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AN INTEGRAL INEQUALITY FOR CONVEX FUNCTIONS AND APPLICATIONS IN NUMERICAL INTEGRATION*

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Abstract

A general integral inequality for convex functions is derived. New error bounds for the midpoint, trapezoid, averaged midpoint-trapezoid and Simpson's quadrature rules are obtained. Applications in numerical integration are also given.

1 Introduction

In recent years a number of authors have considered an error analysis for some known and some new quadrature formulas. They used an approach from an inequalities point of view. For example, the midpoint quadrature rule is considered in [1], [3], [15], the trapezoid rule is considered in [4], [5], [15], the averaged midpoint-trapezoid quadrature rule is considered in [8], [15], [16] and Simpson's rule is considered in [2], [5], [13]. In most cases estimations of errors for these quadrature rules are obtained by means of derivatives of integrands.

In this paper we first derive a general integral inequality for convex functions. Then we apply this inequality to obtain new error bounds for the above mentioned quadrature rules. Finally, we give applications in numerical integration.

The main property of the obtained error bounds is that they are expressed in terms function values of integrand f, which has to be a convex function. Hence, we can apply these quadrature rules (with the obtained error bounds) to integrands which are not differentiable functions. In composite quadrature formulas we use the same data for finding an approximate value of integral and for finding an estimation of error. An illustrative example is given that shows how accurate the obtained estimations can be.

2 A General Inequality

We begin with some elementary facts. Let $f : [a, b] \to R$ be a given function. We say that f is an even function with respect to the point $t_0 = (a+b)/2$ if $f(a+b-t) = f(t)$

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for $t \in [a,b]$. We say that f is an odd function with respect to the point $t_0 = (a+b)/2$ if $f(a+b-t) = -f(t)$ for $t \in [a,b]$. Here we use the term even (odd) function for a given function $f : [a,b] \to R$ if f is even (odd) with respect to the point $t_0 = (a+b)/2$.

Each function $f : [a,b] \to R$ can be represented as a sum of one even and one odd function,

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x) = \frac{f(x)+f(a+b-x)}{2}$ is an even function and $f_2(x) = \frac{f(x)-f(a+b-x)}{2}$ is an odd function.

It is not difficult to verify the following facts. If f is an odd function then $|f|$ is an even function. If f, g are even or odd functions then fg is an even function. If f is an even function and g is an odd function then fg is an odd function.

We now show that if f is an integrable and odd function then $\int_a^b f(x)dx = 0$. We have

$$\int_a^b f(x)dx = -\int_a^b f(a+b-x)dx = \int_a^b f(u)du = -\int_a^b f(x)dx.$$

Thus, $2\int_a^b f(x)dx = 0$ and we proved the above assertion.

If f is an integrable and even function then we have

$$\int_a^b f(x)dx = 2\int_{(a+b)/2}^b f(x)dx = 2\int_a^{(a+b)/2} f(x)dx.$$

We have

$$\int_a^b f(x)dx = \int_a^{(a+b)/2} f(x)dx + \int_{(a+b)/2}^b f(x)dx$$

and

$$\int_{(a+b)/2}^b f(x)dx = \int_{(a+b)/2}^b f(a+b-x)dx = -\int_{(a+b)/2}^a f(u)du = \int_a^{(a+b)/2} f(x)dx.$$

Thus the above assertion holds.

We now show that the function

$$K(a,b,x) = \begin{cases} x-\alpha & x \in [a,(a+b)/2) \\ 0 & x = (a+b)/2 \\ x-\beta & x \in ((a+b)/2, b] \end{cases} \tag{1}$$

is an odd function if $\alpha + \beta = a + b$. For $x \in [a,(a+b)/2)$ we have

$$K(a,b,a+b-x) = a+b-x-\beta = a+b-x-(a+b-\alpha) = -x+\alpha = -K(a,b,x).$$

For $x \in ((a+b)/2, b]$ we have

$$K(a,b,a+b-x) = a+b-x-\alpha = a+b-x-(a+b-\beta) = -x+\beta = -K(a,b,x).$$

Finally, for $x=(a+b)/2$, $K(a,b,a+b-(a+b)/2)=0=-K(a,b,(a+b)/2)$. Hence, $K(a,b,x)$ is an odd function.

THEOREM 1. Let $f \in C^2(a,b)$ and $f''(x) \geq 0$, $x \in [a,b]$, i.e. f is a convex function. Let $K(a,b,x)$ be defined by (1). Then

$$\begin{aligned} & \left|(\alpha-a)f(a)+(\beta-\alpha)f((a+b)/2)+(b-\beta)f(b)-\int_a^b f(x)dx\right| \\ \leq\ & \|K\|_\infty \left[f(a)+f(b)-2f((a+b)/2)\right], \end{aligned} \tag{2}$$

where $\|K\|_\infty = \max_{x\in[a,b]} |K(a,b,x)|$.

PROOF. Integrating by parts, we obtain

$$\begin{aligned} \int_a^b K(a,b,x)f'(x)dx &= \int_a^{(a+b)/2} (x-\alpha)f'(x)dx + \int_{(a+b)/2}^b (x-\beta)f'(x)dx \\ &= (\alpha-a)f(a)+(\beta-\alpha)f\left(\frac{a+b}{2}\right) \\ &\quad +(b-\beta)f(b)-\int_a^b f(x)dx. \end{aligned} \tag{3}$$

If we introduce the notations

$$f_1'(x)=\frac{f'(x)+f'(a+b-x)}{2},\quad f_2'(x)=\frac{f'(x)-f'(a+b-x)}{2}$$

then we have

$$f'(x)=f_1'(x)+f_2'(x)$$

and $K(a,b,x)f_1'(x)$ is an odd function while $|f_2'(x)|$ and $K(a,b,x)f_2'(x)$ are even functions. Thus, we have

$$\begin{aligned} \int_a^b K(a,b,x)f'(x)dx &= \int_a^b K(a,b,x)\left[f_1'(x)+f_2'(x)\right]dx \\ &= \int_a^b K(a,b,x)f_2'(x)dx \end{aligned}$$

and

$$\begin{aligned} \left|\int_a^b K(a,b,x)f'(x)dx\right| &= \left|\int_a^b K(a,b,x)f_2'(x)dx\right| \leq \|K\|_\infty \int_a^b |f_2'(x)|\,dx \\ &\leq 2\|K\|_\infty \int_{(a+b)/2}^b |f_2'(x)|\,dx \\ &= \|K\|_\infty \int_{(a+b)/2}^b |f'(x)+f'(a+b-x)|\,dx. \end{aligned} \tag{4}$$

Note now that $f'(x)$ is an increasing function, since $f''(x) \geq 0$, $x \in [a,b]$. Thus,

$$\begin{aligned} \int_{(a+b)/2}^{b} |f'(x) + f'(a+b-x)|\, dx &= \int_{(a+b)/2}^{b} (f'(x) + f'(a+b-x))\, dx \\ &= f(b) + f(a) - 2f(\frac{a+b}{2}). \end{aligned} \tag{5}$$

From (3)-(5) we easily get (2).

3 Applications to Quadrature Formulas

We have the following results.

PROPOSITION 1. (Midpoint inequality) Let $f \in C^2(a,b)$ and $f''(x) \geq 0$, $x \in [a,b]$, i.e. f is a convex function. Then we have

$$\left| f(\frac{a+b}{2})(b-a) - \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left[f(a) + f(b) - 2f(\frac{a+b}{2}) \right]. \tag{6}$$

PROOF. We choose $\alpha = a$, $\beta = b$ in (1). Then $\|K\|_\infty = \frac{b-a}{2}$. From this fact and (2) we see that (6) holds.

PROPOSITION 2. (Trapezoid inequality) Under the assumptions of Proposition 1 we have

$$\left| \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left[f(a) + f(b) - 2f(\frac{a+b}{2}) \right]. \tag{7}$$

PROOF. We choose $\alpha = \beta = \frac{a+b}{2}$ in (1). Then $\|K\|_\infty = \frac{b-a}{2}$. From the last fact and (2) we see that (7) holds.

PROPOSITION 3. (Averaged midpoint-trapezoid inequality) Under the assumptions of Proposition 1 we have

$$\left| \left[f(a) + 2f(\frac{a+b}{2}) + f(b) \right] \frac{b-a}{4} - \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left[f(a) + f(b) - 2f(\frac{a+b}{2}) \right]. \tag{8}$$

PROOF. We choose $\alpha = \frac{3a+b}{4}$, $\beta = \frac{a+3b}{4}$ in (1). Then $\|K\|_\infty = \frac{b-a}{4}$. From this fact and (2) we see that (8) holds.

PROPOSITION 4. (Simpson's inequality) Under the assumptions of Proposition 1 we have

$$\left| \left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right] \frac{b-a}{6} - \int_a^b f(x)dx \right| \leq \frac{b-a}{3} \left[f(a) + f(b) - 2f(\frac{a+b}{2}) \right]. \tag{9}$$

PROOF. We choose $\alpha = \frac{5a+b}{6}$, $\beta = \frac{a+5b}{6}$ in (1). Then $\|K\|_\infty = \frac{b-a}{3}$. From the last fact and (2) we see that (9) holds.

REMARK 1. The inequalities obtained in Propositions 1-4 can also be derived using the well-known Hermite-Hadamard inequalities. Furthermore, if f is a convex function then we have

$$0 \leq \int_a^b f(x)dx - f(\frac{a+b}{2})(b-a) \leq \frac{b-a}{2}\left[f(a)+f(b)-2f(\frac{a+b}{2})\right]$$

and

$$\frac{b-a}{2}\left[2f(\frac{a+b}{2}) - f(a) - f(b)\right] \leq \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) \leq 0.$$

4 Applications in Numerical Integration

Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, $i = 0, 1, ..., n-1$. We define

$$\sigma_n(f) = \sum_{i=0}^{n-1} h_i \left[f(x_i) + f(x_{i+1}) - 2f\left(\frac{x_i + x_{i+1}}{2}\right)\right]. \tag{10}$$

THEOREM 2. Let $f \in C^2(a,b)$ and $f''(x) \geq 0$, $x \in [a,b]$. Let π be a given subdivision of the interval $[a,b]$. Then

$$\int_a^b f(x)dx = A_M(\pi, f) + R_M(\pi, f),$$

where

$$A_M(\pi, f) = \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right)$$

and

$$|R_M(\pi, f)| \leq \frac{1}{2}\sigma_n(f).$$

PROOF. Apply Proposition 1 to the intervals $[x_i, x_{i+1}]$ and sum.

THEOREM 3. Under the assumptions of Theorem 2 we have

$$\int_a^b f(x)dx = A_T(\pi, f) + R_T(\pi, f),$$

where

$$A_T(\pi, f) = \frac{1}{2}\sum_{i=0}^{n-1} h_i \left[f(x_i) + f(x_{i+1})\right]$$

and

$$|R_T(\pi, f)| \leq \frac{1}{2}\sigma_n(f).$$

PROOF. Apply Proposition 2 to the intervals $[x_i, x_{i+1}]$ and sum.

THEOREM 4. Under the assumptions of Theorem 2 we have

$$\int_a^b f(x)dx = A_{MT}(\pi, f) + R_{MT}(\pi, f),$$

where

$$A_{MT}(\pi, f) = \frac{1}{4}\sum_{i=0}^{n-1} h_i \left[f(x_i) + 2f(\frac{x_i + x_{i+1}}{2}) + f(x_{i+1})\right]$$

and

$$|R_{MT}(\pi, f)| \leq \frac{1}{4}\sigma_n(f).$$

PROOF. Apply Proposition 3 to the intervals $[x_i, x_{i+1}]$ and sum.

THEOREM 5. Under the assumptions of Theorem 2 we have

$$\int_a^b f(x)dx = A_S(\pi, f) + R_S(\pi, f),$$

where

$$A_S(\pi, f) = \frac{1}{6}\sum_{i=0}^{n-1} h_i \left[f(x_i) + 4f(\frac{x_i + x_{i+1}}{2}) + f(x_{i+1})\right]$$

and

$$|R_S(\pi, f)| \leq \frac{1}{3}\sigma_n(f).$$

PROOF. Apply Proposition 3 to the intervals $[x_i, x_{i+1}]$ and sum.

EXAMPLE. Let us now consider the integral $\int_0^1(-\sqrt{x})dx$. Note that $f(x) = -\sqrt{x}$ is a convex function on the interval $[0, 1]$. Note also that we cannot apply the classical estimations of error (expressed in terms of the first, second, ... derivatives), since f', f'', ... are unbounded on the interval $[0, 1]$. The exact value is

$$\int_0^1 (-\sqrt{x})dx = -0.66666666666666.$$

If we use formulas given in theorems 2-5 with $h_i = h = \frac{1}{n}$, $n = 1000$, then we get

$$\begin{array}{rl}
A_M(\pi, f) = -0.66666857129568 & |R_M(\pi, f)| \leq 0.84369E - 05 \\
A_T(\pi, f) = -0.66666013439368 & |R_T(\pi, f)| \leq 0.84369E - 05 \\
A_{MT}(\pi, f) = -0.66666435284468 & |R_{MT}(\pi, f)| \leq 0.42184E - 05 \\
A_S(\pi, f) = -0.66666575899501 & |R_S(\pi, f)| \leq 0.56246E - 05
\end{array}$$

and the exact errors are

$$|R_M(\pi, f)| = 0.19046E - 05,$$

$$|R_T(\pi, f)| = 0.65322E - 05,$$

$$|R_{MT}(\pi, f)| = 0.23138E - 05,$$

$$|R_S(\pi, f)| = 0.90767E - 06.$$

We see that the estimations are very accurate for this example.

REMARK 2. Note that in all error inequalities we use the same values $f(x_i)$ to calculate the approximation of the integral $\int_a^b f(t)dt$ and to obtain the error bound and recall that function evaluations are generally considered the computationally most expensive part of quadrature algorithms. On the other hand, the usual way to estimate the errors is to find $\left\|f^{(k)}\right\|_\infty$, $k \in \{1, 2, ...\}$. Hence, the presented way of estimation of the errors is very simple and effective. We have only one restriction: the integrand has to be a convex function.

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