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# SHARP INEQUALITIES FOR PERIODIC FUNCTIONS *

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**Abstract**

Sharp inequalities for periodic functions are established which can help to improve many existence criteria for solutions of differential equations.

## 1 Introduction

Let $C_T$, where $T > 0$, be the space of all real continuous $T$-periodic functions of the form $x : R \to R$ and endowed with the usual linear structure as well as the norm $\|x\|_0 = \max_{0\le t\le T} |x(t)|$. For any $x \in C^{(1)}(R,R) \cap C_T$ and any $\xi \in [0,T]$, by the fundamental theorem of Calculus,

$$\|x\|_0 \le |x(\xi)| + \int_0^T |x'(s)|\, ds. \tag{1}$$

In particular, let $C_T^0$ be the set of all real functions of the form $y \in C_T$ such that $y(\xi_y) = 0$ for some $\xi_y \in [0,T]$. Then for any $y \in C^{(1)}(R,R) \cap C_T^0$,

$$\|y\|_0 \le \int_0^T |y'(s)|\, ds. \tag{2}$$

Such inequalities have been used, among many things, for finding a priori bounds for $T$-periodic solutions of differential equations. By means of such a priori bounds, we may then look for $T$-periodic solutions by means of fixed point theorems such as the continuation theorems (see e.g. [1]) which are popular (see for examples [1-18]). However, since (1) and (2) were applied to find the a priori bounds in these references, and since they are not sharp inequalities (as will be seen below), the corresponding existence criteria cannot be sharp neither.

In [19], it is noted that (1) can easily be extended to

$$\|x\|_0 \le |x(\xi)| + \frac{1}{2}\int_0^T |x'(s)|\, ds, \tag{3}$$

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which can be used for deriving existence criteria for periodic solutions. The major objective of this paper is, among other things, to show further that the inequalities (3) is sharp. To illustrate their use, we will show how some of the existence criteria can be improved in straightforward manners.

## 2 Sharp Inequalities

We begin by improving the inequality (1).

THEOREM 1. Suppose $x = x(t) \in C^{(1)}(R, R) \cap C_T$ and $\xi \in [0, T]$. Then

$$\|x\|_0 \leq |x(\xi)| + \frac{1}{2}\int_0^T |x'(s)|\, ds, \tag{4}$$

where the constant factor $1/2$ is the best possible.

PROOF. Let $x = x(t) \in C^{(1)}(R, R) \cap C_T$ and $\xi \in [0, T]$. Then for any $t \in [\xi, \xi + T]$, we have

$$x(t) = x(\xi) + \int_\xi^t x'(s)\, ds \tag{5}$$

and

$$x(t) = x(\xi + T) + \int_{\xi+T}^t x'(s)\, ds = x(\xi) - \int_t^{\xi+T} x'(s)\, ds. \tag{6}$$

From (5) and (6), we see that for any $t \in [\xi, \xi + T]$,

$$2x(t) = 2x(\xi) + \int_\xi^t x'(s)\, ds - \int_t^{\xi+T} x'(s)\, ds, \tag{7}$$

that is

$$x(t) = x(\xi) + \frac{1}{2}\left\{\int_\xi^t x'(s)\, ds - \int_t^{\xi+T} x'(s)\, ds\right\}. \tag{8}$$

Thus for any $t \in [\xi, \xi + T]$

$$|x(t)| \leq |x(\xi)| + \frac{1}{2}\int_\xi^{\xi+T} |x'(s)|\, ds, \tag{9}$$

so that

$$\begin{aligned} \|x\|_0 &= \max_{\xi \leq t \leq \xi+T} |x(t)| \leq |x(\xi)| + \frac{1}{2}\int_\xi^{\xi+T} |x'(s)|\, ds. \\ &\leq |x(\xi)| + \frac{1}{2}\int_0^T |x'(s)|\, ds. \end{aligned} \tag{10}$$

Now we assert that if $\alpha$ is a constant and $\alpha < 1/2$, then there are $x \in C^{(1)}(R, R) \cap C_T$ and $\xi \in [0, T]$ such that

$$\|x\|_0 > |x(\xi)| + \alpha \int_0^T |x'(s)|\, ds. \tag{11}$$

Indeed, let $x(t) = 2 - \cos \frac{2\pi}{T} t$ and $\xi = 0$. Then $\|x\|_0 = 3$ and

$$|x(\xi)| + \alpha \int_0^T |x'(s)|\, ds = 1 + \alpha \frac{2\pi}{T} \int_0^T \left| \sin \frac{2\pi}{T} s \right| ds = 1 + 4\alpha < \|x\|_0$$

as required. This shows that the constant 1/2 in (4) is the best possible. The proof is complete.

THEOREM 2. Let $y \in C^{(1)}(R, R) \cap C_T^0$. Then

$$\|y\|_0 \leq \frac{1}{2} \int_0^T |y'(s)|\, ds, \tag{12}$$

and the constant factor 1/2 is the best possible.

PROOF. Let $y \in C^{(1)}(R, R) \cap C_T^0$. Then $y \in C^{(1)}(R, R) \cap C_T$ and there is $\xi \in [0, T]$ such that $y(\xi) = 0$. From Theorem 1, we have

$$\|y\|_0 \leq \frac{1}{2} \int_0^T |y'(s)|\, ds. \tag{13}$$

Now we assert that if $\beta$ be a constant and $\beta < 1/2$, then there is $y \in C^{(1)} \cap C_T^0$ such that

$$\|y\|_0 > \beta \int_0^T |y'(s)|\, ds. \tag{14}$$

Indeed, let $y(t) = 1 - \cos \frac{2\pi}{T} t$. Then $y(0) = 0$, so $y \in C^{(1)}(R, R) \cap C_T^0$, $\|y\|_0 = 2$ and

$$\beta \int_0^T |y'(s)|\, ds = \beta \frac{2\pi}{T} \int_0^T \left| \sin \frac{2\pi}{T} s \right| ds = 4\beta < \|y\|_0 \tag{15}$$

as required. Thus the constant factor 1/2 in (12) is the best possible. The proof is complete.

Next let $x \in C^{(n)}(R, R) \cap C_T$ where $n \geqslant 2$. For $i = 1, 2, .., n-1$, note that $x^{(i-1)}(0) = x^{(i-1)}(T)$, so that is $\xi \in [0, T]$ such that $x^{(i)}(\xi) = 0$. Thus from Theorem 2, we have

COROLLARY 1. Let $x \in C^{(n)}(R, R) \cap C_T$ where $n \geqslant 2$. Then

$$\left\| x^{(i)} \right\|_0 \leq \frac{1}{2} \int_0^T \left| x^{(i+1)}(s) \right| ds, \quad i = 1, 2, ..., n-1. \tag{16}$$

# 3 Applications

By means of the sharp inequalities derived above, we can improve many existence criteria for periodic solutions of delay differential equations in the literature. We will demonstrate our ideas by improving the results in several recent papers.

First, in [3], the authors consider the existence of $2\pi$-periodic solutions of Rayleigh equations of the form

$$x''(t) + f(x'(t)) + g(x(t - \tau(t))) = p(t), \tag{17}$$

where $f$ and $g$ are real continuous functions defined on $R$, $f(0) = 0$, $\tau$ and $p$ are real continuous functions defined on $R$ with period $2\pi$, and $\int_0^{2\pi} p(t)\,dt = 0$.

We will show that the condition $4\pi[r_1 + (2\pi + 1) r_2] < 1$ in Theorem 1 of [3] can be replaced by the weaker condition $2\pi[r_1 + (\pi + 1) r_2] < 1$. More precisely, we have the following existence criteria.

THEOREM 3. Suppose there exist constants $r_1, r_2 \geqslant 0$, $K > 0$ and $D > 0$ such that

[A$_1$] $|f(y)| \le r_1 |y| + K$ for $y \in R$,

[A$_2$] $xg(x) > 0$ and $|g(x)| > r_1 |x| + K$ for $|x| > D$, and

[A$_3$] $\lim_{x \to -\infty} \frac{g(x)}{x} \le r_2$.

Then for $2\pi[r_1 + (\pi + 1) r_2] < 1$, (17) has a $2\pi$-periodic solution.

PROOF. We let

$$x''(t) + \lambda f(x'(t)) + \lambda g(x(t - \tau(t))) = \lambda p(t), \tag{18}$$

where $\lambda \in (0, 1)$. In view of the proof of Theorem 1 in [3], it suffices to prove that for any $2\pi$-periodic solution $x(t)$ of (18), there exist constants $M_0$ and $M_1$, which are independent from $x(t)$ and $\lambda$, such that

$$\|x\|_0 \le M_0 \text{ and } \|x'\|_0 \le M_1. \tag{19}$$

First of all, as in the proof of Theorem 1 in [3], we may show that there is a $t^* \in [0, 2\pi]$ such that

$$|x(t^*)| \le \|x'\|_0 + D. \tag{20}$$

Then by (4) and (20), we get

$$\|x\|_0 \le |x(t^*)| + \frac{1}{2} \int_0^{2\pi} |x'(s)|\,ds < (\pi + 1) \|x'\|_0 + D. \tag{21}$$

In view of the condition $2\pi[r_1 + (\pi + 1) r_2] < 1$, we may find a positive number $\varepsilon$ such that

$$2\pi[r_1 + (\pi + 1)(r_2 + \varepsilon)] < 1. \tag{22}$$

From condition [A$_3$], there is a $\rho > D$ such that for $x < -\rho$,

$$\frac{g(x)}{x} < r_2 + \varepsilon. \tag{23}$$

Let

$$E_1 = \{t \mid t \in [0, 2\pi], x(t - \tau(t)) > \rho\},$$

$$E_2 = \{t \mid t \in [0, 2\pi], x(t - \tau(t)) < -\rho\},$$

$$E_3 = \{t \mid t \in [0, 2\pi], |x(t - \tau(t))| \le \rho\},$$

and $g_\rho = \max_{|x| \le \rho} |g(x)|$. As in the proof of Theorem 1 in [3], we have

$$\int_{E_2} |g(x(t - \tau(t)))| \, dt \le 2\pi (r_2 + \varepsilon) \|x\|_0, \tag{24}$$

$$\int_{E_3} |g(x(t - \tau(t)))| \, dt \le 2\pi g_\rho, \tag{25}$$

$$\begin{aligned} \int_{E_1} |g(x(t - \tau(t)))| \, dt \quad &\le \quad \int_{E_2} |g(x(t - \tau(t)))| \, dt \\ &\quad + \int_{E_3} |g(x(t - \tau(t)))| \, dt + \int_0^{2\pi} |f(x'(t))| \, dt, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \int_0^{2\pi} |x''(s)| \, ds \quad &\le \quad \int_0^{2\pi} |f(x'(t))| \, dt + \int_{E_1} |g(x(t - \tau(t)))| \, dt \\ &\quad + \int_{E_2} |g(x(t - \tau(t)))| \, dt + \int_{E_3} |g(x(t - \tau(t)))| \, dt + 2\pi \|p\|_0. \end{aligned} \tag{27}$$

By (26) and (27), we see that

$$\begin{aligned} \int_0^{2\pi} |x''(s)| \, ds \quad &\le \quad 2\pi \|p\|_0 + 2\int_0^{2\pi} |f(x'(t))| \, dt \\ &\quad + 2\int_{E_2} |g(x(t - \tau(t)))| \, dt + 2\int_{E_3} |g(x(t - \tau(t)))| \, dt. \end{aligned} \tag{28}$$

From (18), (21), (24), (25), (27) and condition [$A_1$], we have

$$\begin{aligned} \int_0^{2\pi} |x''(s)| \, ds \quad &\le \quad 2\left\{\int_0^{2\pi} |f(x'(t))| \, dt + 2\pi (r_2 + \varepsilon) \|x\|_0 + 2\pi g_\rho\right\} + 2\pi \|p\|_0 \\ &\le \quad 2\left\{2\pi r_1 \|x'\|_0 + 2\pi K + 2\pi (r_2 + \varepsilon) [(\pi + 1) \|x'\|_0 + D] + 2\pi g_\rho\right\} + 2\pi \|p\|_0 \\ &\le \quad 4\pi [r_1 + (\pi + 1)(r_2 + \varepsilon)] \|x'\|_0 + \sigma_1 \\ &\le \quad 2\pi [r_1 + (\pi + 1)(r_2 + \varepsilon)] \int_0^{2\pi} |x''(s)| \, ds + \sigma_1 \\ &= \quad \sigma \int_0^{2\pi} |x''(s)| \, ds + \sigma_1, \end{aligned} \tag{29}$$

where the fourth inequality follows from (16), and $\sigma = 2\pi[r_1 + (\pi + 1)(r_2 + \varepsilon)]$ and $\sigma_1 = 4\pi(r_2 + \varepsilon) D + 4\pi g_\rho + 4\pi K + 2\pi \|p\|_0$. It follows that

$$\int_0^{2\pi} |x''(s)|\, ds \le D_1, \tag{30}$$

where $D_1 = \frac{\sigma_1}{1-\sigma}$. By (16), we have

$$\|x'\|_0 \le \frac{1}{2}\int_0^{2\pi} |x''(s)|\, ds \le M_1 \tag{31}$$

where $M_1 = \frac{1}{2}D_1$. In view of (21) and (31), we have

$$\|x\|_0 < (\pi + 1)\|x'\|_0 + D \le M_0, \tag{32}$$

where $M_0 = (\pi + 1) M_1 + D$. The proof is complete.

We remark that the same reasoning shows that the condition $4\pi[r_1 + (2\pi + 1) r_2] < 1$ in Theorem 2 of [3] can be replaced by the weaker condition $2\pi[r_1 + (\pi + 1) r_2] < 1$.

In [4], the authors studied the existence of $T$-periodic solutions of equations of the form

$$x''(t) + f(t, x(t), x(t - \tau_0(t)))\, x'(t) + \beta(t)\, g(x(t - \tau_1(t))) = p(t), \tag{33}$$

where $f$ is a real continuous functions defined on $R^3$ with positive period $T$, $g$ is a real continuous function defined on $R$, and $\beta, \tau_0, \tau_1$ as well as $p$ are real continuous functions defined on $R$ with period $T$. By replacing appropriate inequalities in [4] with ours, it is not difficult to see that the conditions $f_1 < \frac{1}{T}$ and $r < \frac{1-f_1T}{\beta_1 T^2}$ in Theorem 1 of [4] can be replaced by the weaker conditions $f_1 < \frac{2}{T}$ and $r < 2\left(\frac{2-f_1T}{\beta_1T^2}\right)$, and the condition $r < \frac{\sigma}{\beta_1 T}$ in Theorem 2 of [4] by the weaker condition $r < \frac{2\sigma}{\beta_1 T}$.

Similar principles can also be applied to other first order delay differential equations.

For insatnce, in [20], the authors studied the existence of $T$-periodic solutions of equations of the form

$$x'(t) = f(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_m(t))), \tag{34}$$

where $f = f(t, u_0, ..., u_m)$ is a real continuous function defined on $R^{m+2}$ and is $T$-periodic in $t$ for fixed $u_0, ..., u_m$. We assume as in [20] that $a_0, ..., a_m, \tau_1, ..., \tau_m, p \in C_T$ and $\tau_1, ..., \tau_m, p$ are nonnegative functions and $\tau_0(t) \equiv 0$ on $R$.

THEOREM 4. Suppose that (i) there exists $\rho_0 > 0$ such that $f(t, u_0, ..., u_m) > 0$ $(< 0)$ for $u_0, ..., u_m > \rho_0$, and $f(t, u_0, ..., u_m) < 0$ (respectively $> 0$) for $u_0, ..., u_m < -\rho_0$, and (ii)

$$|f(t, u_0, u_1, ..., u_3)| \le |a_0(t)|\,|u_0| + |a_1(t)|\,|u_1| + ... + |a_m(t)|\,|u_m| + p(t) \tag{35}$$

for $t \in R$. If

$$\left\{\sum_{i=0}^{m} \int_0^T |a_i(t)|\, dt\right\} < 2, \tag{36}$$

then the equation (34) has at least one $T$-periodic solution.

PROOF. We first remark that the condition (36) is much weaker than the origianl condition

$$T\left\{\sum_{i=0}^{m}\max_{0\le t\le T}|a_i(t)|\right\}<1.$$

We let

$$x'(t)=\lambda f\left(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_m(t))\right), \tag{37}$$

where $\lambda\in(0,1)$. In view of the proof of Theorem 1 in [20]. It suffices to prove that for any $T$-periodic solution $x(t)$ of (37), there exist constants $R_1$, which is independent from $x(t)$ and $\lambda$, such that

$$\|x\|_0\le\rho_1. \tag{38}$$

From (37), we see that

$$\int_0^T f\left(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_m(t))\right)dt=0. \tag{39}$$

In view of our assumption (i) and (39) and the fact that $x$ is $T$-periodic, there is a $t_0\in[0,T]$ such that

$$|x(t_0)|\le\rho_0. \tag{40}$$

It is easy to see from (4) and (40) that

$$\|x\|_0\le\rho_0+\frac{1}{2}\int_0^T|x'(s)|\,ds. \tag{41}$$

From (35), (37) and (41), we get

$$\begin{aligned}
\int_0^T|x'(t)|\,dt &\le \int_0^T\left|f\left(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_m(t))\right)\right|dt\\
&\le \sum_{i=0}^{m}\int_0^T|a_i(t)|\,|x(t-\tau_i(t))|\,dt+\int_0^T p(t)\,dt\\
&\le \sum_{i=0}^{m}\int_0^T|a_i(t)|\left(\rho_0+\frac{1}{2}\int_0^T|x'(s)|\,ds\right)dt+T\|p\|_0\\
&= \frac{1}{2}\left(\sum_{i=0}^{m}\int_0^T|a_i(t)|\,dt\right)\int_0^T|x'(s)|\,ds+T\|p\|_0+\rho_0\sum_{i=0}^{m}\int_0^T|a_i(t)|\,dt\\
&= \frac{1}{2}\left(\sum_{i=0}^{m}\int_0^T|a_i(t)|\,dt\right)\int_0^T|x'(s)|\,ds+C_1,
\end{aligned} \tag{42}$$

for some positive constant $C_1$. By (36), we see that there is some positive constant $C_2$ such that

$$\int_0^T |x'(t)|\, dt \leq C_2. \tag{43}$$

In view of (41) and (43), we know that

$$\|x\|_0 \leq \rho_0 + \frac{1}{2} C_2. \tag{44}$$

The proof is complete.

To close this note, we mention that in [21], the authors studied the existence of $T$-periodic solutions of equations of the form

$$x'(t) = r(t) - a(t)x(t-\sigma) - b(t)x'(t-\tau), \tag{45}$$

where $\tau$ and $\sigma$ are positive constants, $b \in C_T \cap C^{(1)}(R, R)$ such that $a(t+\sigma) - b'(t+\tau) \neq 0$ for $t \in R$. By means of the same principle illustrated above, it is not difficult to see that the condition

$$\max_{0 \leq t \leq T} |b(t)| + T \max_{0 \leq t \leq T} |a(t)| < 1$$

in the main Theorem of [21] can be replaced by the weaker condition $\max_{0 \leq t \leq T} |b(t)| + \frac{1}{2} \int_0^T a(t) dt < 1$.

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