Applied Mathematics E-Notes, 6(2006), 10-16 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On Rolle's Theorem For Polynomials Over The Complex Numbers*

Luis H. Gallardo†

Received 22 October 2004

Abstract

Let $P(z)$ be a polynomial with complex coefficients of degree d having 2 distinct roots. We prove that its derivative $P'(z)$ has a root "in between" two given roots of $P(z)$ in the following cases: (i) $d \leq 4$, and (ii) $P(z)$ has at most 3 distinct roots. Moreover, the result also holds for any d under some mild conditions.

1 Introduction

Let $R(z)$ be a polynomial with complex coefficients. Let a, b be two distinct complex numbers. We say that $R(z)$ has a root r "in between" a and b if $\pi(r)$ is in the segment $\{a+t(b-a) : 0 < t < 1\}$ where π is the orthogonal projection over the line determined by a and b.

It is not known whether or not the derivative of a polynomial $P(z)$ with complex coefficients and having 2 distinct roots, can have a root "in between" two given distinct roots of $P(z)$.

However, we have:

a) The result follows immediately from Rolle's Theorem when $P(z)$ has all its roots on a line.

b) If for some roots $a \neq b$ of $P(z)$ all other roots of $P(z)$ are in between a and b then P' has some root in between a and b. This holds by Lucas's Theorem (see e.g. [3], p. 22).

c) If $P(z) = z(z-1)Q(z)$, where $Q(0) \neq 0$, $Q(1) \neq 0$ and all zeros z of Q satisfy $|z| \leq 1$ then P' has some root in between 0 and 1. This holds by a special proved case (see [4], p. 270) of a Conjecture of Goodman-Rahman-Ratti and Schmeisser (see [1], [7]).

*Mathematics Subject Classifications: 12D10, 11C08.

†Department of Mathematics, University of Brest, 6, Avenue Le Gorgeu, C.S. 93837, 29238 Brest Cedex 3, France.

See also the comprehensive papers of Marden ([2], [3], [4], and [5]).

We let $n(z) = |z|^2 = z\bar{z}$, denote the square of the norm, $Tr(z) = z + \bar{z}$ denote the trace of a complex z, and $Re(z) = Tr(z)/2$ denote the real part of z.

Observe that we may assume that $P(z)$ has the form:

$$P(z) = (z^2 - 1/4)\, R(z)$$

in which $R(z)$ is some monic polynomial. Thus we will search for roots of the derivative P' in the region

$$S = \{z \in C \mid -1/2 < Re(z) < 1/2\}.$$

2 Some Results For General d

They are presented in the following Proposition:

PROPOSITION 1. Let n be a positive integer. Assume that $d = n + 1$ so that $\deg(R(z)) = n - 1$. Write $R(z) = \epsilon z + Q(z)$ where $Q'(0) = 0$; i.e., ϵ is the coefficient of z in $R(z)$. Thus, $P(z) = (z^2 - 1/4)(\epsilon z + Q(z))$. It is convenient to write $P'(z)$ as:

$$P'(z) = (n + 1)(z - z_1) \dots (z - z_n),$$

$$P'(z) = (n + 1) \sum_{k=0}^{n} (-1)^{n-k} s_{n-k} z^k.$$

(a) If $\epsilon = 0$ then $z = 0$ is a root of P'.

(b) If $|\epsilon| < (n + 1)/2^{n-2}$ then for some integer i we have $|Tr(z_i)| < 1$.

(c) If there is a k such that $|s_k/s_n| > \begin{pmatrix} n \\ k \end{pmatrix} 2^{n-k}$ then for some i we have $|Tr(z_i)| < 1$.

(d) If P' has only one root a then $Tr(a) = 0$.

PROOF. We obtain (a) since z divides $Q'(z)$ when $\epsilon = 0$. Observe that $P'(0) = -\epsilon/4$. Thus $|\epsilon| = 4(n + 1)|z_1| \dots |z_n|$. Assume that $|Tr(z_i)| \geq 1$ holds for all i. From the inequality $|z_i| \geq 1/2|Tr(z_i)|$ we obtain

$$|\epsilon| \geq (n + 1)/2^{n-2}.$$

This proves (b). The proof of (c) is similar. To prove (d) assume that $P' = (n + 1)(z - a)^n$ for some complex a. It follows that $P(z) = (z - a)^{n+1} + b$ for some complex constant b. Since $1/2$ and $-1/2$ are roots of $P(z)$ we obtain

$$(1/2 - a)^{n+1} = (-1/2 - a)^{n+1}.$$

Thus $a \notin \{1/2, -1/2\}$ and

$$1/2 - a = \rho(-1/2 - a) \tag{1}$$

for some complex $\rho \neq 1$ that satisfy $\rho^{n+1} = 1$. It follows that $n(\rho) = 1$ and that $Tr(\rho) \neq 2$. In other words (1) may be written as

$$a = (-1/2)\frac{\rho + 1}{\rho - 1}. \tag{2}$$

From (2) we obtain

$$Tr(a) = (-1/2)\frac{\bar{\rho} - \rho + \rho - \bar{\rho}}{2 - Tr(\rho)} = 0.$$

This completes the proof of the proposition.

3 Case When P Has At Most 3 Different Roots

First of all we recall that an involution f is an operator of period 2, i.e. $f \circ f$ gives the identity. Assume that for some positive integers α, β; for some non negative integer γ, and for some complex number $a \notin \{-1/2, 1/2\}$

$$P(z) = (z - 1/2)^{\alpha}(z + 1/2)^{\beta}(z - a)^{\gamma}.$$

The solution of our problem shall be reduced to prove that some involution h of the Riemann sphere attached to these numbers transforms complex numbers z, with $|Tr(z)| \geq 1$, into complex numbers $w = h(z)$ with $|Tr(w)| < 1$:

LEMMA 1. Given positive integers α, β, γ let h be the involution of the Riemann sphere S_2 defined over a complex z by:

$$w = h(z) = \frac{Az + B}{Cz + D}, \tag{3}$$

where

(a) $A = 2(-\alpha^2 + \beta^2 - (\alpha - \beta)\gamma)$,

(b) $B = -((\alpha - \beta)^2 + (\alpha + \beta)\gamma)$,

(c) $C = 4((\alpha + \beta)^2 + (\alpha + \beta)\gamma)$,

(d) $D = -A = 2(\alpha^2 - \beta^2 + (\alpha - \beta)\gamma)$.

Then $|Tr(z)| \geq 1$ implies $|Tr(w)| < 1$.

PROOF. After some computation it becomes clear that $Tr(w) < 1$ is equivalent to the condition

$$C_1 > 0$$

where

$$C_1 = 8\alpha s((\beta-\alpha)(\beta-\alpha-\gamma) - 2(\beta^2-\alpha^2-\alpha\gamma)t + 4(\alpha+\beta)sn) \qquad (4)$$

and

$$s = \alpha+\beta+\gamma, \ \ t = Tr(z), \ \ n = n(z).$$

One has the inequality

$$4n \geq t^2.$$

Thus, it suffices to prove that

$$K_1 = 8\alpha s((\beta-\alpha)(\beta-\alpha-\gamma) - 2(\beta^2-\alpha^2-\alpha\gamma)t + (\alpha+\beta)st^2) > 0;$$

i.e., it suffices to prove that $K > 0$ where

$$K = K_1/(8\alpha(\alpha+\beta)s^2).$$

In other words, we need to prove that

$$K = (t-t_1)(t-t_2) > 0,$$

where

$$t_1 = \frac{\beta-\alpha}{\beta+\alpha}, \ \ t_2 = \frac{\beta-\alpha-\gamma}{\beta+\alpha+\gamma}.$$

But, this holds since $|t| \geq 1$ while trivially $-1 < t_1\,, t_2 < 1$.

Similarly, we may prove that $Tr(w) > -1$ since it shall suffice to prove that

$$C_2 = 8\beta s L > 0,$$

where

$$L = (\beta-\alpha)(\beta+\gamma-\alpha) - 2(\gamma\beta+\beta^2-\alpha^2)t + 4(\alpha+\beta)sn.$$

This completes the proof of the lemma.

We are ready to present our main theorem.

THEOREM 1. Let a be a complex number, let α, β be positive integers and let γ be a non negative integer. Let $P(z)$ be a polynomial defined by

$$P(z) = (z-1/2)^{\alpha}(z+1/2)^{\beta}(z-a)^{\gamma}.$$

Then P' has a root r that satisfy

$$|Tr(r)| < 1.$$

PROOF. When $a \in \{-1/2, 1/2\}$ we may assume that $\gamma = 0$. From the equation

$$P'(z)/P(z) = \alpha/(z-1/2) + \beta/(z+1/2) = 0,$$

we obtain

$$z = \frac{\beta - \alpha}{2(\beta + \alpha)}.$$

From this it follows readily that

$$|Tr(z)| < 1,$$

completing the proof in this case.

The equation

$$P'(z)/P(z) = \alpha/(z - 1/2) + \beta/(z + 1/2) + \gamma/(z - a) = 0,$$

may be written in the form

$$(z - z_1)(z - z_2) = 0,$$

where

(a) $z_1 + z_2 = \frac{(1/2+a)\beta - (1/2-a)\alpha}{\alpha+\beta+\gamma}$,

(b) $z_1\, z_2 = \frac{a(\beta-\alpha)/2 - \gamma/4}{\alpha+\beta+\gamma}$.

From these 2 equalities, we obtain after some computation

$$z_2 = h(z_1)$$

where h is defined in Lemma 1. The result follows immediately from the same lemma.

See ([4], p. 268) and ([3], p. 9) for a geometric interpretation of the position in the complex plane of the zeros z_1, z_2 above.

4 Case Of $\deg(P) \leq 4$

The resolution of this case comes essentially from a result of Grace and Heawood reported by Polya:

LEMMA 2. Let $a \neq b$ be two distinct complex numbers. Let d be a positive integer. Let $P(z)$ be a polynomial of degree d with complex coefficients such that $P(a) = 0 = P(b)$. Then P' has a root r that satisfies

$$|r - (a + b)/2| \leq |(a - b)| \,/\, (2\, \tan(\pi/d)).$$

PROOF. See [6] Problem 150, p. 60 and p. 238.

REMARK. Observe that Proposition 1 part (b) and also Lemma 2 imply Rolle's Theorem for

$$\deg(P) \leq 3.$$

Indeed, it remains only some polynomials to study:

LEMMA 3. Let $P(z)$ be a polynomial of degree $d \leq 4$ with complex coefficients such that $P(1/2) = 0 = P(-1/2)$ and such that $\gcd(P', z^2 - 1/4) \neq 1$ then P' has a root r that satisfy

$$|Tr(r)| < 1.$$

PROOF. Following the Remark we may assume that $n = 4$. Observe that if the result holds for $P(z)$ then it also holds for $-P(-z)$. So, it suffices to study the following polynomials:

(a) $((z^2 - 1/4)^2)' = (2z - 1)(2z + 1)z$,

(b) $((z^2 - 1/4)(z - 1/2)^2)' = (1/4)(4z + 1)(2z - 1)^2$,

(c) $P(z) = (z^2 - 1/4)(z - 1/2)(z - a)$ where $a \notin \{-1/2, 1/2\}$.

The cases (a) and (b) are also easily deduced from Proposition 1 part (b). Observe that $P(z)$ has 3 distinct roots in the case (c) above. Thus the result follows from Theorem 1.

We are ready to present the main result of the section:

THEOREM 2. Let $P(z)$ be a polynomial of degree $d \leq 4$ with complex coefficients such that $P(1/2) = 0 = P(-1/2)$ then P' has a root r that satisfy

$$|Tr(r)| < 1.$$

PROOF. The result follows immediately from Lemmata 2 and 3.

When $P(z)$ has 4 distinct roots, and $d \geq 5$, the problem seems non trivial.

Acknowledgments. I discovered the problem in an old version of the Web page of Denis Simon. We thank the Mathematics Department of the Penn State University at University Park campus, for hospitality. This helped us to complete this paper while working with Leon Vaserstein in our main subject (Waring's problem for polynomials).

Thanks also to the referee that improved our former proof of Lemma 3 by suggesting the Remark.

References

[1] A. W. Goodman, Q. I. Rahman and J. S. Ratti, On the zeros of a polynomial and its derivative, Proc. Amer. Math. Soc., 21(1969), 273–274.

[2] M. Marden, The location of the zeros of the derivative of a polynomial, Amer. Math. Monthly, 42(5)(1935), 277–286.

[3] M. Marden, Geometry of Polynomials, Math. Surveys no. 3, American Mathematical Society, 1966.

[4] M. Marden, Conjectures on the critical points of a polynomial, Amer. Math. Monthly, 90(4)(1983), 267–276.

[5] M. Marden, The search for a Rolle's Theorem in the complex domain, Amer. Math. Monthly, 92(9)(1985), 643–650.

[6] G. Pólya and G. Szegö, Problems and Theorems in Analysis, Volume II, Springer-Verlag, 1976.

[7] G. Schmeisser, Bemerkungen zu einer Vermutung von Ilieff, Math. Z., 111(1969), 121–25.