**Applied Mathematics E-Notes, 5(2005), 157-163 ©** ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

# FURTHER EXTENDED SINH-COSH AND SIN-COS METHODS AND NEW NON-TRAVELING WAVE SOLUTIONS OF THE (2+1)-DIMENSIONAL DISPERSIVE LONG WAVE EQUATIONS*

Deng-Shan Wang†, Yu-Jie Ren‡§, Hong-Qing Zhang¶

Received 11 September 2004

**Abstract**

In this paper, based on symbolic computation and the further extended sinh-cosh and sin-cos methods, families of non-travelling wave solutions of the (2+1)-dimension dispersive long wave equations are obtained. These solutions include Jacobi elliptic function solution, soliton-like solution, and so on.

## 1 Introduction

In recent years, nonlinear evolution and wave equations (NEWEs) have attracted considerable attention. As is known to all, there exist many methods for obtaining solutions of NEWEs, such as the inverse scattering method [1], Darboux transformation method [2-4], Hirota bilinear method [5-6], algebro-geometric method [7-9], tanh function method [10-11], Backlund transformation method[12-13], sin-cos method [14-15], and so on.

The integrable dispersive long wave equations, especially the higher dimensional ones, are of current interest in both physics and mathematics. Since the 1960's, many dispersive long wave equations of one dimension have been proposed to model the water wave propagation in certain infinitely-long channels of finite constant depth and narrow width. These equations are found integrable, to have some soliton solutions and plenty of mathematical properties associated with the infinite dimensional completely integrable Hamiltonian systems. Lately, improvement has been made in the stability theory for solitary wave solutions of model equations for long waves.

*Mathematics Subject Classifications: 35Q51, 35Q53, 35Q99.

†Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P. R. China
‡Department of Mathematics and Physics, Dalian Institute of Light Industry, Dalian 116034, P. R. China
§Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P. R. China
¶Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P. R. China

In this paper, with the help of symbolic computational software, we further improve the extended sinh-cosh and sin-cos methods in [14-19] and obtain some new non-travelling wave solutions of the (2+1)-dimension dispersive long wave equations in [20]

$$\begin{cases} u_{yt} + v_{xx} + \frac{1}{2}(u^2)_{xy} = 0, \\ v_t + (uv + u + u_{xy})_x = 0, \end{cases} \tag{1}$$

which were obtained by Boiti et al. as a compatibility condition for a weak Lax pair.

# 2 Methods

Consider a given NEWE with independent variables $x = (t, x_1, x_2, \cdots)$ and dependent variable $u(x)$. The following formal solution of the given NEWE will be sought by the following ansätz

$$u(x) = A_0 + \sum_{i=1}^{n} \cosh^{i-1}(\omega)[A_i \sinh(\omega) + B_i \cosh(\omega)], \tag{2}$$

where $n$ is an integer which is determined by balancing the highest order derivative term with the highest order nonlinear term in the given NEWE, and $A_0 = A_0(x), ..., A_n = A_n(x)$, $B_1 = B_1(x), ..., B_n = B_n(x)$, $\omega = \omega(\mu)$, $\mu = \mu(x)$ are all differentiable functions. $\omega$ satisfies

$$\left(\frac{d\omega}{dx}\right)^2 = \sinh^2(\omega(\mu)) + c, \tag{3}$$

where $c = 1 - m^2$ and $m$ is the modulus of the Jacobi elliptic function.

Equation (3) has the following solutions:

$$\sinh(\omega) = \mathrm{cs}(\mu, m) = \frac{\mathrm{cn}(\mu, m)}{\mathrm{sn}(\mu, m)},$$

$$\cosh(\omega) = \mathrm{ns}(\mu, m) = \frac{1}{\mathrm{sn}(\mu, m)}, \tag{4}$$

where $\mathrm{sn}(\mu, m), \mathrm{cn}(\mu, m)$ are the Jacobian elliptic sine function and the Jacobian elliptic cosine function respectively. It is well known that when $m \to 1$, $\mathrm{cs}(\mu, m) \to \mathrm{csch}(\mu)$, $\mathrm{ns}(\mu, m) \to \coth(\mu)$; and when $m \to 0$, $\mathrm{cs}(\mu, m) \to \cot(\mu), \mathrm{ns}(\mu, m) \to \csc(\mu)$.

We can also seek NEWE's solution in the following form:

$$u(x) = A_0 + \sum_{i=1}^{n} \cos^{i-1}(\omega)[A_i \sin(\omega) + B_i \cos(\omega)], \tag{5}$$

where $n, \omega, A_0, A_i, B_i,\ i = 1, 2, \cdots, n$, are all defined as in (2) and $\omega$ satisfies

$$\left(\frac{d\omega}{dx}\right)^2 = -\sin^2(\omega(\mu)) + c, \tag{6}$$

where $c = m^2$ and $m$ is the modulus of the Jacobi elliptic function.

Equation (6) has the following solutions:

$$\sin(\omega) = m\mathrm{sn}(\mu, m),$$

$$\cos(\omega) = \mathrm{dn}(\mu, m), \tag{7}$$

where $\mathrm{dn}(\mu, m)$ is the Jacobian elliptic function of the third kind. We remark that when $m \to 1$, $\mathrm{sn}(\mu, m) \to \tanh(\mu), \mathrm{dn}(\mu, m) \to \mathrm{sech}(\mu)$, and when $m \to 0$, $\mathrm{sn}(\mu, m) \to \sin(\mu), \mathrm{dn}(\mu, m) \to 1$.

**Remark:** We may assume the coefficients of the ansätz (2) and ansätz (5) are undetermined functions and obtain more general solutions of the NEWEs.

# 3 New Exact Non-travelling Wave Solutions

In this section, we use the methods described in Section 2 to find new non-travelling wave solutions of the (2+1)-dimension dispersive long wave equations.

## 3.1 The application of the further extended sinh-cosh method

Now we apply the further extended sinh-cosh method to (1). By the balancing procedure, we suppose (1) has the following solution:

$$\left\{ \begin{array}{l} u = A_0 + A_1 \sinh(\omega) + B_1 \cosh(\omega), \\ v = B_0 + A_2 \sinh(\omega) + B_2 \cosh(\omega) + \cosh(\omega)\left(A_3 \sinh(\omega) + B_3 \cosh(\omega)\right), \end{array} \right. \tag{8}$$

where $\omega = \omega(\mu), \mu = \alpha x + p + q$, and $\alpha$ is a nonzero constant, $A_0, B_0, A_1, B_1, A_2, B_2, A_2, B_2, A_3, B_3$ are functions of $(y, t)$, $p$ is the function of $y$, $q$ is a function of $t$.

With the help of Maple, substitution of (8) into (1) with $\omega$ satisfying (3), yields a differential equation about $\sinh^i(\omega)\cosh^j(\omega)(\sinh^2(\omega) + c)^{\frac{k}{2}}$, $i = 1, 2, ...; j = 0, 1; k = 0, 1$. Setting the coefficients of $\sinh^i(\omega)\cosh^j(\omega)(\sinh^2(\omega) + c)^{\frac{k}{2}}$, $i = 1, 2, ...; j = 0, 1; k = 0, 1$, to zero, we can deduce a series of over-determined partial differential equations about $p, q, A_0, B_0, A_1, B_1, A_2, B_2, A_2, B_2, A_3, B_3$.

Solving the over-determined equations with Maple, we derive the solutions of the over-determined partial differential equations :

Case 1.1

$$A_0 = F_3(t), A_1 = -2\alpha, A_2 = 0, A_3 = 0, B_2 = 0, q = \int -F_3(t)\,\alpha\, dt + C_3, \tag{9}$$

$$B_3 = F_1(y), B_0 = -1 + \frac{1}{2}F_1(y)\,c - \frac{1}{2}F_1(y), p = \int -\frac{F_1(y)}{2\alpha}dy, B_1 = 0,$$

Case 1.2

$$A_0 = F_3(t), A_1 = 0, B_0 = -1 + \frac{1}{2}F_1(y)\,c - F_1(y), q = \int -F_3(t)\,\alpha dt + C_3, \tag{10}$$

$$A_2 = 0, B_1 = -2\,\alpha, B_2 = 0, A_3 = 0, B_3 = F_1\left(y\right), p = \int -\frac{F_1\left(y\right)}{2\alpha}dy,$$

where $F_1(y)$ is an arbitrary function of $y$, $F_3(t)$ is an arbitrary function of $t$, and $C_3$ is an arbitrary constant.

From (8) and Case 1.1-Case 1.2, we derive the following non-travelling wave solutions of (1):

Family 1.

$$\left\{\begin{array}{l} u_{11} = F_3\left(t\right) - 2\,\alpha\,\mathrm{cs}\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3, m\right), \\ v_{11} = -1 + \frac{F_1(y)(c-1)}{2} + F_1\left(y\right)\mathrm{ns}^2\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3, m\right). \end{array}\right. \tag{11}$$

Family 2.

$$\left\{\begin{array}{l} u_{12} = F_3\left(t\right) - 2\,\alpha\,\mathrm{ns}\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3, m\right), \\ v_{12} = -1 + \frac{F_1(y)(c-1)}{2} + F_1\left(y\right)\mathrm{ns}^2\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3, m\right). \end{array}\right. \tag{12}$$

When $m \to 1, \mathrm{cs}(\mu, m) \to \mathrm{csch}(\mu)$ and $\mathrm{ns}(\mu, m) \to \coth(\mu), c \to 0$. So under the limit condition, we obtain the following soliton-like solutions of (1):

Family 3.

$$\left\{\begin{array}{l} u_{13} = F_3\left(t\right) - 2\,\alpha\,\mathrm{csch}\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3\right), \\ v_{13} = -1 - \frac{F_1(y)}{2} + F_1\left(y\right)\coth^2\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3\right). \end{array}\right. \tag{13}$$

Family 4.

$$\left\{\begin{array}{l} u_{14} = F_3\left(t\right) - 2\,\alpha\,\coth\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3\right), \\ v_{14} = -1 - \frac{F_1(y)}{2} + F_1\left(y\right)\coth^2\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3\right). \end{array}\right. \tag{14}$$

When $m \to 0, cs(\mu, m) \to cot(\mu)$ and $ns(\mu, m) \to csc(\mu), c \to 1$. So under the limit condition, we obtain the following trigonometric function solutions of (1):

Family 5.

$$\left\{\begin{array}{l} u_{15} = F_3\left(t\right) - 2\,\alpha\cot\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3\right), \\ v_{15} = -1 + F_1\left(y\right)\csc^2\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3\right). \end{array}\right. \tag{15}$$

Family 6.

$$\left\{\begin{array}{l} u_{16} = F_3\left(t\right) - 2\,\alpha\,csc\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3\right), \\ v_{16} = -1 + F_1\left(y\right)csc^2\left(\alpha\,x - \int \frac{F_1(y)}{2\alpha}dy - \int F_3\left(t\right)\alpha dt + C_3\right). \end{array}\right. \tag{16}$$

### 3.2 The application of the further extended sin-cos method

Now we apply the further extended sine-cos method to (1). By the balancing procedure, we suppose (1) has the following solution:

$$\left\{\begin{array}{l} u = A_0 + A_1 \sin(\omega) + B_1 \cos(\omega), \\ v = B_0 + A_2 \sin(\omega) + B_2 \cos(\omega) + \cos(\omega)\left(A_3 \sin(\omega) + B_3 \cos(\omega)\right), \end{array}\right. \tag{17}$$

where $\omega = \omega(\mu), \mu = \alpha x + q$, and $\alpha$ is a nonzero constant, $q, A_0, B_0, A_1, B_1, A_2, B_2, A_2, B_2, A_3, B_3$ are functions of $(y, t)$ to be determined later.

With the help of Maple, substitution of (17) into (1) with $\omega$ satisfying (6), yields a differential equation about $\sin^i(\omega)\cos^j(\omega)(-\sin^2(\omega)+c)^{\frac{k}{2}}$, $i = 1, 2, ...; j = 0, 1; k = 0, 1$. Setting the coefficients of $\sin^i(\omega)\cos^j(\omega)(-\sin^2(\omega)+c)^{\frac{k}{2}}$, $i = 1, 2, ...; j = 0, 1; k = 0, 1$, to zero, we can deduce a series of over-determined partial differential equations about $q, A_0, B_0, A_1, B_1, A_2, B_2, A_2, B_2, A_3, B_3$.

Solving the over-determined equations with Maple, we derive the solutions of the over-determined partial differential equations:
Case 2.1

$$q = \int \frac{F_1(y)}{2\alpha} dy + F_3(t), B_0 = \frac{1}{2} F_1(y) c - 1 - \frac{1}{2} F_1(y),$$

$$B_3 = F_1(y), A_2 = 0, B_2 = 0, B_1 = 0, A_3 = 0, A_0 = -\frac{\frac{d}{dt} F_3(t)}{\alpha}, A_1 = -2\alpha, \tag{18}$$

Case 2.2

$$q = \int \frac{F_1(y)}{2\alpha} dy + F_3(t), B_3 = F_1(y), A_2 = 0, B_2 = 0, A_3 = 0,$$

$$A_0 = -\frac{\frac{d}{dt} F_3(t)}{\alpha}, A_1 = 0, B_0 = -F_1(y) + \frac{1}{2} F_1(y) c - 1, B_1 = -2 i\alpha, \tag{19}$$

where $F_1(y)$ is an arbitrary function of $y$, $F_3(t)$ is an arbitrary function of $t$.

From (17) and Case 2.1-Case 2.2, we obtain the following non-travelling wave solutions of (1):

Family 1.

$$\left\{\begin{array}{l} u_{21} = -\frac{\frac{d}{dt} F_3(t)}{\alpha} - 2\alpha m \operatorname{sn}\left(\alpha x + \int \frac{F_1(y)}{2\alpha} dy + F_3(t), m\right), \\ v_{21} = \frac{1}{2} F_1(y) c - 1 - \frac{1}{2} F_1(y) + F_1(y) \operatorname{dn}^2\left(\alpha x + \int \frac{F_1(y)}{2\alpha} dy + F_3(t), m\right). \end{array}\right. \tag{20}$$

Family 2.

$$\left\{\begin{array}{l} u_{22} = -\frac{\frac{d}{dt} F_3(t)}{\alpha} - 2 i\alpha \operatorname{dn}\left(\alpha x + \int \frac{F_1(y)}{2\alpha} dy + F_3(t), m\right), \\ v_{22} = -F_1(y) + \frac{1}{2} F_1(y) c - 1 + F_1(y) \operatorname{dn}^2\left(\alpha x + \int \frac{F_1(y)}{2\alpha} dy + F_3(t), m\right). \end{array}\right. \tag{21}$$

When $m \to 1, \mathrm{sn}(\mu, m) \to \tanh(\mu)$ and $\mathrm{dn}(\mu, m) \to \mathrm{sech}(\mu), c \to 1$. So under the limit condition, we can obtain the following soliton-like solutions of (1):

Family 3.

$$\begin{cases} u_{23} = -\frac{\frac{d}{dt}F_3(t)}{\alpha} - 2\,\alpha\,\tanh\left(\alpha\,x + \int \frac{F_1(y)}{2\alpha}dy + F_3\,(t)\right), \\ v_{23} = -1 + F_1\,(y)\,\mathrm{sech}^2\left(\alpha\,x + \int \frac{F_1(y)}{2\alpha}dy + F_3\,(t)\right). \end{cases} \tag{22}$$

Family 4.

$$\begin{cases} u_{24} = -\frac{\frac{d}{dt}F_3(t)}{\alpha} - 2\,i\alpha\,\mathrm{sech}\left(\alpha\,x + \int \frac{F_1(y)}{2\alpha}dy + F_3\,(t)\right), \\ v_{24} = -\frac{1}{2}\,F_1\,(y) - 1 + F_1\,(y)\,\mathrm{sech}^2\left(\alpha\,x + \int \frac{F_1(y)}{2\alpha}dy + F_3\,(t)\right). \end{cases} \tag{23}$$

These solutions with arbitrary functions involved imply that these solutions have rich local structures. Some of the properties of these new non-travelling wave solutions of (1) are shown by means of figures as follows: Figure 1 and Figure 2 show the properties of $u_{13}, v_{13}$ and $u_{15}, v_{15}$, respectively, where we select the parameters as follows:

$$F_1\,(y) = e^y, F_3\,(t) = e^t, \alpha = 1/2, C_3 = 1/2$$

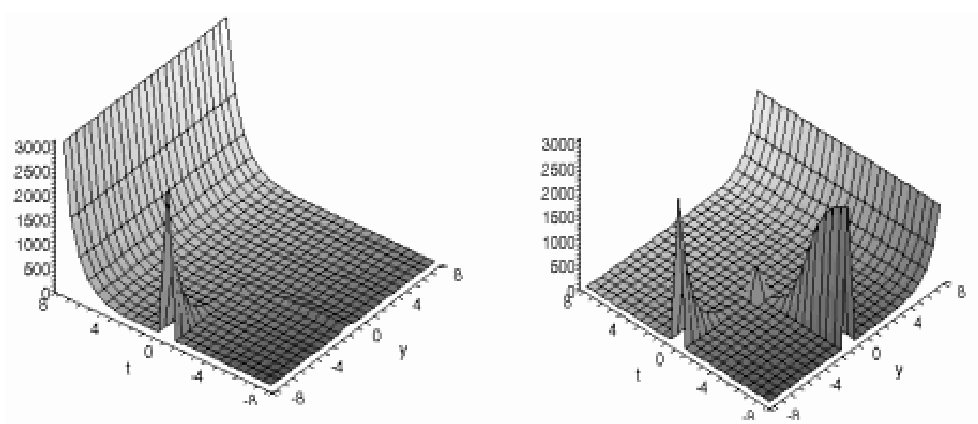


Figure 1. New soliton-like solutions $u_{13}$ and $v_{13}$ of the (2+1)dimension dispersive long wave equations are shown at $x = 0$.

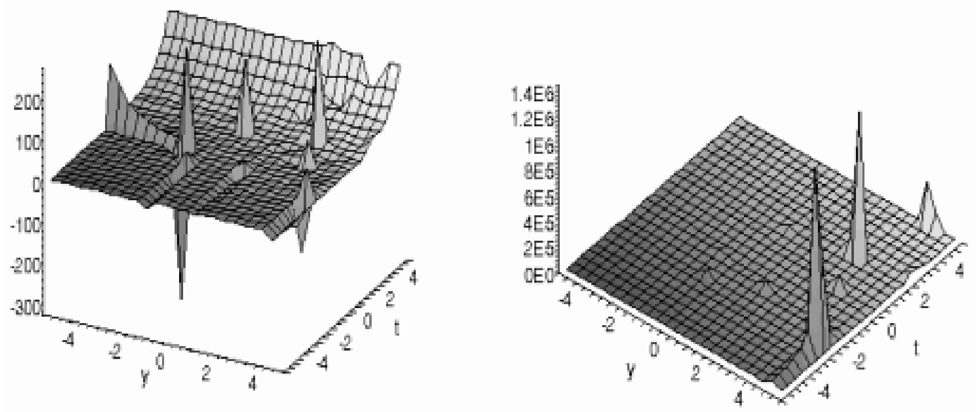


Figure 2. New trigonometric function solutions $u_{15}$ and $v_{15}$ of the (2+1)dimension dispersive long wave equations are shown at $x = 0$.

In summary, with the help of Maple and by using the further extended sinh-cosh and sin-cos methods, we obtained families of non-travelling wave solutions of the (2+1)dimension dispersive long wave equations (1). The two methods can also be applied to solve many other NEWEs.

# References

[1] M. J. Ablowitz and H. Segur, Soliton and the Inverse Scattering Transformation, Philadelphia, 1981.

[2] V. B. Matveev and M. A. Salle, Darboux Transformation and Solitons, Berlin, Springer, 1991.

[3] P. G. Esteveza, Darboux transformation and solutions for an equation in (2+1) dimensions, J. Math Phys., 40(1999), 1406–1419.

[4] V. G. Dubrousky and B. G. Konop, $\partial$ dressing and exact solutions for the (2+1)-dimensional Harry Dym equation, J. Phys. A, 27(1994), 4619–4628.

[5] R. Hirota and J. Sat, Soliton solution of a coupled KdV equation, Phys. Lett. A 85(1981), 407–408.

[6] H. W. Tam, W. X. Ma, X. B. Hu and D. L. Wang, The Hirota-Satsuma coupled KdV equation and a coupled Ito system revisited, J. Phys. Soc. Jpn., 69(2000), 45–51.

[7] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'Skii, A. R. Its and V. B. Matveev, Algebro-Geometrical Approach to Nonlinear Integrable Equations, Berlin, Springer-Verlag, 1994.

[8] M. S. Alber and Y. N. Fedorov, Algebraic geometrical solutions for certain evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians, Inverse Probl., 17(2001), 1017–1042.

[9] C. W. Cao, X. G. Geng and H. Y. Wang, Algebro-geometric solution of the (2+1) dimensional Burgers equation with a discrete variable, J. Math. Phys., 43(2002), 621–643.

[10] W. A. Malfliet, Solitary wave solutions of nonlinear wave equations, Amer. J. Phys., 60(1992), 650–654.

[11] E. J. Parkes, Exact solutions to the two-dimensional Korteweg-de Vries-Burgers equation, J. Phys. A, 27(1994), 497-502.

[12] M. R. Miura, Backlund Transformation, Berlin, Springer, 1978.

[13] Z. Y. Yan and H. Q. Zhang, Symbolic computation and new families of exact soliton-like solutions to the integrable BK equations in (2+1)-dimensional spaces, J. Phys. A, 34(2001), 1785–1792.

[14] C. T. Yan, A simple transformation for nonlinear waves, Phys. Lett. A, 224(1996), 77–84.

[15] X. D. Zheng, T. Xia and H. Zhang, New exact traveling wave solutions for compound KdV-Burgers equations in mathematical physics, Appl. Math. E-Notes, 2(2002), 45-50.

[16] T. Xia, B. Li and H. Q. Zhang, New explicit and exact solutions for the Nizhnik-Novikov-Vesselov Equation, Appl. Math. E-Notes, 1(2001), 139–142.

[17] F. D. Xie, Z. Yan and H. Q. Zhang, Explicit and exact traveling wave solutions of Whitham-Broer-Kaup shallow water equations, Phys. Lett. A, 285(2001), 76–80.

[18] Z. T. Fu, S. K. Liu and S. D. Liu, New transformations and new approach to find exact solutions to nonlinear equations, Phys. Lett. A, 299(2002), 507-512.

[19] Z. Y. Yan and H. Q. Zhang, New explicit and exact travelling wave solutions for a system of variant Boussinesq equation in mathematical physics, Phys. Lett. A, 252(1999), 291–296.

[20] M. Boiti, J. Leon and F. Pempinelli, Spectral transform for a two spatial dimension extension of the dispersive long wave equation, Inverse Problem, 3(1987), 371–387.