Applied Mathematics E-Notes, 5(2005), 129-137 © ISSN 1607-2510
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ON THE BEZIER VARIANT OF SRIVASTAVA-GUPTA OPERATORS*

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Received 27 August 2004

Abstract

In the present paper, we introduce the Bezier variant of a general sequence of linear positive operators introduced by Srivastava and Gupta [8] and estimate the rate of convergence of these operators for functions of bounded variation.

1 Introduction

To approximate integrable functions on the interval $[0,\infty)$, Srivastava and Gupta [8] introduced a general sequence of linear positive operators $G_{n,c}$ and estimated the rate of convergence of $G_{n,c}(f;x)$ by means of the decomposition technique for functions of bounded variation. Let $B_r[0,\infty)$ be the class of bounded variation functions satisfying the growth condition

$$|f(t)| \le M(1+t)^r,\ M>0, r\ge 0, t\to\infty.$$

For a function $f\in B_r[0,\infty)$, the operators are $G_{n,c}$ defined by

$$G_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)f(t)dt + p_{n,0}(x;c)f(0) \tag{1}$$

where

$$p_{n,k}(x;c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}^{(k)}(x) = \begin{cases} e^{-nx}, & c=0 \\ (1+cx)^{-n/c}, & c=1,2,3,... \end{cases}.$$

Here $\{\phi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of functions, defined on the interval $[0,b]$, $b>0$, satisfying the following properties: For each $n\in N$,

*Mathematics Subject Classifications: 41A25, 41A30.

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(i) $\phi_{n,c}(x) \in C^{\infty}[a,b]$, $b > a \geq 0$.
(ii) $\phi_{n,c}(0) = 1$.
(iii) $\phi_{n,c}(x)$ is completely monotone so that $(-1)^k \phi_{n,c}^{(k)}(x) \geq 0$ for $k \in N$.
(iv) There exists an integer c such that

$$\phi_{n,c}^{(k+1)}(x) = -n\phi_{n+c,c}^{(k)}(x),\ n > \max\{0, -c\}; x \in [a,b].$$

Zeng and Gupta [9] introduced the Bezier variants of the well known Baskakov operators, after this Gupta and collaborators in [1-4] have introduced and studied the rate of convergence for different summation-integral type operators. The general sequence of operators $G_{n,c}$, introduced by Srivastava and Gupta [8], have many interesting properties in approximation theory, also Bezier curves play an important role in computer aided geometric design which have many applications in applied mathematics and computer sciences. This motivated us to study further in this direction and we now introduce the Bezier variant of $G_{n,c}$ operators. We define, for $f \in B_r[0,\infty)$ and for each $\alpha \geq 1$, the Bezier variant of the operators $G_{n,c}$ as

$$G_{n,c,\alpha}(f;x) = n\sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)f(t)dt + Q_{n,0}^{(\alpha)}(x;c)f(0) \tag{2}$$

where $Q_{n,k}^{(\alpha)}(x;c) = J_{n,k}^{\alpha}(x;c) - J_{n,k+1}^{\alpha}(x;c)$ and $J_{n,k}(x;c) = \sum_{j=k}^{\infty} p_{n,j}(x;c)$.

The aim of this study is to estimate the rate of convergence of the operators $G_{n,c,\alpha}$ for functions of bounded variation. It is obvious that $G_{n,c,\alpha}$ are linear positive operators and $G_{n,c,\alpha}(1;x) = 1$. In the special case $\alpha = 1$, the operators $G_{n,c,\alpha}$ reduce to the operators $G_{n,c}$ defined by (1). We have the following particular cases:

Case 1. If $\alpha = 1$ and $c = 0$, then the operators $G_{n,c,\alpha}$ reduce to the Phillips operators

$$G_{n,0,1}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;0)\int_0^{\infty} p_{n,k-1}(t;0)f(t)dt + p_{n,0}(x;0)f(0)$$

where

$$p_{n,k}(x;0) = e^{-nx}\frac{(nx)^k}{k!}.$$

The Phillips operators are introduced in [6,7].

Case 2. If $\alpha = 1$ and $c = 1$, then the operators $G_{n,c,\alpha}$ reduce to summation-integral type operators

$$G_{n,1,1}(f;x) = n\sum_{k=1}^{\infty} P_{n,k}(x;1)\int_0^{\infty} p_{n+1,k-1}(t;1)f(t)dt + P_{n,0}(x;1)f(0)$$

where

$$p_{n,k}(x;1) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k(1+x)^{-(n+k)}.$$

These operators has been studied by Gupta et al. in [5].

Therefore, for the operators $G_{n,c,\alpha}$ defined by (2), in the case of $c = 0$ and $c = 1$, we obtain Bezier type of the Phillips operators given *case 1* and Bezier type of the summation integral type operators given *case 2*, respectively.

Alternatively we may rewrite the operators (2) as

$$G_{n,c,\alpha}(f;x) = \int_0^\infty K_{n,c,\alpha}(x;t)dt$$

where

$$K_{n,c,\alpha}(x;t) = n\sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)}(x;c)p_{n+c,k-1}(t;c) + Q_{n,0}^{(\alpha)}(x;c)\delta(t) \tag{3}$$

and $\delta(t)$ is the Dirac delta function.

2 Auxiliary Results

In order to prove our main result we require following lemmas.

LEMMA 1 ([3,8]). For all $x \in (0,\infty)$ and $k \in N$,

$$p_{n,k}(x;c) = \begin{cases} \frac{1}{\sqrt{2enx}}, & c = 0 \\ \sqrt{\frac{1+cx}{2enx}}, & c = 1,2,3,... \end{cases}$$

where the coefficient $1/\sqrt{2e}$ and the estimate order $n^{-1/2}$ are best possible.

LEMMA 2 ([8]). Let

$$\mu_{n,m}(x;c) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^\infty p_{n+c,k-1}(t;c)(t-x)^m dt + (-x)^m p_{n,0}(x;c).$$

Then

$$\mu_{n,0}(x;c) = 1, \mu_{n,1}(x;c) = \frac{cx}{n-c}, \mu_{n,2}(x;c) = \frac{x(1+cx)(2n-c)+(1+3cx)cx}{(n-c)(n-2c)}$$

and

$$\mu_{n,m}(x;c) = O(n^{-[(m+1)/2]}).$$

LEMMA 3. For all $x \in (0,\infty)$, we have

$$n\int_x^\infty p_{n,k}(t;c)dt = \sum_{j=0}^{k} p_{n,j}(x;c).$$

LEMMA 4. Let $x \in (0,\infty)$ and $K_{n,c,\alpha}(x;t)$ be defined by (3). Then, for $\lambda > 2$ and for sufficiently large n,

$$(i)\ \beta_{n,c,\alpha}(x;y) = \int_0^y K_{n,c,\alpha}(x;t)dt \le \frac{\alpha\lambda x(1+cx)}{n(x-y)^2},\ 0 \le y < x,$$

and

$$(ii)\ 1-\beta_{n,c,\alpha}(x;z) = \int_z^\infty K_{n,c,\alpha}(x;t)dt \le \frac{\alpha\lambda x(1+cx)}{n(z-x)^2},\ x < z < \infty.$$

PROOF. First, we prove (i). In view of Lemma 2 and the inequality $|a^\alpha - b^\alpha| \le \alpha\,|a-b|$ which is valid for $0 \le a, b \le 1$ and $\alpha \ge 1$, we have

$$\beta_{n,c,\alpha}(x;y) \le \int_0^y (\frac{x-t}{x-y})^2 K_{n,c,\alpha}(x;t)dt \le \frac{\alpha}{(x-y)^2}\mu_{n,2}(x;c) \le \frac{\alpha\lambda x(1+cx)}{n(x-y)^2}.$$

The proof of (ii) is similar.

3 Rate of Convergence

We have the following result.

THEOREM 1. Let f be an element of $B_r[0,\infty)$. If $\alpha \ge 1, r \in N$ and $\lambda > 2$ are given, then there exists a constant $C(f,\alpha,r;x)$ such that for n sufficiently large

$$\begin{aligned} &\left|G_{n,c,\alpha}(f;x) - \left[\frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-)\right]\right| \\ \le\ & 2\alpha\sqrt{\frac{1+cx}{2enx}}\,|f(x+)-f(x-)| \\ &+\frac{6\alpha\lambda x(1+cx)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + C\alpha 2^r \frac{(1+x)^r}{x^r}O(n^{-r}). \end{aligned}$$

where

$$g_x(t) = \begin{cases} f(t)-f(x-), & 0 \le t < x \\ 0, & t = x \\ f(t)-f(x+), & t > x \end{cases},$$

$V_a^b(g_x)$ is the total variation of g_x on $[a,b]$ and $O(n^{-r}) = \mu_{n,2r}(x;c)$.

PROOF. For any bounded variation function, it is know that

$$\begin{aligned} f(t) \ =\ & \frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-) + \frac{f(x+)-f(x-)}{2}\left(sgn_x(t) + \frac{\alpha-1}{\alpha+1}\right) \\ &+g_x(t) + \delta(t)\left[f(x) - \frac{1}{2}f(x+) - \frac{1}{2}f(x-)\right] \end{aligned}$$

where

$$sgn_x(t) = \begin{cases} -1, & 0 \le t < x \\ 0, & t = x \\ 1, & t > x \end{cases} \quad \text{and } \delta(t) = \begin{cases} 1, & t = x \\ 0, & t \ne x \end{cases}.$$

It follows that

$$\begin{aligned} & \left| G_{n,c,\alpha}(f(t);x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right] \right| \\ \le \; & \frac{1}{2} \left| f(x+) - f(x-) \right| \left| G_{n,c,\alpha}(sgn_x(t);x) + \frac{\alpha-1}{\alpha+1} \right| + \left| G_{n,c,\alpha}(g_x(t);x) \right| \\ & + \left| f(x) - \frac{1}{2} f(x+) - \frac{1}{2} f(x-) \right| \left| G_{n,c,\alpha}(\delta(t);x) \right| \end{aligned} \tag{4}$$

For the operators $G_{n,c,\alpha}$, it is obvious that $G_{n,c,\alpha}(\delta(t);x) = 0$. In view of Lemma 3 and Lemma 4, we first estimate $G_{n,c,\alpha}(sgn_x(t);x)$ as

$$\begin{aligned} G_{n,c,\alpha}(sgn_x(t);x) &= -1 + 2n \sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)}(x;c) \int_x^{\infty} p_{n+c,k-1}(t;c)dt \\ &= -1 + 2 \sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)}(x;c) \sum_{j=0}^{k-1} p_{n,j}(x;c) \\ &= -1 + 2 \sum_{j=0}^{\infty} p_{n,k}(x;c) J_{n,j+1}^{\alpha}(x;c). \end{aligned}$$

Therefore, we obtain

$$G_{n,c,\alpha}(sgn_x(t);x) + \frac{\alpha-1}{\alpha+1} = 2 \sum_{j=0}^{\infty} p_{n,k}(x;c) J_{n,j+1}^{\alpha}(x;c) - \frac{2}{\alpha+1} \sum_{j=0}^{\infty} Q_{n,k}^{(\alpha+1)}(x;c)$$

since $\sum_{j=0}^{\infty} Q_{n,k}^{(\alpha+1)}(x;c) = 1$. By the mean value theorem, it follows

$$Q_{n,j}^{(\alpha+1)}(x;c) = J_{n,j}^{\alpha+1}(x;c) - J_{n,j+1}^{\alpha+1}(x;c) = (\alpha+1) p_{n,j}(x;c) \gamma_{n,j}^{\alpha}(x;c)$$

where

$$J_{n,j+1}(x;c) < \gamma_{n,j}(x;c) < J_{n,j}(x;c).$$

Then, using Lemma 1, we obtain

$$\begin{aligned} \left| G_{n,c,\alpha}(sgn_x(t);x) + \frac{\alpha-1}{\alpha+1} \right| &\le 2 \sum_{j=0}^{\infty} p_{n,k}(x;c) \left| J_{n,j+1}^{\alpha}(x;c) - J_{n,j}^{\alpha}(x;c) \right| \\ &\le 2\alpha \sum_{j=0}^{\infty} p_{n,k}(x;c) p_{n,j}(x;c) \\ &\le 2\alpha \sqrt{\frac{1+cx}{2enx}}. \end{aligned} \tag{5}$$

We now estimate $G_{n,c,\alpha}(g_x(t);x)$. By Lebesgue-Stieltjes integral representation, we have

$$G_{n,c,\alpha}(g_x;x) = \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{\infty}\right) K_{n,c,\alpha}(x;t)g_x(t)dt. \qquad (6)$$

Let us denote the three integrals on the right hand side by E_1, E_2 and E_3 respectively. We first estimate E_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|E_2| \leq \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} |g_x(t)|\, K_{n,c,\alpha}(x;t)dt.$$

Since $|g_x(y)| \leq V_y^x(g_x) \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)$ and $\int_a^b K_{n,c,\alpha}(x;t)dt \leq 1$, for $(a,b) \subset [0,\infty)$, we conclude

$$|E_2| \leq \frac{1}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \qquad (7)$$

Next we estimate E_1. Writing $y = x - x/\sqrt{n}$ and using Lebesgue-Stieltjes integration by parts, we have

$$E_1 = \int_0^y g_x(t)d_t(\beta_{n,c,\alpha}(x;t)) = g_x(y)\beta_{n,c,\alpha}(x;y) - \int_0^y \beta_{n,c,\alpha}(x;t)d_t(g_x(t)).$$

Since $|g_x(y)| \leq V_y^x(g_x)$ we conclude that

$$|E_1| \leq V_y^x(g_x)\beta_{n,c,\alpha}(x;y) + \int_0^y \beta_{n,c,\alpha}(x;t)d_t(-V_t^x(g_x)).$$

Lemma 3 implies that

$$|E_1| \leq V_y^x(g_x)\frac{\alpha\lambda x(1+cx)}{n(x-y)^2} + \frac{\alpha\lambda x(1+cx)}{n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)).$$

Integrating the last term by parts, we get

$$|E_1| \leq \frac{\alpha\lambda x(1+cx)}{n}\left[x^{-2}V_0^x(g_x) + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}dt\right].$$

Now replacing the variable y in the last integral by $x - x/\sqrt{u}$, we obtain

$$|E_1| \leq \frac{\alpha\lambda x(1+cx)}{n}\left[x^{-2}V_0^x(g_x) + x^{-2}\frac{1}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x)\right].$$

Hence,

$$|E_1| \leq \frac{2\alpha\lambda(1+cx)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x). \qquad (8)$$

Finally we estimate E_3. We put

$$q_{n,c,\alpha}(x;t) = \begin{cases} 1-\beta_{n,c,\alpha}(x;t), & 0 \le t < 2x \\ 0, & t = 2x \end{cases}$$

and $z = x + x/\sqrt{n}$, then

$$E_3 = \int_z^{2x} g_x(t) d_t(q_{n,c,\alpha}(x;t)) + \int_{2x}^{\infty} -g_x(2x) K_{n,c,\alpha}(x;t)dt + \int_{2x}^{\infty} g_x(t) d_t(\beta_{n,c,\alpha}(x;t)).$$

Let the three integrals on the right hand side de denoted by E_{31}, E_{32} and E_{33} respectively. Then

$$E_{31} = g_x(z) q_{n,c,\alpha}(x;z) + \int_z^{2x} q^{\sim}_{n,c,\alpha}(x;t) d_t(g_x(t))$$

where $q^{\sim}_{n,c,\alpha}(x,z)$ is the normalized form $q_{n,c,\alpha}(x,z)$. Since $q_{n,\alpha}(x,z-) = q^{\sim}_{n,\alpha}(x,z)$ and $g_x(z-) \le V_x^{z-}(g_x)$, we have

$$E_{31} = V_x^{z-}(g_x) q_{n,c,\alpha}(x;z) + \int_z^{2x} q^{\sim}_{n,c,\alpha}(x;t) d_t(-V_x^t(g_x)).$$

Applying Lemma 4

$$\begin{aligned} |E_{31}| &\le V_x^{z-}(g_x) \frac{\alpha\lambda x(1+cx)}{n(z-x)^2} + \frac{\alpha\lambda x(1+cx)}{n} \int_z^{2x-} \frac{1}{(x-t)^2} d_t(-V_x^t(g_x)) \\ &\quad + \frac{1}{2} V_x^{2x-}(g_x) \int_{2x}^{\infty} K_{n,c,\alpha}(x;u)du \\ &\le V_x^{z-}(g_x) \frac{\alpha\lambda x(1+cx)}{n(z-x)^2} + \frac{\alpha\lambda x(1+cx)}{n} \int_z^{2x-} \frac{1}{(x-t)^2} d_t(V_x^t(g_x)) \\ &\quad + \frac{1}{2} V_x^{2x-}(g_x) \frac{\lambda x(1+cx)}{nx^2}. \end{aligned}$$

Thus arguing similarly as in estimate of E_1, we get

$$|E_{31}| \le \frac{2\alpha\lambda(1+cx)}{nx} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{9}$$

Again Lemma 4, we get

$$|E_{32}| \le \frac{\alpha\lambda(1+cx)}{nx} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{10}$$

Finally for $n > r$, we can write

$$|E_{33}| \le M \int_{2x}^{\infty} K_{n,c,\alpha}(x;t)[(1+t)^r + (1+x)^r]dt.$$

For $t \geq 2x$, using the inequalities

$$(1+t)^r \leq 2^r \frac{(1+x)^r}{x^r}(t-x)^{2r},$$

$$(1+x)^r \leq 2^r \frac{(1+x)^r}{x^r}(t-x)^{2r},$$

Lemma 2 and Lemma 4, we obtain

$$|E_{33}| \leq C\alpha 2^r \frac{(1+x)^r}{x^r} O(n^{-r}). \tag{11}$$

Combining the estimates of (5) -(11) we reach the required result. This completes the proof of the theorem.

4 Some Examples

Suppose the Szasz and Baskakov basis functions are defined respectively by

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!} \text{ and } b_{n,k}(x) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k(1+x)^{-(n+k)}.$$

Just like operators defined in (2), we may define some other mixed Bezier variants of summation-integral type operators as:

(i) Szasz-Baskakov-Bezier operators

$$S_{n,\alpha}(f;x) = (n-1)\sum_{k=1}^{\infty} R_{n,k}^{(\alpha)}(x)\int_0^{\infty} b_{n,k}(t)f(t)dt + R_{n,0}^{(\alpha)}(x)f(0), \quad x \in [0,\infty)$$

where $R_{n,k}^{(\alpha)}(x) = \left(\sum_{j=k}^{\infty} s_{n,j}(x)\right)^{\alpha} - \left(\sum_{j=k+1}^{\infty} s_{n,j}(x)\right)^{\alpha}$.

(ii) Baskakov-Szasz-Bezier operators

$$B_{n,\alpha}(f;x) = n\sum_{k=1}^{\infty} L_{n,k}^{(\alpha)}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt + L_{n,0}^{(\alpha)}(x)f(0), \quad x \in [0,\infty)$$

where $L_{n,k}^{(\alpha)}(x) = \left(\sum_{j=k}^{\infty} b_{n,j}(x)\right)^{\alpha} - \left(\sum_{j=k+1}^{\infty} b_{n,j}(x)\right)^{\alpha}$.

The approximation properties of the above operators are similar to the operators (2), but there are problems to find the estimation of $S_{n,\alpha}(sgn_x(t);x)$ and $B_{n,\alpha}(sgn_x(t);x)$ explicitly whenever we wish to obtain the rate of convergence for functions of bounded variation. Actually due to different basis functions the above method fails as one cannot find the exact relationship between the summation of one basis function with the integration of the different basis functions. The author [1] was also not able to give the answer of such problem. Still such type of problem may be considered as open problems.

Acknowledgments. The authors are thankful to the reviewers for making valuable suggestions, leading to a better presentation of the paper.

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