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An Inverse Transient Thermoelastic Problem Of A Thin Annular Disc*

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Abstract

This paper is concerned with an inverse transient thermoelastic problem in which we need to determine the unknown temperature, displacement and stress function on the outer curved surface of a thin annular disc when the interior heat flux is known. Finite Marchi-Fasulo transform and Laplace transform techniques are used.

1 Introduction

The inverse thermoelastic problem consists of the determination of the temperature of the heating medium and the heat flux of a solid when the conditions of the displacement and stresses are known at some points of the solid under consideration. This inverse problem is relevant to different industries where machinery such as the main shaft of lathe and turbine and roll of a rolling mill is subject to heating.

In [1] and [2], one-dimensional transient thermoelastic problems are considered and the heating temperature and the heat flux on the surface of an isotropic infinite slab are derived. The direct problems of thermoelasticity of a thin circular plate are considered in [5, 7, 9] and inverse problems of thermoelasticity of a thin annular disc are considered in [3] and [4].

In the present problem an attempt is made to study the inverse transient thermoelastic problem to determine the unknown temperature, displacement and stress functions of the disc occupying the space $D = \{(x, y, z) \in R^3 : a \leq (x^2 + y^2)^{1/2} \leq b, -h \leq z \leq h\}$ with known interior heat flux. Finite Marchi-Fasulo integral transform and Laplace transform techniques are used to find the solution of the problem. Numerical estimate for the temperature distribution on the outer curved surface is obtained.

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2 Statement Of The Problem

Consider a thin annular disc of thickness $2h$ occupying the space D. The differential equation governing the displacement function $U(r,z,t)$, where $r=(x^2+y^2)^{1/2}$ is

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=(1+\upsilon)a_t T \tag{1}$$

with

$$U_r=0 \text{ at } r=a \text{ and } r=b, \tag{2}$$

where υ and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the disc respectively and T is the temperature of the disc satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{k}\frac{\partial T}{\partial t}, \tag{3}$$

where $k=\frac{K}{\rho c}$ is the thermal diffusibility of the material of the disc, K is the conductibility of the medium and ρ is its calorific capacity (which is assumed to be constant), subject to the initial condition

$$T(r,z,0)=0 \text{ for all } a\le r\le b \text{ and } -h\le z\le h, \tag{4}$$

the interior condition

$$\frac{\partial T(\xi,z,t)}{\partial r}=f(z,t) \quad \text{for all } a\le \xi\le b,\ -h\le z\le h \text{ and } t>0 \tag{5}$$

and the boundary conditions

$$\frac{\partial T(a,z,t)}{\partial r}=u(z,t) \quad \text{for all } -h\le z\le h \text{ and } t>0, \tag{6}$$

$$T(b,z,t)=g(z,t) \quad \text{for all } -h\le z\le h \text{ and } t>0, \tag{7}$$

$$[T(r,z,t)+k_1\frac{\partial T}{\partial z}(r,z,t)]_{z=h}=F_1(r,t) \quad \text{for all } a\le r\le b \text{ and } t>0, \tag{8}$$

$$[T(r,z,t)+k_2\frac{\partial T}{\partial z}(r,z,t)]_{z=-h}=F_2(r,t) \quad \text{for all } a\le r\le b \text{ and } t>0. \tag{9}$$

The functions $F_1(r,t)$ and $F_2(r,t)$ are known constants and they are set to be zero here as in other literatures [3-4, 6, 9] so as to obtain considerable mathematical simplicities. The constants k_1 and k_2 are the radiation constants on the two plane surfaces. The function $f(z,t)$ is assumed to be known while the function $g(z,t)$ is not.

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r} \frac{\partial U}{\partial r} \tag{10}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \tag{11}$$

where μ is the Lamé elastic constant, while each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the disc in the plane state of stress.

The equations (1) to (11) constitute the mathematical formulation of the problem under consideration [5].

3 Solution Of The Problem

Applying the finite Marchi-Fasulo integral transform defined in [6] to the equations (3) to (7), (5) and using (8) as well as (9), we obtain

$$\frac{d^2 \bar{T}}{dr^2} + \frac{1}{r} \frac{d\bar{T}}{dr} - a_n^2 \bar{T} = \frac{1}{k} \frac{d\bar{T}}{dt} \tag{12}$$

the initial condition

$$\bar{T}(r, n, 0) = 0, \tag{13}$$

the boundary conditions

$$\frac{d\bar{T}(a, n, t)}{dr} = \bar{u}(n, t), \tag{14}$$

$$\bar{T}(b, n, t) = \bar{g}(n, t), \tag{15}$$

the interior condition

$$\frac{d\bar{T}(\xi, n, t)}{dr} = \bar{f}(n, t) \tag{16}$$

where $\bar{T}$ denotes the Marchi-Fasulo transform of T and n is the Marchi-Fasulo transform parameter, a_n are the solutions of the equation

$$\begin{aligned} & [\alpha_1 a \cos(ah) + \beta_1 \sin(ah)] \times [\beta_2 \cos(ah) + \alpha_2 a \sin(ah)] \\ = \; & [\alpha_2 a \cos(ah) - \beta_2 \sin(ah)] \times [\beta_1 \cos(ah) - \alpha_1 a \sin(ah)], \end{aligned}$$

α_1, α_2, β_1 and β_2 are constants and

$$\bar{f}(n, t) = \int_{-h}^{h} f(z, t) P_n(z) dz,$$

$$\bar{u}(n,t) = \int_{-h}^{h} u(z,t) P_n(z) dz,$$

$$P_n(z) = Q_n \cos(a_n z) - W_n \sin(a_n z),$$

$$Q_n = a_n(\alpha_1 + \alpha_2)\cos(a_n h) + (\beta_1 - \beta_2)\sin(a_n h),$$

$$W_n = (\beta_1 + \beta_2)\cos(a_n h) + (\alpha_2 - \alpha_1) a_n \sin(a_n h).$$

Applying the Laplace transform defined in [8] to the equations (12), (14) to (16) and using (13), we obtain

$$\frac{d^2\bar{T}^*}{dr^2} + \frac{1}{r}\frac{d\bar{T}^*}{dr} - q^2\bar{T}^* = 0, \; q^2 = a_n^2 + \frac{s}{k}, \tag{17}$$

the boundary conditions

$$\frac{d\bar{T}^*(a,n,s)}{dr} = \bar{u}^*(n,s), \tag{18}$$

$$\bar{T}^*(b,n,s) = \bar{g}^*(n,s), \tag{19}$$

and the interior condition

$$\frac{d\bar{T}^*(\xi,n,s)}{dr} = \bar{f}^*(n,s), \tag{20}$$

where $\bar{T}^*$ denotes the Laplace transform of $\bar{T}$ and s is a Laplace transform parameter.

The equation (17) is a Bessel equation whose solution is given by

$$\bar{T}^*(r,n,s) = AI_0(qr) + BK_0(qr) \tag{21}$$

where A, B are the constants depending on n, and $I_0(qr)$, $K_0(qr)$ are modified Bessel's functions of first and second kind of order zero respectively and as r tends to zero, $K_0(qr)$ tends to infinity but $\bar{T}^*(r,n,s)$ remains finite.

Using (18) and (20) in (21), we obtain

$$A = \frac{\bar{f}^*(n,s)K_0(qa) - \bar{u}^*(n,s)K_0(q\xi)}{q[K_0(qa)I_0(q\xi) - K_0(q\xi)I_0(qa)]},$$

and

$$B = \frac{-\bar{f}^*(n,s)I_0(qa) + \bar{u}^*(n,s)I_0(q\xi)}{q[K_0(qa)I_0(q\xi) - K_0(q\xi)I_0(qa)]}.$$

Substituting these values in (21) and then inversion of Laplace transform and finite Marchi-Fasulo transform leads to

$$\begin{aligned}
T(r,z,t) \;=\;& \sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{[Y_0'(\lambda_m a)][Y_0'(\lambda_m \xi)]}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]}\\
&\times[Y_0'(\lambda_m a)J_0'(\lambda_m r)-Y_0'(\lambda_m r)J_0'(\lambda_m a)]\times\int_0^t \bar{f}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt'\\
&-\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{[Y_0'(\lambda_m \xi)]^2}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]}\\
&\times[Y_0'(\lambda_m \xi)J_0'(\lambda_m r)-Y_0'(\lambda_m r)J_0'(\lambda_m \xi)]\\
&\times\int_0^t \bar{u}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt' \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
g(z,t) \;=\;& \sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{[Y_0'(\lambda_m a)][Y_0'(\lambda_m \xi)]}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]}\\
&\times[Y_0'(\lambda_m a)J_0'(\lambda_m b)-Y_0'(\lambda_m b)J_0'(\lambda_m a)]\times\int_0^t \bar{f}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt'\\
&-\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{[Y_0'(\lambda_m \xi)]^2}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]}\\
&\times[Y_0'(\lambda_m \xi)J_0'(\lambda_m b)-Y_0'(\lambda_m b)J_0'(\lambda_m \xi)]\\
&\times\int_0^t \bar{u}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt', \qquad (23)
\end{aligned}$$

where m, n are positive integers, and λ_m are the positive roots of the transcendental equations

$$[Y_0'(\lambda_m a)J_0'(\lambda_m b)-Y_0'(\lambda_m b)J_0'(\lambda_m a)]=0$$

and

$$\lambda_n=\int_{-h}^{h}P_n^2(z)dz=h[Q_n^2+W_n^2]+\frac{\sin(a_n h)}{2a_n}[Q_n^2-W_n^2]$$

Equations (22) and (23) are the desired solutions of the given problem with $\beta_1=\beta_2=1$ and $\alpha_1=k_1$, $\alpha_2=k_2$.

4 Determination Of Thermoelastic Displacement

Substituting the value of $T(r,z,t)$ from the equation (22) in (1) one obtains the thermoelastic displacement function $U(r,z,t)$:

$$\begin{aligned}
U(r,z,t) &= -(1+\upsilon)a_t \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \sum_{m=1}^{\infty} \frac{[Y_0'(\lambda_m a)][Y_0'(\lambda_m \xi)]}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]} \\
&\times [Y_0'(\lambda_m a)J_0'(\lambda_m r) - Y_0'(\lambda_m r)J_0'(\lambda_m a) \times \int_0^t \bar{f}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt'] \\
&+(1+\upsilon)a_t \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \sum_{m=1}^{\infty} \frac{[Y_0'(\lambda_m \xi)]^2}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]} \\
&\times [Y_0'(\lambda_m \xi)J_0'(\lambda_m r) - Y_0'(\lambda_m r)J_0'(\lambda_m \xi)] \int_0^t \bar{u}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt'.
\end{aligned}$$

5 Determination Of Stress Functions

Using the series expansion of $T(r,z,t)$ in the equations (10) and (11), the stress functions are obtained as

$$\begin{aligned}
\sigma_{rr} &= \frac{2\mu}{r}(1+\upsilon)a_t \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \sum_{m=1}^{\infty} \frac{\lambda_m[Y_0'(\lambda_m a)][Y_0'(\lambda_m \xi)]}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]} \\
&\times [Y_0'(\lambda_m a)J_0''(\lambda_m r) - Y_0''(\lambda_m r)J_0'(\lambda_m a) \times \int_0^t \bar{f}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt'] \\
&-\frac{2\mu}{r}(1+\upsilon)a_t \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \sum_{m=1}^{\infty} \frac{\lambda_m[Y_0'(\lambda_m \xi)]^2}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]} \\
&\times [Y_0'(\lambda_m \xi)J_0''(\lambda_m r) - Y_0''(\lambda_m r)J_0'(\lambda_m \xi)] \int_0^t \bar{u}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt' \quad (24)
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\theta\theta} &= 2\mu(1+\upsilon)a_t \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \sum_{m=1}^{\infty} \frac{\lambda_m^2[Y_0'(\lambda_m a)][Y_0'(\lambda_m \xi)]}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]} \\
&\times [Y_0'(\lambda_m a)J_0'''(\lambda_m r) - Y_0'''(\lambda_m r)J_0'(\lambda_m a) \times \int_0^t \bar{f}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt'] \\
&-2\mu(1+\upsilon)a_t \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \sum_{m=1}^{\infty} \frac{\lambda_m[Y_0'(\lambda_m \xi)]^2}{(\lambda_m^2+a_n^2)[[Y_0'(\lambda_m a)]^2-[Y_0'(\lambda_m \xi)]^2]} \\
&\times [Y_0'(\lambda_m \xi)J_0'''(\lambda_m r) - Y_0'''(\lambda_m r)J_0'(\lambda_m \xi)] \int_0^t \bar{u}(n,t')e^{-k(\lambda_m^2+a_n^2)(t-t')}dt' \quad (25)
\end{aligned}$$

6 Convergence Of The Series Solution

In order for the solution to be meaningful the series expressed in equation (22) should converge for all $a \leq r \leq b$ and $-h \leq z \leq h$, and we should further investigate the conditions which has to be imposed on the functions $u(z,t)$, $g(z,t)$ and $f(z,t)$ so that the convergence of the series expansion for $T(r,z,t)$ is valid. The temperature equation (22) can be expressed as

$$\begin{aligned} T(r,z,t) \quad = \quad & \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \sum_{m=1}^{M_0} \frac{[Y_0'(\lambda_m a)][Y_0'(\lambda_m \xi)]}{(\lambda_m^2 + a_n^2)[[Y_0'(\lambda_m a)]^2 - [Y_0'(\lambda_m \xi)]^2]} \\ & \times [Y_0'(\lambda_m a) J_0'(\lambda_m r) - Y_0'(\lambda_m r) J_0'(\lambda_m a)] \times \int_0^t \bar{f}(n,t') e^{-k(\lambda_m^2 + a_n^2)(t-t')} dt' \\ & - \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \sum_{m=1}^{M_0} \frac{[Y_0'(\lambda_m \xi)]^2}{(\lambda_m^2 + a_n^2)[[Y_0'(\lambda_m a)]^2 - [Y_0'(\lambda_m \xi)]^2]} \\ & \times [Y_0'(\lambda_m \xi) J_0'(\lambda_m r) - Y_0'(\lambda_m r) J_0'(\lambda_m \xi)] \int_0^t \bar{u}(n,t') e^{-k(\lambda_m^2 + a_n^2)(t-t')} dt'. \end{aligned}$$

We impose conditions so that both $T(r,n,t)$ and $T_t(r,n,t)$ converge in some generalized sense to $g(r,n)$ and $h(r,n)$ respectively as $t \to 0$ in the Marchi-Fasulo transform domain. As per finite classic Marchi-Fasulo transform $\sin(a_n z)$ and $\cos(a_n z)$ are bounded, thus λ_n converges to a finite limit with $\beta_1 = \beta_2 = 1$ and $\alpha_1 = k_1$, $\alpha_2 = k_2$ and h selected with finite limits for our desired solution. Taking into account of the asymptotic behaviors of $P_n(z)$, positive roots λ_m, and eigenvalues a_n given in [6], it is observed that the series expansion for $T(r,z,t)$ will be convergent if

$$\int_0^t e^{-k(\lambda_m^2 + a_n^2)(t-t')} \left\{ \begin{array}{c} \bar{f}(n,t') \\ \bar{u}(n,t') \end{array} \right\} dt' = O\left(\frac{1}{\lambda_m^2 + a_n^2} \right)^k, \ k > 0. \tag{26}$$

Here $\bar{f}(n,t')$ and $\bar{u}(n,t')$ in equation (26) can be chosen as one of the following functions or their combination involving addition or multiplication of constant functions, $\sin(\omega t')$, $\cos(\omega t')$, $e^{-kt'}$ or polynomials of t'. Thus, $T(r,z,t)$ is convergent to a limit $\{T(r,z,t)\}_{\,r=b,z=h}$ as convergence of a series for $r = b$ implies convergence for all $r \leq b$ at any z.

7 Special Case And Numerical Result

In (23), let

$$f(z,t) = (1 - e^{-t})(z-h)^2(z+h)^2\xi,$$

$$u(z,t) = (1 - e^{-t})(z-h)^2(z+h)^2 a,$$

$$\alpha = 4(k_1 + k_2)\xi, \ a = 1, \ b = 2, \ \xi = 1.5, \ h = 1, \ k = 0.86,$$

$t = 1$ second, and λ_m the positive roots of the transcendental equation

$$[Y_0'(\lambda_m a)J_0'(\lambda_m b) - Y_0'(\lambda_m b)J_0'(\lambda_m a)] = 0.$$

Then

$$\begin{aligned}
\frac{g(z,t)}{\alpha} &= \sum_{n=1}^{\infty}\left[\frac{\cos^2(a_n) - \cos(a_n)\sin(a_n)}{a_n^2\lambda_n}\right]P_n(z) \\
&\quad\times\left\{\sum_{m=1}^{\infty}\frac{[Y_0'(\lambda_m)][Y_0'(1.5\lambda_m)][Y_0'(\lambda_m)J_0'(2\lambda_m) - Y_0'(2\lambda_m)J_0'(\lambda_m)]}{(\lambda_m^2 + a_n^2)[[Y_0'(\lambda_m)]^2 - [Y_0'(1.5\lambda_m)]^2]}\right. \\
&\quad\left.-\sum_{m=1}^{\infty}\frac{[Y_0'(1.5\lambda_m)]^2[Y_0'(1.5\lambda_m)J_0'(2\lambda_m) - Y_0'(2\lambda_m)J_0'(1.5\lambda_m)]}{(\lambda_m^2 + a_n^2)[[Y_0'(\lambda_m)]^2 - [Y_0'(1.5\lambda_m)]^2]}\right\} \\
&\quad\times\int_0^t (1 - e^{-t'})e^{-0.86(\lambda_m^2 + a_n^2)(1-t')}dt'. \qquad (27)
\end{aligned}$$

The (27) is calculated numerically and it is observed that as r increases the value of $\frac{g(z,t)}{\alpha}$ increases.

8 Conclusion

The temperature, displacements and thermal stresses on the outer curved surface of a thin annular disc have been obtained, when the interior heat flux and the other three boundary conditions are known, with the aid of finite Marchi-Fasulo transform and Laplace transform techniques. The results are obtained in terms of Bessel's function in the form of infinite Marchi-Fasulo transform series. The series solutions converge provided we take sufficient number of terms in the series. Since the thickness of annular disk is very small, the series solution given here will be definitely convergent.

Any particular case can be derived by assigning suitable values to the parameters and functions in the series expressions. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

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