*

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Abstract

In this paper a numerical algorithm, based on the Adomian decomposition method and a modified form of this method, is presented for solving a system of second-order boundary value problems associated with obstacle, unilateral, and contact problems. The scheme is shown to be highly accurate, and only a few terms are required to obtain accurate computable solutions.

1 Introduction

In this paper, we use Adomian decomposition method (in short ADM) [1, 2] for obtaining approximate solutions of a system of second-order boundary value problem of the type

$$u'' = \begin{cases} f(x), & a \leq x \leq c, \\ p(x)u(x) + f(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \quad (1)$$

with the boundary conditions

$$u(a) = \alpha_1 \quad \text{and} \quad u(b) = \alpha_2, \quad (2)$$

and the continuity conditions of u and u' at c and d . Here, f and p are continuous functions on $[a, b]$ and $[c, d]$, respectively. The parameters r, α_1 and α_2 are real finite constants. Such type of systems arise in the study of obstacle, unilateral, moving and free boundary value problems and has important applications in other branches of pure and applied sciences, see for example [3-15] and the references therein.

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In 1981, Villaggio [6] used the classical Rayleigh-Ritz method for solving a special form of (1), namely,

$$u'' = \begin{cases} 0, & 0 \leq x \leq \frac{\pi}{4}, \\ u(x) - 1, & \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \\ 0, & \frac{3\pi}{4} \leq x \leq \pi, \end{cases} \quad (3)$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(\pi) = 0, \quad (4)$$

and the continuity conditions of u and u' at $\pi/4$ and $3\pi/4$. After this, Noor and Khalifa [8] have solved problem (3) using collocation method with cubic splines as basis functions. They have shown that this collocation method gives approximate solutions with first-order accuracy. Similar conclusions were pointed out by Noor and Tirmizi [9], Al-Said [10], and Al-Said et al [11], where second- and fourth-order finite difference and spline methods were used to solve problem (1). On the other hand, A-Said [12, 13] has developed and analyzed quadratic and cubic splines methods for solving (1). He proved that both quadratic and cubic spline methods can be used to produce second-order smooth approximations for the solution of (1) and its first derivative over the whole interval $[a, b]$. More recently Al-Said [14] has used cubic spline polynomial functions for solving such a type of second-order system of differential equations associated with obstacle and unilateral problems.

The Adomian decomposition method [1, 2] will be effectively used to approach problem (1) with two point boundary conditions. The Adomian algorithm assumes a series solution for the unknown function $u(x)$. The boundary conditions will be imposed on various approximants of the obtained series solution to complete the determination of the remaining constants. The Adomian decomposition method has many advantages over the classical techniques mainly, it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations.

The convergence of the decomposition series has been investigated by several authors, see [16-19]. They obtained some results about the speed of convergence of this method. Abbaoui and Cherruault [19] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [19].

The paper is organized as follows. In section 2 we extend application of the decomposition method to construct our numerical solutions for of the second-order boundary value problem. In section 3, we present a numerical experiment to illustrate the efficiency and simplicity of the method.

2 Decomposition Method

In this section, we outline the steps to obtain approximate solution of (1) using the ADM. To begin, it is convenient to rewrite the second-order boundary value problem

in the standard operator from

$$L_x u = \begin{cases} f(x), & a \leq x \leq c, \\ p(x)u(x) + f(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \quad (5)$$

where the differential operator L_x is given by

$$L_x = \frac{d^2}{dx^2}. \quad (6)$$

Equation (5) can be expressed in terms of unit step function as

$$L_x u = f(x)(y(x-a) - y(x-c)) + (p(x)u(x) + f(x) + r)(y(x-c) - y(x-d)) + f(x)(y(x-d) - y(x-b)), \quad (7)$$

where $y(x)$ is the Heaviside function.

The inverse operator L_x^{-1} is therefore considered a two-fold integral operator defined by

$$L_x^{-1}(\cdot) = \int_a^x \int_a^x (\cdot) dx dx \quad (8)$$

Operating with L_x^{-1} on (7) and using the boundary conditions at $x = b$ yields

$$u(x) = \alpha_1 + A(x-a) + L_x^{-1} \left[f(x)(y(x-a) - y(x-c)) + (p(x)u(x) + f(x) + r)(y(x-c) - y(x-d)) + f(x)(y(x-d) - y(x-b)) \right] \quad (9)$$

where the constant

$$A = u(b), \quad (10)$$

will be determined later by using the boundary condition at $x = b$.

Now, we decompose the unknown function $u(x)$ a sum of components defined by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \quad (11)$$

The zeroth component is usually taken to be all terms arise from the initial conditions and the integration of the source term in (5), i.e.,

$$u_0 = \alpha_1 + A(x-a) + L_x^{-1} \left[f(x)(y(x-a) - y(x-c)) \right] \quad (12)$$

$$+(f(x) + r)(y(x - c) - y(x - d)) + f(x)(y(x - d) - y(x - b))].$$

The remaining components $u_n(x)$, $n \geq 1$, can be completely determined such that each term is computed by using the previous term. Since u_0 is known,

$$u_n = L_x^{-1} \left[p(x)u_{n-1}(y(x - c) - y(x - d)) \right], \quad n \geq 1. \quad (13)$$

A slight modification to the ADM was proposed by Wazwaz [20] that gives some flexibility in the choice of the zeroth component u_0 to be any simple term and modify the term u_1 accordingly. Since the computation in (13) depends heavily on u_0 the whole computations to find the solution will be simplified considerably. For example an alternative to (13) might be

$$\begin{aligned} u_0 &= \alpha_1 + A(x - a) \\ u_1 &= L_x^{-1} \left[f(x)(y(x - a) - y(x - c)) + (f(x) + r)(y(x - c) - y(x - d)) \right. \\ &\quad \left. + f(x)(y(x - d) - y(x - b)) \right] + L_x^{-1} \left[p(x)u_0(y(x - c) - y(x - d)) \right] \\ u_n &= L_x^{-1} \left[p(x)u_{n-1}(y(x - c) - y(x - d)) \right], \quad n \geq 2. \end{aligned} \quad (14)$$

Finally an N-term approximate solution is given by

$$\Phi_N(x) = \sum_{n=0}^{N-1} u_n(x), \quad N \geq 1, \quad (15)$$

and the exact solution is $u(x) = \lim_{N \rightarrow \infty} \Phi_N$.

To show the effectiveness of the proposed decomposition method and to give a clear overview of the methodology, one example of the second-order obstacle boundary value problems (1) will be discussed in the following section. All the results are calculated by using the symbolic calculus software Mathematica.

3 Applications and Numerical Results

To illustrate the application of the numerical method developed in the previous sections, we consider the second-order obstacle boundary value problem of finding u such that

$$\left. \begin{aligned} -u'' &\geq f(x) && \text{on } \Omega = [0, \pi] \\ u &\geq \psi(x) && \text{on } \Omega = [0, \pi] \\ [u'' - f(x)][u - \psi(x)] &= 0 && \text{on } \Omega = [0, \pi] \\ u(0) &= u(\pi) = 0, \end{aligned} \right\} \quad (16)$$

where $f(x)$ is a given force acting on the string and $\psi(x)$ is the elastic obstacle. We study the problem (16) in the framework of variational inequality approach, it can be shown that, [3, 7, 13], problem (16) is equivalent to the variational inequality problem

$$a(u, v - u) \geq \langle f, v - u \rangle, \quad \text{for all } v \in K, \quad (17)$$

where K is the closed convex set $K = \{v : v \in H_0^1(\Omega) : v \geq \psi \text{ on } \Omega\}$. This equivalence has been used to study the existence of a unique solution of (16) see, for example [3, 7, 13].

Now using the idea of Lewy and Stampacchia [5], the variational inequality (17) may be written as

$$\left. \begin{aligned} u'' - \nu\{u - \psi\}(u - \psi) &= f, & 0 < x < \pi \\ u(0) = u(\pi) &= 0, \end{aligned} \right\}, \quad (18)$$

where

$$\nu(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad (19)$$

is a discontinuous function and is known as the penalty function, and $\psi(x)$ is the given obstacle function defined by

$$\psi(x) = \begin{cases} -1, & \text{for } 0 \leq x \leq \frac{\pi}{4} \text{ and } \frac{3\pi}{4} \leq x \leq \pi, \\ 1, & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}. \end{cases} \quad (20)$$

Equation (18) describes the equilibrium configuration of an obstacle string pulled at the ends and lying over elastic step of constant height 1 and unit rigidity. Since obstacle function ψ is known, so it is possible to find the solution of the problem in the interval $[0, \pi]$.

From equations (18) - (20), we obtain the following system of differential equations:

$$u'' = \begin{cases} f & \text{for } 0 \leq x \leq \frac{\pi}{4} \text{ and } \frac{3\pi}{4} \leq x \leq \pi, \\ u + f - 1 & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \end{cases} \quad (21)$$

with the boundary conditions

$$u(0) = u(\pi) = 0 \quad (22)$$

and the condition of continuity of u and u' at $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$. Note that the problem (21) is a special form of the system (1) with $p(x) = 1$ and $r = -1$.

Example. We consider the system of differential equations (21) when $f = 0$

$$u'' = \begin{cases} 0, & \text{for } 0 \leq x \leq \frac{\pi}{4} \text{ and } \frac{3\pi}{4} \leq x \leq \pi, \\ u - 1, & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \end{cases} \quad (23)$$

with the boundary conditions (22). The analytical solution for this problem is [14]

$$u(x) = \begin{cases} \frac{4}{\gamma_1}x, & 0 \leq x \leq \frac{\pi}{4}, \\ 1 - \frac{4}{\gamma_2} \cosh(\frac{\pi}{2} - x), & \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \\ \frac{4}{\gamma_1}(\pi - x), & \frac{3\pi}{4} \leq x \leq \pi, \end{cases} \quad (24)$$

where $\gamma_1 = \pi + 4 \coth(\pi/4)$ and $\gamma_2 = \pi \sinh(\pi/4) + 4 \cosh(\pi/4)$.

To calculate the components of the decomposition series (11) for $u(x)$ by using the decomposition method outlined in the previous section, we consider the following three cases:

Case I: for $0 \leq x \leq \frac{\pi}{4}$. In this case we implement the Adomian decomposition method and obtain the recursive relation

$$\begin{aligned} u_0 &= \alpha_1 + u'(0)x \\ u_n &= L_x^{-1}(u_{n-1}), \quad n \geq 1. \end{aligned} \quad (25)$$

The initial condition $u'(0)$ is directly taken from the analytical solution (24). Nonetheless, such approach is needed to evaluate the accuracy of the numerical scheme.

Case II: for $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$. To determine the components u_n , $n \geq 0$, the modified decomposition method will be implemented in this case. Although a slight change is made in this recently developed modification [20], it introduces a qualitative tool that facilitates the computational work. In this approach, we split the terms (12) into two parts, one is assigned to the zeroth component $u_0(x)$ and the remaining part is assigned to $u_1(x)$ among other terms. On these identifications, we obtain the recursive relation

$$\begin{aligned} u_0 &= u(\frac{\pi}{4}) \\ u_1 &= u'(\frac{\pi}{4})(x - \frac{\pi}{4}) + L_x^{-1}(-1) + L_x^{-1}(u_0) \\ u_n &= L_x^{-1}(u_{n-1}), \quad n \geq 2. \end{aligned} \quad (26)$$

The initial conditions $u(\frac{\pi}{4})$ and $u'(\frac{\pi}{4})$ are taken directly from the approximate solution obtained in case I.

Case III: for $\frac{3\pi}{4} \leq x \leq \pi$. As in case I, we obtain the recursive relation

$$\begin{aligned} u_0 &= u(\frac{3\pi}{4}) + u'(\frac{3\pi}{4})(x - \frac{3\pi}{4}) \\ u_n &= L_x^{-1}(u_{n-1}), \quad n \geq 1. \end{aligned} \quad (27)$$

The initial condition $u(\frac{3\pi}{4})$ is taken directly from the approximate solution obtained in case II and $u'(\frac{3\pi}{4})$ was determined by using the boundary condition at $x = \pi$.

Table 1 shows the analytical solution, the numerical solution and the absolute errors obtained by using the decomposition method. It is interesting to note that we obtained the numerical solution by using the zeroth component $u_0(x)$ only for case I and III, and by using 6 components of the decomposition series for case II. It is obvious that evaluating more components of $u(x)$ will enhance the numerical solution dramatically.

Table 1: Numerical results for equation (23).

x	Analytical solution	Numerical solution	Absolute error
0.0	0.00000000000	0.00000000000	0.0000E-00
$\pi/6$	0.22663996194	0.22663996194	0.0000E-00
$\pi/4$	0.33995994291	0.33995994291	0.0000E-00
$\pi/3$	0.43183013566	0.43183013566	3.3306E-16
$\pi/2$	0.50170955156	0.50170955156	2.7590E-12
$2\pi/3$	0.43183013566	0.43183013367	1.9910E-09
$3\pi/4$	0.33995994291	0.33995992224	2.0671E-08
$7\pi/8$	0.16997997145	0.16997996112	1.0335E-08
π	0.00000000000	0.00000000000	0.0000E-00

A clear conclusion can be drawn from the numerical results in Table 1 that the Adomian decomposition method provides highly accurate numerical solutions without spatial discretization for the problem. It is also worth noting that the advantage of the decomposition methodology displays a fast convergence of the solutions. The illustrations show the dependence of the rapid convergence depend on the character and behavior of the solutions just as in a closed form solutions.

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