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A Generalized Iterative Algorithm For Generalized Successively Pseudocontractions*

Krishna Kumar and Birendra Kumar Sharma†

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Abstract

The concept of generalized successively pseudocontraction is introduced to apply the generalized iterative algorithm for the solvability of a class of modified nonlinear variational inequality problems (MNVI) involving uniformly L-Lipschitzian generalized successively pseudocontractions on closed convex sets in Hilbert spaces. Certain well-known results are obtained as consequence. The structure of proposed class of Pseudocontraction is supported by an example.

1 Introduction

An offshoot of contraction mapping, the class of pseudocontractive mappings gained important because of its firm connection with nonexpansive mapping on one side and monotone operators on the other. The existence of fixed points and the behavior of the iterative sequences of such nonlinear operators became equally interesting. As result, several type of pseudocontractive mapping were studied and results concerning them exist in the literature dealing with existence and convergence of iterative sequences of such mappings (see [2, 8, 12] and references cited therein).

The standard variational inequality theory was introduced and studied by Stampacchia [15]. Soon it became a powerful tool of applied mathematics. For the reason, variational inequalities were found useful in dealing with nonlinear partial differential equations and hence begun to play an important role in mechanics, optimization and control problems, operations research, management sciences and other branches of mathematical and engineering sciences. It is now a rich source of inspiration for scientists and engineers. (see [4, 5, 6, 7, 9, 10, 11, 13, 15, 17, 19] and references cited therein).

The Mann [14] iterative scheme, introduced in 1953 in different context was found useful to prove the convergence of the sequence of those nonlinear mappings for which the Banach [1] principle failed. Consequently in 1974, Ishikawa [8] devised a new iteration scheme to establish the convergence of a Lipschitzian pseudocontractive map when Mann iteration process also failed to converge. However, a bulk of literature now

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†School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur (Chhattisgarh)-492010, India

exist around the theme of establishing the convergence of the Mann iteration for certain classes of mappings and then showing that the Ishikawa iteration also converges.

It is our purpose in this paper to introduce a concept of generalized successively pseudocontraction mapping. This concept is more general in the sense that it includes several established class of pseudocontractive operators as particular case. This class of mappings is characterized by the fact that generalized successively pseudocontractive mappings are generalized pseudocontractive but converse is not true in general. Next, we study a class of modified nonlinear variational inequality problems and prove some existence theorems of solutions and convergence theorems of the Mann and Ishikawa iterative sequences for the class of modified nonlinear variational inequality problems involving uniformly L-Lipschitzian generalized successively pseudocontractive mappings in Hilbert space. Our Theorems $3.2, 3.3$ and 3.4 not only extend but improve and generalize the Theorem 2.1 and 2.2 of Verma [17] as below:

1. We replace generalized pseudocontractive mappings by more general generalized successively pseudocontractive mappings.
2. We extend the nonlinear variational inequality (NVI) problem (8) to the modified nonlinear variational inequality (MNVI) problem (3).
3. We replace Mann iterative process by Mann type and Ishikawa type iterations processes.

2 Preliminaries

First, we define the concept of generalized successively pseudocontractive mapping as below:

DEFINITION 2.1. An operator $T : H \mapsto H$ is said to be generalized successively pseudocontractive, if for any $x, y \in H$, there exists a constant $k > 0$ such that

$$\|T^n x - T^n y\|^2 \leq k^2 \|x - y\|^2 + \|T^n x - T^n y - k(x - y)\|^2, \tag{1}$$

For all $n \geq 1$. This is mutually equivalent to

$$\langle T^n x - T^n y, x - y \rangle \leq k\|x - y\|^2 \tag{2}$$

The above mapping includes the following important class of mappings as below:

1. If $n = 1$, then T is called generalized pseudocontractive mapping introduced by Verma [16] in Hilbert space.
2. If $k \in (0, 1)$, then T is called successively strongly pseudocontractive mapping introduced by Liu, Kim, Kim [12].
3. If $n = 1$ and $k \in (0, 1)$, then T is called strongly pseudocontractive mapping studied by various researchers (see [3, 5] and references cited therein).

4. If $n = 1$ and $k = 1$, then T is called pseudocontractive mapping introduced by Browder and Petryshyn [2].
5. If $n = 1$ and $k > 0$, then T is called Lipschitzian continuous.
6. If $k > 0$, then T is called uniformly L-Lipschitzian.

We give an example to support that the class of generalized successively pseudocontractions is wider than the class of generalized pseudocontractions.

EXAMPLE. Let $X = R$, the set of real numbers. Define a map $T : R \mapsto R$ by

$$Tx = \frac{x}{2}, \forall \ x \in R.$$

Clearly 0 is the unique fixed point of T. Suppose that $x = 2, y = \frac{1}{3}$ and $k = \frac{1}{3}$. Then we have $Tx = 1, Ty = \frac{1}{6}$ and $T^n x = (\frac{1}{2})^{n-1}, T^n y = \frac{1}{3}(\frac{1}{2})^n$. Consider,

$$\begin{aligned}
&|Tx - Ty|^2 \leq k^2|x - y|^2 + |Tx - Ty - k(x - y)|^2 \\
&\Rightarrow (\tfrac{5}{6})^2 \leq (\tfrac{1}{6})^2(\tfrac{5}{3})^2 + (\tfrac{5}{18})^2 \\
&\Rightarrow \tfrac{9}{36} \leq \tfrac{5}{36},
\end{aligned}$$

which shows that T is not generalized pseudocontraction. Now, we show that T is generalized successively pseudocontraction.

$$\begin{aligned}
&|T^n x - T^n y|^2 \leq k^2|x - y|^2 + |T^n x - T^n y - k(x - y)|^2 \\
&\Rightarrow (\tfrac{5}{3})^2(\tfrac{1}{2})^{2n} \leq (\tfrac{1}{3})^2(\tfrac{5}{3})^2 + (\tfrac{5}{3})^2[(\tfrac{1}{2})^n - \tfrac{1}{3}]^2 \\
&\Rightarrow 0 \leq \tfrac{2}{9} - (\tfrac{1}{2})^{n-1}\tfrac{1}{3},
\end{aligned}$$

which is true for all $n \geq 2$ and hence T is generalized successively pseudocontractive mappings.

DEFINITION 2.2. Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $\langle x, y\rangle$ and $\|x\|$ denote, respectively the inner product and norm on H. Let P_K be the projection of H onto K. Let $T : K \mapsto H$ be a nonlinear mapping.

Now, we define the following modified nonlinear variational inequality problem (I): Find an element $x \in H$ such that

$$\langle (I - T^n)x, \ \ y - x\rangle \geq 0 \tag{3}$$

for all $y \in H$ and $\forall \ n \geq 1$, where I is the identity mapping.

Next, we recall the following iterative processes given by Mann [14] and Ishikawa [8]: Let X be a Banach space. The **modified Mann iteration process [14]** is defined by

$$x_0 \in X, \ \ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, n \geq 0, \tag{4}$$

where the $\alpha_n \in (0, 1)$ for all $n \geq 0$.

The **modified Ishikawa iteration process [8]** is defined by

$$\begin{aligned}
&x_0 \in X, \\
&x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\
&y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \ \ n \geq 0,
\end{aligned} \tag{5}$$

where $0 \leq \alpha_n \leq \beta_n \leq 1,\ \ n \geq 0$. In specific situations, additional conditions are placed on $\{\alpha_n\}$ and $\{\beta_n\}$.

In the sequel we use the following Lemmas.

LEMMA 2.3.[9] Let K be a nonempty closed convex subset of a real Hilbert space H. Then for an element z in H, an element x in K satisfies

$$\langle x - z,\ \ y - x\rangle \geq 0 \tag{6}$$

for all y in K iff $x = P_K z$, where P_K is projection onto K.

LEMMA 2.4.[9] $P_K : H \mapsto K$ is nonexpansive, i.e.

$$\|P_K(x) - P_K(y)\| \leq \|x - y\| \tag{7}$$

for all x and y in H.

LEMMA 2.5.[17] Let K be a nonempty closed convex subset of a Hilbert space H. Then nonlinear variational inequality problem: find $x \in H$ such that

$$\langle (I - T)x, y - x\rangle \geq 0 \tag{8}$$

for all $y \in K$, has a solution x in K iff x in K satisfies the relation

$$x = P_K[x - t(x - Tx)], \tag{9}$$

where $t > 0$ is arbitrary.

LEMMA 2.6.[18] Let $\{a_n\}$ be a nonnegative sequence which satisfies the following inequality

$$a_{n+1} \leq (1 - \lambda_n)a_n + \sigma_n$$

where $\lambda_n \in (0, 1), \forall\ \ n \in N,\ \ \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = 0(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

3 Main Result

Now we give main results of this paper.

THEOREM 3.1. Let K be a nonempty closed convex subset of a Hilbert space H. Then modified nonlinear variational inequality problem (3) has a solution z in K iff z in K satisfies the relation

$$z = P_K[z - t(z - T^n z)],\ \ \forall\ \ n \geq 1,$$

where $t > 0$ is a constant.

PROOF. Suppose $z \in K$ is a solution of the modified nonlinear variational inequality problem (3). Then $z \in K$ such that

$$\langle z - T^n z,\ \ y - z\rangle \geq 0\ \ \forall\ \ y \in K,\ \ \forall\ \ n \geq 1. \tag{10}$$

Since $t > 0$, it follows that

$$\langle z - (z - t(z - T^n z)),\ \ y - z\rangle \geq 0\ \ \forall\ y \in K,\ \ \forall\ n \geq 1. \tag{11}$$

Now from Lemma 2.5, we find that

$$z = P_K[z - t(z - T^n z)]\ \ \forall\ n \geq 1. \tag{12}$$

Conversely, if z satisfies the relation

$$z = P_K[z - t(z - T^n z)]\ \ \forall\ n \geq 1,$$

then $z \in K$ and hence by Lemma 2.5, we have

$$\begin{aligned}&\langle z - (z - t(z - T^n z)),\ \ y - z\rangle \geq 0,\\ &t\langle (z - T^n z),\ \ y - z\rangle \geq 0,\end{aligned}$$

$\forall\ y \in K,\ \forall\ n \geq 1$. Since $t > 0$, it implies that

$$\langle z - (z - t(z - T^n z)),\ \ y - z\rangle \geq 0\ \ \forall\ y \in K,\ \ \forall\ n \geq 1.$$

Hence z is a solution of the MNVI problem (3).

THEOREM 3.2. Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \mapsto H$ be uniformly L-Lipschitzian generalized successively pseudocontractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two increasing sequences in [0, 1) satisfying the following conditions: (i) $\alpha_n \to 0,\ \beta_n \to 0$ as $n \to \infty$, and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \forall\ n \geq 0$. Then for every $x_0 \in H$, the sequence $\{x_n\}$ generated iteratively by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K[(1 - t)x_n + tT^n y_n]\ \ \forall\ n \geq 0, \tag{13}$$

$$y_n = (1 - \beta_n)x_n + \beta_n P_K[(1 - t)x_n + tT^n x_n]\ \ \forall\ n \geq 0, \tag{14}$$

converges to a unique solution of the modified nonlinear variational inequality (MNVI) problem (3).

PROOF. Let z be a solution of the MNVI problem (3). Now we first prove that z is a unique solution of the MNVI problem (3). Let z_1 be another solution of the MNVI problem (3). Then we have,

$$\langle (I - T^n)z, y - z\rangle \geq 0\ \ \forall\ y \in K$$

and

$$\langle (I - T^n)z_1, y - z_1\rangle \geq 0\ \ \forall\ y \in K.$$

These can be written as

$$\langle (I - T^n)z, z - y\rangle \leq 0\ \ \forall\ y \in K$$

and

$$-\langle (I-T^n)z_1, y-z_1\rangle \leq 0 \ \ \forall \ \ y \in K.$$

Adding above we get,

$$\begin{aligned} &\langle (I-T^n)z-(I-T^n)z_1, z-z_1\rangle \leq 0 \\ &\Rightarrow \langle z-z_1, z-z_1\rangle - \langle T^nz-T^nz_1, z-z_1\rangle \leq 0 \\ &\Rightarrow \|z-z_1\|^2 - k\|z-z_1\| \leq 0 \\ &\Rightarrow (1-k)\|z-z_1\|^2 \leq 0, \end{aligned} \tag{15}$$

where $k \in (0,1)$. So $\|z-z_1\| = 0$ i.e. $z = z_1$. Hence z is the unique solution of the MNVI problem (3).

Let $x, y \in H$, then from Lemma 2.4 and (13) we have,

$$\begin{aligned} \|x_{n+1}-z\| &= \|(1-\alpha_n)x_n + \alpha_n P_K[(1-t)x_n + tT^ny_n] - z\| \\ &= \|(1-\alpha_n)(x_n-z) + \alpha_n\{P_K[(1-t)x_n + tT^ny_n] - z\}\| \\ &= \|(1-\alpha_n)(x_n-z) + \alpha_n\{P_K[(1-t)x_n + tT^ny_n] \\ &\quad -P_K[(1-t)z + tT^nz]\}\| \\ &\leq (1-\alpha_n)\|x_n-z\| + \alpha_n\|(1-t)(x_n-z) + t(T^ny_n - T^nz)\| \\ &\leq (1-t\alpha_n)\|x_n-z\| + tL\alpha_n\|y_n-z\|. \end{aligned} \tag{16}$$

Now consider,

$$\begin{aligned} \|y_n-z\| &= \|(1-\beta_n)x_n + \beta_n P_K[(1-t)x_n + tT^nx_n] - z\| \\ &= \|(1-\beta_n)(x_n-z) + \beta_n\{P_K[(1-t)x_n + tT^nx_n] - z\}\| \\ &= \|(1-\beta_n)(x_n-z) + \beta_n\{P_K[(1-t)x_n + tT^nx_n] \\ &\quad -P_K[(1-t)z + tT^nz]\}\| \\ &\leq (1-\beta_n)\|x_n-z\| + \beta_n\|(1-t)(x_n-z) + t(T^nx_n - T^nz)\| \end{aligned} \tag{17}$$

From (2), it follows that

$$\begin{aligned} &\|(1-t)(x_n-z) + t(T^nx_n - T^nz)\|^2 \\ &= (1-t)^2\|x_n-z\|^2 + 2t(1-t)\langle T^nx_n - T^nz, \ \ x_n - z\rangle + t^2\|T^nx_n - T^nz\|^2 \\ &\leq (1-t)^2\|x_n-z\|^2 + 2t(1-t)k\|x_n-z\|^2 + t^2L^2\|x_n-z\|^2 \\ &= [(1-t)^2 + 2t(1-t)k + t^2L^2]\|x_n-z\|^2. \end{aligned} \tag{18}$$

From (17) and (18) we have,

$$\begin{aligned} \|y_n-z\| &\leq (1-\beta_n)\|x_n-z\| + \beta_n[(1-t)^2 + 2t(1-t)k + t^2L^2]^{1/2}\|x_n-z\| \\ &\leq [1-\beta_n + \beta_n((1-t)^2 + 2t(1-t)k + t^2L^2)^{1/2}]\|x_n-z\| \\ &= [1-(1-((1-t)^2 + 2t(1-t)k + t^2L^2)^{1/2})\beta_n]\|x_n-z\| \\ &= [1-(1-\delta)\beta_n]\|x_n-z\|, \end{aligned} \tag{19}$$

where $0 < \delta = [(1-t)^2 + 2t(1-t)k + t^2L^2] < 1$ for all t such that $0 < t < 2(1-r)/(1-2r+L^2), r < 1$ and $L \geq 1$. Now from (16) and (19) we have,

$$\begin{aligned} \|x_{n+1}-z\| &\leq (1-t\alpha_n)\|x_n-z\| + tL\alpha_n[1-(1-\delta)\beta_n]\|x_n-z\| \\ &= [1-t\alpha_n + tL\alpha_n(1-(1-\delta)\beta_n)]\|x_n-z\| \\ &= [1-\{1-L(1-(1-\delta)\beta_n)\}t\alpha_n]\|x_n-z\| \\ &\leq [1-\lambda]\|x_n-z\| \end{aligned} \tag{20}$$

where we choose t and α_n such that $0 < \lambda = [1 - L(1-(1-\delta)\beta_n)]\alpha_n t < 1$. Hence from **Lemma 2.6**, the sequence $\{x_n\}$ converges to strongly to the unique solution z.

THEOREM 3.3. Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \mapsto H$ be Lipschitzian generalized pseudocontractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two increasing sequences in [0, 1) satisfying the following conditions: (i) $\alpha_n \to 0,\ \beta_n \to 0$ as $n \to \infty$, and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \forall\ n \geq 0$. Then for every $x_0 \in H$ the sequence $\{x_n\}$ generated iteratively by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[(1-t)x_n + tTy_n]\ \forall\ n \geq 0, \tag{21}$$

$$y_n = (1-\beta_n)x_n + \beta_n P_K[(1-t)x_n + tTx_n]\ \forall\ n \geq 0, \tag{22}$$

converges to a unique solution of the nonlinear variational inequality (NVI) problem (8).

PROOF. Taking $n = 1$ in Theorem 3.2, the conclusion of Theorem 3.3 can be obtained from Theorem 3.2.

THEOREM 3.4. Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \mapsto H$ be uniformly L-Lipschitzian generalized successively pseudocontractive mapping. Let $\{\alpha_n\}$ be an increasing sequence in [0, 1) satisfying the following conditions: (i) $\alpha_n \to 0$ as $n \to \infty$, and $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \forall\ n \geq 0$. Then for every $x_0 \in H$ the sequence $\{x_n\}$ generated iteratively by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[(1-t)x_n + tT^n x_n]\ \forall\ n \geq 0, \tag{23}$$

converges to a unique solution of the modified nonlinear variational inequality (MNVI) problem (3).

PROOF. Taking $\beta_n = 0$ in Theorem 3.2, the conclusion of Theorem 3.4 can be obtained from Theorem 3.2.

COROLLARY 3.5 ([17]). Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \mapsto H$ be Lipschitzian generalized pseudocontractive mapping. Let $\{\alpha_n\}$ be an increasing sequence in [0, 1) satisfying the following conditions: (i) $\alpha_n \to 0$ as $n \to \infty$, and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \forall\ n \geq 0$. Then for every $x_0 \in H$ the sequence $\{x_n\}$ generated iteratively by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[(1-t)x_n + tTx_n]\ \forall\ n \geq 0, \tag{24}$$

converges to a unique solution of the nonlinear variational inequality (NVI) problem (8).

PROOF. Taking $\beta_n = 0$ in Theorem 3.3, the conclusion of our Corollary can be obtained from Theorem 3.3.

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