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# Almost Sure Tracking For Linear Discrete-Time Periodic Systems With Independent Random Perturbations*

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**Abstract**

In this paper we solve the tracking problem for linear discrete-time periodic systems with independent random perturbations in Hilbert spaces. Under stabilizability and either uniform observability or detectability conditions we find the optimal control which minimize the cost function associated to this problem. We extend the results from [1] to the general case where the stochastic perturbations act both on the state and control variables. We also establish the connection between the uniform observability and the tracking problem.

## 1 Statement Of The Problem

Let $H, V, U$ be separable real Hilbert spaces and let us denote by $L(H,V)$ (resp. $L(H)$) the Banach space of all bounded linear operators which transform $H$ into $V$ (resp. $H$). We write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements and operators. If $A \in L(H)$ then $A^*$ is the adjoint operator of $A$. The operator $A \in L(H)$ is said to be nonnegative and we write $A \geq 0$, if $A$ is self-adjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in H$. If $H$ is a Hilbert space, we will denote by $\mathcal{H}$ the cone of all nonnegative operators from $L(H)$. We denote by $I$ the identity operator on $H$. The sequence $L_n \in L(H,V), n \in \mathbf{Z}$ is bounded on $\mathbf{Z}$ if $\sup_{n\in\mathbf{Z}} \|L_n\| < \infty$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We will denote by $\mathcal{B}(H)$ the Borel $\sigma$-field of $H$. If $\xi$ is a real or $H$- valued random variable on $\Omega$, we write $E(\xi)$ for mean value (expectation) of $\xi$. We will use the notation $L^p(H) = L^p(\Omega, \mathcal{F}, P, H)$, $p \in \mathbf{N}^*$ for the space of all equivalence class of $H$-valued random variables $\xi$ such that $E\|\xi\|^p < \infty$.

DEFINITION 1. A sequence $\{\eta_n\}, n \in \mathbf{Z}$ of $H$-valued random variables is $\tau$-periodic, $\tau \in \mathbf{N}^*$ if

$$P\{\eta_{n_1+\tau} \in A_1, ..., \eta_{n_m+\tau} \in A_m\} = P\{\eta_{n_1} \in A_1, ..., \eta_{n_m} \in A_m\}, \quad (1)$$

for all $n_1, n_2, ..., n_m \in \mathbf{Z}$ and all $A_p \in \mathcal{B}(H)$, $p = 1,..,m$.

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Let $\xi_n \in L^4(\mathbf{R}), n \in \mathbf{Z}$ be real and independent random variables, which satisfy the condition $E(\xi_n) = 0$ and let $\mathcal{F}_n$, $n \in \mathbf{Z}$, be the $\sigma$-algebra generated by $\{\xi_i, i \leq n-1\}$. We will denote $L_n^p(H) = L^p(\Omega, \mathcal{F}_n, P, H), p \in \mathbf{N}^*$.

Let us consider the system with control, denoted $\{A : D, B : H\}$

$$x_{n+1} = A_n x_n + \xi_n B_n x_n + (D_n + \xi_n H_n)\, u_n \tag{2}$$

and the output

$$y_n = C_n x_n \tag{3}$$

where $A_n, B_n \in L(H), D_n, H_n \in L(U, H), C_n \in L(H, V), n \in \mathbf{Z}$ and the control $\{.., u_n, u_{n+1}, ...\}$ belongs to the class $\widetilde{U}$ defined by the property that $u_n \in L_n^4(U)$, $n \in \mathbf{Z}$ and $\sup_{n \in \mathbf{Z}} E \|u_n\|^4 < \infty$. If $D_n = H_n = 0$ for $n \in \mathbf{Z}$, we will use the notation $\{A, B\}$ for the stochastic system $\{A : 0, B : 0\}$.

We need the following hypotheses:

$H_0$: The sequences $A_n, B_n \in L(H), D_n, H_n \in L(U, H), C_n \in L(H, V), K_n \in L(U)$, $r_n \in H, E\left(\xi_n^4\right) \in \mathbf{R}_+$ are bounded on $\mathbf{Z}$ and

$$K_n \geq \delta I, \delta > 0 \text{ for all } n \in \mathbf{Z}. \tag{4}$$

$H_1$ : The sequences $A_n, B_n, D_n, C_n, K_n,\ r_n, b_n = E\left(\xi_n^2\right), \xi_n,\ n \in \mathbf{Z}$ introduced above are $\tau$-periodic, $\tau \in \mathbf{Z}$.

Assume that $H_1$ and (4) hold and let us denote by $U_{k,\eta}$ the subset of admissible controls from $\widetilde{U}$ with the property that $x_n = x(n,k)\eta$, the solution of (2) with the initial condition $x_k = \eta \in L_k^4(H)$, satisfies the condition $\sup_{n \geq k} E \|x_n\|^4 < \infty$.

The tracking problem consist in finding a feedback control $u$ in a suitable class of controls such us the solution $x_n = x(n,k)\eta$ of the controlled system (2) is "as close as possible" to a given $\tau$-periodic signal $r_n$.

So, for every $k \in \mathbf{Z}$, we look for an optimal "pair" $(\eta, u), u \in U_{k,\eta},\ \eta \in L_k^4(H)$, which minimize the following quadratic cost

$$I_k(\eta, u) = \overline{\lim}_{q \to \infty} \frac{1}{q-k} \sum_{n=k}^{q-1} \left[ \|C_n (x_n - r_n)\|^2 + \langle K_n u_n, u_n \rangle \right], \tag{5}$$

where $x_n$ is the solution of (2) with the initial condition $x_k = \eta \in L_k^4(H)$, for all $n \in \mathbf{Z}$, $n \geq k$. Under stabilizability and uniform observability (or detectability) conditions (see Theorem 11, for the periodic case and Theorem 13 for the time invariant case) we will find the optimal cost and the optimal control.

## 2 Bounded Solutions Of Affine Discrete Time Systems

We denote by $X(n,k), n \geq k \geq 0$, the random evolution operator associated to the stochastic system $\{A, B\}$ that is $X(k,k) = I$ and for all $n > k$,

$$X(n,k) = (A_{n-1} + \xi_{n-1} B_{n-1}) \cdots (A_k + \xi_k B_k).$$

The stochastic system $\{A, B\}$ with the initial condition $x_k = \eta \in L^2(H)$ has a unique solution $x_n = x_n(k, \eta)$ given by $x_n = X(n,k)\eta$. If $B_n = 0$ for all $n \in \mathbf{Z}$, we will denote the deterministic system $x_{n+1} = A_n x_n$ by $\{A\}$.

DEFINITION 2. a) The stochastic system $\{A, B\}$ is uniformly exponentially stable iff there exist $\beta \geq 1, a \in (0,1)$ such that, for all $n \geq k$, $n, k \in \mathbf{Z}$ and $x \in H$, we have

$$E \|X(n,k)x\|^2 \leq \beta a^{n-k} \|x\|^2 . \tag{6}$$

b) The deterministic system $\{A\}$ is uniformly exponentially stable iff there exist $\beta \geq 1$, $a \in (0,1)$ such that, for all $n \geq k$, $n, k \in \mathbf{Z}$, we have

$$\|A_{n-1}A_{n-2}\cdots A_k\|^2 \leq \beta a^{n-k}.$$

REMARK 3. It is not difficult to see that $\{A, B\}$ is uniformly exponentially stable if and only if (6) holds for all $\eta \in L_k^2(H)$.

The following result is known [4], [1].

PROPOSITION 4. If $H_0$ holds, $f_n$ is a bounded on $\mathbf{Z}$ sequence and $\{A\}$ is uniformly exponentially stable then the system

$$y_n = A_n^* y_{n+1} + f_n \tag{7}$$

has a unique, bounded on $\mathbf{Z}$, solution. Moreover, if $H_1$ holds and $f_n$ is $\tau$- periodic, then this solution is $\tau$-periodic.

Arguing as in the proof of Proposition 14 from [4] we can establish the following result.

PROPOSITION 5. Assume that $H_0$ holds and the sequences $q_n, v_n \in H, n \in \mathbf{Z}$ are bounded on $\mathbf{Z}$. If $\{A, B\}$ is uniformly exponentially stable then the system

$$x_{n+1} = A_n x_n + \xi_n B_n x_n + v_n + \xi_n q_n \tag{8}$$

(without initial condition) has a unique solution in $L^2(H)$,

$$x_n = \sum_{p=-\infty}^{n-1} X(n, p+1) \left(v_p + \xi_p q_p\right)$$

which is mean square bounded on $\mathbf{Z}$, that is, there exists $M > 0$ such that $E \|x_n\|^2 < M$ for all $n \in \mathbf{Z}$. Moreover, if $H_1$ is satisfied and $q_n, v_n, n \in \mathbf{Z}$ are $\tau$-periodic, then (8) has a unique solution, which is mean square bounded on $\mathbf{Z}$ and $\tau$-periodic.

## 3 Discrete-Time Riccati Equation Of Stochastic Control And Uniform Observability

We will denote by $\{A, B; C\}$ the discrete time stochastic system formed by the stochastic system $\{A, B\}$ and the observation relation $z_n = C_n x_n$.

DEFINITION 6.(see Definition 6 in [2]) We say that $\{A, B; C\}$ is uniformly observable if there exist $n_0 \in \mathbf{N}$ and $\rho > 0$ such that

$$\sum_{n=k}^{k+n_0} E \left\| C_n X(n,k) x \right\|^2 \geq \rho \left\| x \right\|^2$$

for all $k \in \mathbf{Z}$ and $x \in H$.

We consider the mappings

$$\begin{aligned}
&\mathcal{D}_n : \mathcal{H} \to \mathcal{U}, \mathcal{D}_n(S) = D_n^* S D_n + b_n H_n^* S H_n, \\
&\mathcal{V}_n : \mathcal{H} \to L(H,U), \mathcal{V}_n(S) = D_n^* S A_n + b_n H_n^* S B_n, \\
&\mathcal{G}_n : \mathcal{H} \to \mathcal{H}, \mathcal{G}_n(S) = (\mathcal{V}_n(S))^* \left(K_n + \mathcal{D}_n(S)\right)^{-1} \mathcal{V}_n(S).
\end{aligned}$$

We introduce the following Riccati equation

$$R_n = A_n^* R_{n+1} A_n + b_n B_n^* R_{n+1} B_n + C_n^* C_n - \mathcal{G}_n(R_{n+1}) \tag{9}$$

on $\mathcal{H}$ connected with the quadratic cost (5). A sequence $\{R_n\}_{n \in \mathbf{Z}}, R_n \in \mathcal{H}$ such as (9) holds is said to be a *solution of the Riccati equation* (9).

DEFINITION **7**. [2] a) The system (2) is stabilizable if there exists a bounded on $\mathbf{Z}$ sequence $F = \{F_n\}_{n \in \mathbf{Z}}, F_n \in L(H,U)$ such that $\{A + DF, B + HF\}$ is uniformly exponentially stable. b) A solution $R = (R_n)_{n \in \mathbf{Z}}$ of (9) is said to be stabilizing for (2) if $\{A + DF, B + HF\}$ with

$$F_n = -(K_n + \mathcal{D}_n(R_{n+1}))^{-1} \mathcal{V}_n(R_{n+1}),\ n \in \mathbf{Z} \tag{10}$$

is uniformly exponentially stable. c) The system $\{A, B, C\}$ is detectable if there exists a bounded on $\mathbf{Z}$ sequence $J = \{J_n\}_{n \in \mathbf{Z}}, J_n \in L(V,H)$ such that, the stochastic system, $\{A + JC, B\}$ is uniformly exponentially stable.

The following result concerning the Riccati equation is known (see [3]).

THEOREM 8. Assume $H_0$ holds and 1) system (2) is stabilizable; and 2) system $\{A, B; C\}$ is either uniformly observable or detectable. Then the Riccati equation (9) admits a unique nonnegative, bounded on $\mathbf{Z}$ and stabilizing solution.

REMARK 9. It is known (see [3]) that the stochastic observability does not imply detectability. Hence the results obtained in this paper under uniform observability conditions are different from the ones obtained under detectability conditions (see [1]).

THEOREM 10. If $H_1$ and the hypotheses of the Theorem 8 hold, then the Riccati equation (9) admits a unique nonnegative, $\tau$-periodic solution, which is stabilizing for the stochastic system with control (2).

PROOF. From Theorem 8 it follows that the Riccati equation (9) admits a unique nonnegative, bounded on $\mathbf{Z}$ and stabilizing solution. We only have to prove that this solution is $\tau$-periodic. Let us consider the sequence $R(M,M) = 0 \in \mathcal{H}$,

$$R(M,n) = A_n^* R(M,n+1) A_n + b_n B_n^* R(M,n+1) B_n + C_n^* C_n - \mathcal{G}_n(R(M,n+1))$$

for all $n \leq M-1$. If $H_1$ holds, then it is clear that $R(M,n) = R(M+\tau, n+\tau)$ for all $n \leq M-1$. It is known that if the system (2) is stabilizable, then the solution of the Riccati equation (9) is given by $R_n x = \lim_{M \to \infty} R(M,n)x$, for all $x \in H$ and $n \in \mathbf{Z}$ (see [3]). Now it is easy to see that $R_{n+\tau} = R_n$ for all $n \in \mathbf{Z}$ and the conclusion follows.

# 4 Main Results

Let us denote $f_n = A_n r_n - r_{n+1}$ and $p_n = B_n r_n$. The following theorem gives the optimal control, which minimize the cost function (5).

THEOREM 11. Assume that $H_1$ and the hypotheses 1) and 2) of the Theorem 8 hold. Let $R_n$ and $h_n$ be the unique $\tau$-periodic solution of the Riccati equation (9) respectively of the Lyapunov equation

$$h_n = (A_n + D_n F_n)^* h_{n+1} + b_n (B_n + H_n F_n)^* R_{n+1} p_n + R_n f_{n-1} \tag{11}$$

with $F_n$ given by (10). If $\widetilde{x}_n, n \in \mathbf{Z}$ is the unique $\tau$-periodic solution of (2), where

$$\widetilde{u}_n = F_n \widetilde{x}_n - F_n r_n - V_n^{-1} (D_n^* h_{n+1} + b_n H_n^* R_{n+1} p_n), \tag{12}$$

and if $E \|\widetilde{x}_k\|^4 < \infty$ then the optimal cost is

$$\begin{aligned} I_k(\widetilde{x}_k, \widetilde{u}) = \frac{1}{\tau} \sum_{i=0}^{\tau} \Big[ & b_i \langle R_{i+1} p_i, p_i \rangle + 2 \langle h_{i+1}, f_i \rangle - \langle R_{i+1} f_i, f_i \rangle \\ & - \left\| V_i^{-1/2} [D_i^* h_{i+1} + b_i H_i^* R_{i+1} p_i] \right\|^2 \Big] \end{aligned} \tag{13}$$

where $V_n = K_n + \mathcal{D}_n(R_{n+1})$.

PROOF. First, we note that, since the solution of the Riccati equation (9) is stabilizing for (2), we can apply Proposition 4 to deduce that the equation (11) has a unique $\tau$-periodic solution. Let us consider the function

$$v_n : H \to R, v_n(x) = \langle R_n x, x \rangle + 2 \langle h_n - R_n f_{n-1}, x \rangle .$$

If $x_n$ is the solution of the system (2), then $\overline{x}_n = x_n - r_n$ is the solution of the following system

$$\begin{cases} \overline{x}_{n+1} = A_n \overline{x}_n + \xi_n B_n \overline{x}_n + D_n u_n + \xi_n H_n u_n + f_n + \xi_n p_n \\ \overline{x}_k = x - r_k \in H, \, k \in \mathbf{Z} \end{cases} \tag{14}$$

By a simple computation we obtain

$$\begin{aligned} v_{n+1}(\overline{x}_{n+1}) &= \langle R_{n+1} \overline{x}_{n+1}, \overline{x}_{n+1} \rangle + 2 \langle h_{n+1} - R_{n+1} f_n, \overline{x}_{n+1} \rangle \\ &= v_n(\overline{x}_n) - \|C_n \overline{x}_n\|^2 - \langle K_n u_n, u_n \rangle \\ &\quad + \left\| V_n^{1/2} \left( u_n - F_n \overline{x}_n + b_n V_n^{-1} H_n^* R_{n+1} p_n + V_n^{-1} D_n^* h_{n+1} \right) \right\|^2 \\ &\quad - \left\| V_n^{-1/2} (b_n H_n^* R_{n+1} p_n + D_n^* h_{n+1}) \right\|^2 \\ &\quad - \langle R_{n+1} f_n, f_n \rangle + b_n \langle R_{n+1} p_n, p_n \rangle + 2 \langle h_{n+1}, f_n \rangle + c_n + d_n, \end{aligned} \tag{15}$$

where $c_n = \left(\xi_n^2 - b_n\right) \langle R_{n+1}(B_n \overline{x}_n + H_n u_n + p_n), B_n \overline{x}_n + H_n u_n + p_n \rangle$ and

$$d_n = 2\xi_n \langle R_{n+1}(A_n \overline{x}_n + D_n u_n) + h_{n+1}, B_n \overline{x}_n + H_n u_n + p_n \rangle .$$

Since $\{A+DF, B+HF\}$ is uniformly exponentially stable, we can use Proposition 5 to deduce that $\overline{x}_n$, the unique, bounded on $\mathbf{Z}$ solution of the system (14), where

$$\overline{u}_n = F_n\overline{x}_n - V_n^{-1}\left(D_n^* h_{n+1} + b_n H_n^* R_{n+1} p_n\right).$$

is $\tau$-periodic and there exists $\gamma > 0$ such that $E\left\|\overline{x}_n\right\|^4 < \gamma$ for all $n \in \mathbf{Z}$. Consequently $\overline{u}_n$ is $\tau$-periodic and $\overline{u} \in U_{k,\overline{x}_k}$.

If $\widetilde{x}_n$ is the unique $\tau$-periodic solution of (2) with $\widetilde{u}_n$ given by (12) ($\widetilde{u}_n = \overline{u}_n$) then $\overline{x}_n = \widetilde{x}_n - r_n$ and using (15) we get

$$\begin{aligned}\frac{1}{n-k}\sum_{i=k}^{n-1}\left\|C_i\left(\widetilde{x}_i - r_i\right)\right\|^2 + \langle K\widetilde{u}_i, \widetilde{u}_i\rangle = \frac{1}{n-k}\sum_{i=k}^{n-1}\Big[b_i\left\langle R_{i+1}p_i, p_i\right\rangle + \\ +2\left\langle h_{i+1}, f_i\right\rangle - \left\langle R_{i+1}f_i, f_i\right\rangle - \left\|V_i^{-1/2}\left[D_i^* h_{i+1} + b_i H_i^* R_{i+1} p_i\right]\right\|^2\Big] \\ -\frac{1}{n-k}\left[v_{n+1}\left(\overline{x}_{n+1}\right) - v_k(\overline{x}_k)\right] + \frac{1}{n-k}\sum_{i=k}^{n-1}\left(c_i + d_i\right). \qquad (16)\end{aligned}$$

We recall Chebyshev's inequality: $P(\|\xi\| > \varepsilon) < \frac{E(f(\|\xi\|))}{f(\varepsilon)}$ for all nonnegative and nondecreasing functions $f : \mathbf{R}_+ \to \mathbf{R}_+$ and $\xi$ a $H$- valued or real random variable. Since $E\left\|\overline{x}_n\right\|^4 < \gamma$ for all $n \in \mathbf{Z}$, we have

$$\begin{aligned}P\left(\frac{1}{n-k}\left[\left\langle R_{n+1}\overline{x}_{n+1}, \overline{x}_{n+1}\right\rangle + \left\langle h_{n+1} - R_{n+1}f_n, \overline{x}_{n+1}\right\rangle\right] > \varepsilon\right) \\ < \frac{2E\left\langle R_{n+1}\overline{x}_{n+1}, \overline{x}_{n+1}\right\rangle^2 + 2\gamma_1 E\left\|\overline{x}_{n+1}\right\|^2}{(n-k)^2\varepsilon^2} \le \frac{\gamma_2}{(n-k)^2\varepsilon^2}.\end{aligned}$$

If $A_n = \left\{\omega \in \Omega / \frac{1}{n-k}\left(\left\langle R_{n+1}\overline{x}_{n+1}, \overline{x}_{n+1}\right\rangle + 2\left\langle h_{n+1} - R_{n+1}f_n, \overline{x}_{n+1}\right\rangle\right)(\omega) > \varepsilon\right\}$, it follows from the Borel-Cantelli lemma that

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_n\right) = 0.$$

Thus as $n \to \infty$, $\frac{1}{n-k}v_{n+1}\left(\overline{x}_{n+1}\right) \to 0$ $P.a.s.$ Analogously it follows that as $n \to \infty$, $\frac{1}{n-k}v_k(\overline{x}_k) \to 0$ $P.a.s.$ Now we will prove that $\frac{1}{n-k}\sum_{i=k}^{n-1}\left(c_i + d_i\right) \to 0$ $P.a.s.$ It is easy to see that $L_n = \sum_{i=k}^{n-1}\left(c_i + d_i\right)$ is a martingale. We have

$$\sum_{i=k}^{\infty}\frac{E\left|L_{i+1} - L_i\right|^2}{i^2} = \sum_{i=k}^{\infty}\frac{E\left|c_i + d_i\right|^2}{i^2} \le \sum_{i=k}^{\infty}\frac{2\left[(a_i + b_i^2)\gamma_3 + b_i\gamma_4\right]}{i^2},$$

where $a_i = E\left\|\overline{x}_i\right\|^4$. Since $a_n, b_n, n \in \mathbf{Z}$ are bounded on $\mathbf{Z}$ we deduce that there exists $\gamma_5 \in \mathbf{R}_+^*$ such as $\sum_{i=k}^{\infty}\frac{E\left|L_{i+1}-L_i\right|^2}{i^2} < \gamma_5\sum_{i=k}^{\infty}\frac{1}{i^2} < \infty$. By Corollary 2, given by Shiryaev in [5], it follows that as $n \to \infty$, $\frac{1}{n-k}L_n \to 0$ $P.a.s.$ As $n \to \infty$ in (16), it follows

$$I_k(\widetilde{x}_k,\widetilde{u}) = \overline{\lim}_{n\to\infty}\frac{1}{n-k}\sum_{i=k}^{n-1}\Big[-\left\|V_i^{-1/2}\left[D_i^* h_{i+1}+b_i H_i^* R_{i+1}p_i\right]\right\|^2 - \langle R_{i+1}f_i,f_i\rangle + b_i\langle R_{i+1}p_i,p_i\rangle + 2\langle h_{i+1},f_i\rangle\Big] \quad (17)$$

Thus

$$\min_{(\eta,u),u\in U_{k,\eta}} I_k(\eta,u) \le I_k(\widetilde{x}_k,\widetilde{u}).$$

If $\eta \in L_k^4(H)$ and $u \in U_{k,\eta}$ and reasoning as above it is not difficult to deduce that

$$I_k(\eta,u) \ge I_k(\widetilde{x}_k,\widetilde{u}).$$

Thus $\min_{(\eta,u),u\in U_{k,\eta}} I_k(\eta,u) = I_k(\overline{x}_k,\widetilde{u})$ and the optimal cost does not depend on the initial value $\overline{x}_k$. Since all the sequences in $I_k(\overline{x}_k,\widetilde{u})$ are $\tau$-periodic, we take successively $n = p\tau + k$ and $p\to\infty$ in (17) and we get

$$I(\widetilde{u}) = \frac{1}{\tau}\sum_{i=1}^{\tau}\Big[-\left\|V_i^{-1/2}\left[D_i^* h_{i+1}+b_i H_i^* R_{i+1}p_i\right]\right\|^2 - \langle R_{i+1}f_i,f_i\rangle + b_i\langle R_{i+1}p_i,p_i\rangle + 2\langle h_{i+1},f_i\rangle\Big].$$

The proof is complete.

REMARK 12. Assume that the hypotheses of the above theorem fulfilled. If $\widetilde{u}_n$ is given by (12) then the unique $\tau$-periodic solution of (2), according Proposition 5, is

$$\widetilde{x}_n = -\sum_{i=-\infty}^{n-1}\widetilde{X}(n,i+1)\left(D_i+\xi_i H_i\right)\left\{V_i^{-1}\left[D_i^*\left(h_{i+1}\right)+b_i H_i^* R_{i+1}p_i\right]+F_i r_i\right\} \quad (18)$$

where $\widetilde{X}(n,k)$ is the random evolution operator associated to the system $\{A+DF, B+HF\}$.

# 5 Time Invariant Case

We have the following result.

THEOREM 13. Assume that $H_1$ holds for $\tau = 1$, the hypotheses of the Theorem 11 are fulfilled and $\{\xi_n\}, n\in\mathbf{Z}$ is 1-periodic.

i) Then the following algebraic Riccati equation, respectively Lyapunov equation

$$R = A^*RA + b^*BRB + C^*C - F^*(K + D^*RD + bH^*RB)F$$
$$h = (A+DF)^*h + R(Ar - r) + b\,(B+HF)^*\,RBr$$

have unique solutions. Moreover

$$h=\sum_{p=0}^{\infty}\left(A^{*}+F^{*}D^{*}\right)^{p}\left[R(Ar-r)+b\left(B+HF\right)^{*}RBr\right]$$

ii) If $R$ and $h$ are the solutions of the above algebraic equations, then the optimal cost is

$$\begin{aligned} I(\widetilde{u}) &= b\left\langle RBr,Br\right\rangle+2\left\langle h,Ar-r\right\rangle-\left\|V^{-1/2}\left[D^{*}h+bH^{*}RBr\right]\right\|^{2} \\ &\quad -\left\langle R\left(Ar-r\right),\left(Ar-r\right)\right\rangle, \end{aligned}$$

and an optimal control is $\widetilde{u}_n=F\widetilde{x}_n-Fr-\left[D^{*}h+bH^{*}RBr\right]$, where $\widetilde{x}_n$ is given by (18) and we assume that $E\left\|\widetilde{x}_n\right\|^{4}<\infty$.

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