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SOLVING DYNAMICAL SYSTEMS WITH CUBIC TRIGONOMETRIC SPLINES*

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Abstract

Special nonlinear Dynamical Systems, which are playing an important role in celestial mechanics, are solved here numerically, using Cubic Trigonometric Splines. The existence of the unique spline approximation is proved and the convergence order of the method is shown to be cubic. The choice of the trigonometric splines is made, because in general they give better results in approximating periodical functions.

1 Introduction

Consider the dynamical system

$$\mathbf{y}'' = \mathbf{f}(x, \mathbf{y}),\ a \le x \le b,\ \mathbf{y}:\ \mathbf{R} \to \mathbf{R}^k \tag{1}$$

with the initial values $\mathbf{y}(a) = \mathbf{y}_a$ and $\mathbf{y}'(a) = \mathbf{y}'_a$, where $\mathbf{y}(x) = (y_1(x), y_2(x), \dots, y_k(x))^\dagger$ and $\mathbf{f} = (f_1, f_2, \dots, f_k)^\dagger$. Assume also $f \in C^3$ in T where $T = \{(x, \mathbf{y})/x \in [a, b]\}$ and $\mathbf{f}$ satisfies the Lipschitz condition:

$$\exists L > 0\ \forall x \in [a, b]\ \forall \mathbf{y}_1, \mathbf{y}_2 \in \mathbf{R}^k\ ||\mathbf{f}(x, \mathbf{y}_1) - \mathbf{f}(x, \mathbf{y}_2)|| \le L||\mathbf{y}_1 - \mathbf{y}_2||, \tag{2}$$

where $||\cdot||$ denotes here the euclidian norm. This is satisfied if the partial derivatives of $\mathbf{f}$ are bounded in T.

Consider also the differential operator $L_4 y := y^{(4)} + \frac{5}{2}y'' + \frac{9}{16}y$ with the following fundamental system:

$$N_L = \operatorname{span}\left\{\sin\frac{x}{2}, \cos\frac{x}{2}, \sin\frac{3x}{2}, \cos\frac{3x}{2}\right\}.$$

The extended Taylor formula for the operator L_4 and $y \in C^4[a, b]$ is: (see [10], p. 425)

$$y(x) = u_y(x) + \int_a^x \frac{4}{3}\sin^3\frac{x - \xi}{2} L_4 y(\xi)\, d\xi,\ x \in [a, b], \tag{3}$$

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with

$$\begin{aligned} u_y(x) &= \frac{1}{8}[9y(a)+4y''(a)]\cos\frac{x-a}{2}+\frac{1}{4}[9y'(a)+4y'''(a)]\sin\frac{x-a}{2} \\ &\quad -\frac{1}{8}[y(a)+4y''(a)]\cos\frac{3(x-a)}{2}-\frac{1}{12}[y'(a)+4y'''(a)]\sin\frac{3(x-a)}{2}. \end{aligned}$$

The function $u_y \in N_L$ satisfies the following conditions: $u_y(a) = y(a)$, $u'_y(a) = y'(a)$, $u''_y(a) = y''(a)$ and $u'''_y(a) = y'''(a)$. We divide the interval $[a, b]$ in n subintervals and let $\Omega_n = \{x_{-3}, x_{-2}, x_{-1}, x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}, x_{n+3}\}$ the set of extended equally spaced knots with $x_i = a + ih$ and $h = \frac{b-a}{n}$. The cubic trigonometric splines with respect to Ω_n are functions in $C^2[a, b]$ which in every subinterval $[x_i, x_{i+1}]$, for $i = 0, ..., n-1$ belong in N_L. For the computation of the splines the expression

$$s(x) = \sum_{i=-3}^{n-1} \alpha_i TB_i^3(x) \tag{4}$$

is preferred, where $TB_i^3(x)$ are the **T**rigonometric **B**asic splines, denoted in the following with TB-splines, (see [5] for a recursion formula) and α_i are the coefficients that must be computed. For the cubic trigonometric splines we have the restriction that $x_{i+4} - x_i = 4h < 2\pi$.

The exact formula for the computation of the cubic TB-spline is given here:

$$TB_i^3(x) := \theta \begin{cases} \sin^3(\frac{x-x_i}{2}) & \text{in } I_i, \\ \sin^2(\frac{x-x_i}{2})\sin(\frac{x_{i+2}-x}{2})+ & \\ +\sin(\frac{x-x_i}{2})\sin(\frac{x_{i+3}-x}{2})\sin(\frac{x-x_{i+1}}{2})+ & \\ +\sin^2(\frac{x-x_{i+1}}{2})\sin(\frac{x_{i+4}-x}{2}) & \text{in } I_{i+1}, \\ \sin^2(\frac{x_{i+3}-x}{2})\sin(\frac{x-x_i}{2})+ & \\ +\sin(\frac{x_{i+4}-x}{2})\sin(\frac{x-x_{i+1}}{2})\sin(\frac{x_{i+3}-x}{2})+ & \\ +\sin^2(\frac{x_{i+4}-x}{2})\sin(\frac{x-x_{i+2}}{2}) & \text{in } I_{i+2}, \\ \sin^3(\frac{x_{i+4}-x}{2}) & \text{in } I_{i+3}, \\ 0 & \text{else}, \end{cases}$$

where

$$I_i := [x_i, x_{i+1})$$

and

$$\theta = \frac{1}{\sin(h/2)\sin(h)\sin(3h/2)}.$$

For the error analysis of the method the extended Taylor formula (3) is used, as in a recent article [8] for solving ODE with quadratic trigonometric splines and also for interpolation in [3, 7]. More about splines can be found in [1, 2, **?**, 9, 10].

2 Presentation of the Method and Existence of the Numerical Solution

Consider the system

$$\begin{cases} y_1''(x) = f_1(x, y_1(x), y_2(x)), \\ y_2''(x) = f_2(x, y_1(x), y_2(x)), \end{cases} \tag{5}$$

where for simplicity reasons the method is presented for 2×2 systems, with the initial values $y_1(a) = y_{1a}$, $y_2(a) = y_{2a}$, $y_1'(a) = y_{1a}'$, $y_2'(a) = y_{2a}'$ and $\mathbf{F}(\mathbf{X}) = (f_1(\mathbf{X}), f_2(\mathbf{X}))$. Let $\tilde{y}_1(x) = \sum_{i=-3}^{n-1} \alpha_i TB_i^3(x)$ and $\tilde{y}_2(x) = \sum_{i=-3}^{n-1} \beta_i TB_i^3(x)$ the spline approximations to the solutions $y_1(x)$, $y_2(x)$ of (5). We require that these approximations satisfy the initial conditions in a and the system (5) in the other knots of $[a, b]$, i.e. $\tilde{y}_1(a) = y_{1a}$, $\tilde{y}_2(a) = y_{2a}$, $\tilde{y}_1'(a) = y_{1a}'$, $\tilde{y}_2'(a) = y_{2a}'$, $\tilde{y}_1''(a) = y_1''(a) = f_1(a, y_1(a), y_2(a))$, $\tilde{y}_2''(a) = y_2''(a) = f_2(a, y_1(a), y_2(a))$ and

$$\begin{cases} \tilde{y}_1''(a + kh) = f_1(a + kh, \tilde{y}_1(a + kh), \tilde{y}_2(a + kh)), \\ \tilde{y}_2''(a + kh) = f_2(a + kh, \tilde{y}_1(a + kh), \tilde{y}_2(a + kh)) \end{cases} \tag{6}$$

for $k = 1 \ldots n$. This is a modification of a method due to F. R. Loscalzo and T. Talbot [6]. The first three coefficients $\alpha_{-3}, \alpha_{-2}, \alpha_{-1}$, and $\beta_{-3}, \beta_{-2}, \beta_{-1}$ are computed easily from the initial conditions. The rest of them ($\alpha_k =: z_1$ and $\beta_k =: z_2$) are computed iteratively from the system

$$\begin{cases} z_1 = -\frac{d_2}{d_1}\alpha_{k-1} - \alpha_{k-2} + m f_1 \left(x_{k+1}, \frac{z_1 + 4\cos\frac{h}{2}\alpha_{k-1} + \alpha_{k-2}}{2\cos\frac{h}{2}[4\cos^2\frac{h}{2} - 1]}, \frac{z_2 + 4\cos\frac{h}{2}\beta_{k-1} + \beta_{k-2}}{2\cos\frac{h}{2}[4\cos^2\frac{h}{2} - 1]} \right), \\ z_2 = -\frac{d_2}{d_1}\beta_{k-1} - \beta_{k-2} + m f_2 \left(x_{k+1}, \frac{z_1 + 4\cos\frac{h}{2}\alpha_{k-1} + \alpha_{k-2}}{2\cos\frac{h}{2}[4\cos^2\frac{h}{2} - 1]}, \frac{z_2 + 4\cos\frac{h}{2}\beta_{k-1} + \beta_{k-2}}{2\cos\frac{h}{2}[4\cos^2\frac{h}{2} - 1]} \right), \end{cases}$$

where $m_0 = 2\cos\frac{h}{2}[4\cos^2\frac{h}{2} - 1]$, $d_1 = \frac{3}{2}[1 - \frac{3}{2}\sin^2\frac{h}{2}]$, $d_2 = 2\cos\frac{h}{2}[-\frac{7}{8}\cos\frac{h}{2} - \frac{5}{8} - \frac{1}{4}\sin^2\frac{h}{2}]$ and $m = \frac{\sin h \sin(3h/2)}{d_1}$, which means a system of the form

$$\begin{cases} z_1 = \varphi_1(z_1, z_2), \\ z_2 = \varphi_2(z_1, z_2). \end{cases} \tag{7}$$

The previous system (7) can also be written as $\mathbf{Z} = \Phi(\mathbf{Z})$ with $\mathbf{Z} = (z_1, z_2)^T$, $\Phi(\mathbf{Z}) = (\varphi_1(\mathbf{Z}), \varphi_2(\mathbf{Z}))$.

PROPOSITION 1. If $h < \sqrt{\frac{1}{2L}}$, then the system (7) has a unique solution.

PROOF. We are going to prove that for $h < \sqrt{\frac{1}{2L}}$ the function Φ is a strong contraction mapping. From Banach's Theorem follows, that Φ has a unique fixed point. Since the solutions of the systems (7) are the coefficients of the spline approximations, it means that these approximations exist and are uniquely determined. $\mathbf{F}$ satisfies condition (2) from which we obtain

$$\begin{aligned} \left\| \Phi(\mathbf{Z}) - \Phi(\tilde{\mathbf{Z}}) \right\| &= m \left\| \mathbf{F}\left(x_{k+1}, \frac{z_1 + A}{m_0}, \frac{z_2 + B}{m_0} \right) - \mathbf{F}\left(x_{k+1}, \frac{\tilde{z}_1 + A}{m_0}, \frac{\tilde{z}_2 + B}{m_0} \right) \right\| \\ &\leq L \left| \frac{m}{m_0} \right| \left\| \mathbf{Z} - \tilde{\mathbf{Z}} \right\|, \end{aligned}$$

where $A = 4\cos(\frac{h}{2})\alpha_{k-1} + \alpha_{k-2}$ and $B = 4\cos(\frac{h}{2})\beta_{k-1} + \beta_{k-2}$. Since $x_{i+4} - x_i = 4h < 2\pi \iff \frac{h}{2} < \frac{\pi}{4}$ it follows, $\sin^2(\frac{h}{2}) < \frac{1}{2}$ and $\cos^2(\frac{h}{2}) > \frac{1}{2}$, so $\left|\frac{m}{m_0}\right| < 2h^2$. Consequently

$$\left\|\Phi(\mathbf{Z}) - \Phi(\tilde{\mathbf{Z}})\right\| < 2h^2 L \left\|\mathbf{Z} - \tilde{\mathbf{Z}}\right\| < \left\|\mathbf{Z} - \tilde{\mathbf{Z}}\right\|.$$

3 Error Estimates

We are going to derive error estimates in the one dimensional case. (It is easy to generalize these results in the n-dimensional case.) Let

$$y'' = f(x, y), \quad x \in [a, b] \tag{8}$$

the differential equation, where f satisfies the Lipschitz condition (2) and $s(x)$ the spline approximation of the solution $y(x)$. Let also $s_i(x) := s(x)|_{[x_i, x_{i+1}]}$ where $s(x)$ satisfies the initial conditions in a i.e. $s_a := s(a) = y_a$, $s'_a := s'(a) = y'_a$, $s''_a := s''(a) = y''_a$ also $s(x)$ satisfies (8) in the rest of the knots $x_1, x_2, \ldots, x_n$ i.e. $s''_i = f(x, s_i)$ where $s_i := s(x_i)$ and $s''_i := s''(x_i)$. We define $e_i := s_i - y_i$ the error at x_i. At the beginning $e_1 = s_1 - y_1$ is computed, where for s_1, y_1 the Taylor formula (3) is applied. So

$$s_1 = s_0(x_1) = \frac{1}{8}(9s_0 + 4s''_0)\cos\frac{h}{2} + \frac{1}{4}(9s'_0 + 4s'''_0)\sin\frac{h}{2} -$$

$$\frac{1}{8}(s_0 + 4s''_0)\cos\frac{3h}{2} - \frac{1}{12}(s'_0 + 4s'''_0)\sin\frac{3h}{2}, \quad \text{where} \quad s'''_0 := s'''_0(x_0 + 0)$$

and

$$\begin{aligned} y_1 \quad &= \quad y(x_1) = \frac{1}{8}(9y_0 + 4y''_0)\cos\frac{h}{2} + \frac{1}{4}(9y'_0 + 4y'''_0)\sin\frac{h}{2} \\ &\quad -\frac{1}{8}(y_0 + 4y''_0)\cos\frac{3h}{2} - \frac{1}{12}(y'_0 + 4y'''_0)\sin\frac{3h}{2} + K_1, \end{aligned}$$

with $K_1 = \int_{x_0}^{x_1} \frac{4}{3}\sin^3\frac{x_1 - \xi}{2} L_4 y(\xi)\, d\xi$. Subtracting the two equations we obtain

$$e_1 = e'''_0 \frac{4}{3}\sin^3\frac{h}{2} - K_1. \tag{9}$$

Differentiating twice the Taylor formula (3) and applying it for $s(x_1)$ and $y(x_1)$ we obtain

$$e'''_0 = \frac{-e''_1 - K''_1}{\sin\frac{h}{2}(-2 + 3\sin^2\frac{h}{2})} \tag{10}$$

where $K''_1 = \int_{x_0}^{x_1} \sin\frac{x_1 - \xi}{2}\left[3\cos^2\frac{x_1 - \xi}{2} - 1\right] L_4 y(\xi)\, d\xi$. Substituting (10) in (9) gives

$$e_1 = -\frac{4}{3}e''_1 \frac{\sin^2\frac{h}{2}}{-2 + 3\sin^2\frac{h}{2}} - \tilde{K}, \tag{11}$$

where

$$\begin{aligned}\tilde{K} \quad : \quad &= K_1 + \frac{4}{3}\frac{K_1'' \sin^2 \frac{h}{2}}{(-2+3\sin^2\frac{h}{2})} \\ &= \frac{8}{3(3\cos^2\frac{h}{2}-1)}\int_{x_0}^{x_1} \sin\frac{x_1-\xi}{2}\sin\frac{x_0-\xi}{2}\sin\frac{x_2-\xi}{2}L_4 y(\xi)\,d\xi.\end{aligned}$$

Applying the mean value theorem for integrals, the above becomes

$$\tilde{K} = L_4 y(\gamma)\left[-\frac{1}{24}h^4 + O(h^6)\right], \quad \text{for a} \quad \gamma \in [x_0, x_1].$$

The Lipschitz condition (2) gives

$$|e_1''| = |s_1'' - y_1''| = |f(x_1, s_1) - f(x_1, y_1)| \leq L|s_1 - y_1| = L|e_1|$$

and (11) gives $|e_1| \leq |e_1|O(h^2) + O(h^4)$ i.e. $|e_1| = O(h^4)$. This also means that $|e_1''| = O(h^4)$. From equation (10) we obtain that $|e_0'''| = O(h)$. Finally differentiating the Taylor formula (3) one and three times follows respectively that

$$e_1' = 2e_0''' \cos\frac{h}{2}\sin^2\frac{h}{2} - K_1' = O(h^3), \tag{12}$$

and

$$e_1''' = \frac{1}{8}e_0'''\big(9\cos\frac{3h}{2} - \cos\frac{h}{2}\big) - K_1''' = O(h), \tag{13}$$

where $e_1''' := s_1''' - y_1'''$, $s_1''' = s_0'''(x_1 - 0)$, $K_1' = \int_{x_0}^{x_1} 2\sin^2\frac{x_1-\xi}{2}\cos\frac{x_1-\xi}{2}L_4 y(\xi)\,d\xi$ and $K_1''' = \int_{x_0}^{x_1}[-\frac{5}{4}\sin(x_1-\xi)\sin\frac{x_1-\xi}{2} + \cos(x_1-\xi)\cos\frac{x_1-\xi}{2}]L_4 y(\xi)\,d\xi$.

Following the same analysis

$$\begin{aligned} e_2 \quad = \quad & \frac{1}{8}e_1\left(9\cos\frac{h}{2} - \cos\frac{3h}{2}\right) + e_1'\left(\frac{9}{4}\sin\frac{h}{2} - \frac{1}{12}\sin\frac{3h}{2}\right) \\ & + 2e_1''\cos\frac{3h}{2}\sin^2\frac{h}{2} + \frac{4}{3}e_1'''\sin^3(\frac{3h}{2}) + K_2,\end{aligned}$$

with $K_2 = \int_{x_1}^{x_2}\frac{4}{3}\sin^3\frac{x_2-\xi}{2}L_4 y(\xi)\,d\xi$, which means $e_2 = e_1 + O(h^4)$.

Generalizing the above results in $[x_k, x_{k+1}]$ we conclude $e_k = e_{k-1} + O(h^4)$ which means that $e_n = nO(h^4) = O(h^3)$ and that $e_k' = O(h^3)$, $e_k'' = O(h^3)$ (Lipschitz) and that $e_k''' := s_k''' - y_k''' = O(h)$, where $s_k''' = s_k'''(x_k + 0)$.

For the estimation of the global error in $[x_i, x_{i+1}]$ the Taylor formula (3) is applied to $s(x)$, $y(x)$. So

$$\begin{aligned} s_i(x) - y(x) \quad = \quad & \frac{1}{8}e_i\left[9\cos\frac{x-x_i}{2} - \cos\frac{3(x-x_i)}{2}\right] \\ & + e_i'\left[\frac{9}{4}\sin\frac{x-x_i}{2} - \frac{1}{12}\sin\frac{3(x-x_i)}{2}\right] \\ & + 2e_i''\cos\frac{3(x-x_i)}{2}\sin^2\frac{x-x_i}{2} + \frac{4}{3}e_i'''\sin^3\frac{3(x-x_i)}{2} + K,\end{aligned}$$

for $x \in [x_i, x_{i+1}]$, where $s_i''' = s_i'''(x_i+0)$ and $K = \int_{x_i}^{x} \frac{4}{3} \sin^3 \frac{x-\xi}{2} L_4 y(\xi)\, d\xi$, which means that $\|s(x) - y(x)\|_{\infty,[a,b]} = O(h^3)$.

Therefore we have the following Proposition

PROPOSITION 2. The convergence order of the spline approximations of the solution of the differential equation (8) is cubic, i.e. that $\|s(x) - y(x)\|_{\infty,[a,b]} = O(h^3)$.

4 Numerical Results

Applying the method to the following Dynamical Systems

$$\begin{cases} \ddot{x} + \alpha x - \varepsilon x^2 = 0 \\ \ddot{y} + \beta y - 2\varepsilon x y = 0 \end{cases} \tag{Σ_1}$$

$$\begin{cases} \ddot{x} + \alpha x - \varepsilon z^2 = 0 \\ \ddot{y} + \beta y - \eta z^2 = 0 \\ \ddot{z} + c z - 2z(\varepsilon x + \eta y) = 0 \end{cases} \tag{Σ_2}$$

which play as already mentioned a special role in celestial mechanics, numerical results are calculated and presented.

The system (Σ_2) is also solved numerically in [11], with the method of rational Fourier series approximations. The results are listed here for comparison.

t	$y_{rational}$	$z_{rational}$
0.0	0.0001603	0
0.5	0.0001417	0.2347
1.0	0.0001031	0.4120
1.5	0.0000799	0.4883
2.0	0.0000935	0.4450
2.5	0.0001313	0.2925
3.0	0.0001584	0.0683
3.5	0.0001500	-0.1726
4.0	0.0001137	-0.3713
4.5	0.0000830	-0.4788
5.0	0.0000862	-0.4690
5.5	0.0001203	-0.3443
6.0	0.0001541	-0.1352

Using our method for $t \in [0, 6]$ and dividing this interval in 120 subintervals, the same initial values, in particular $x(0) = 0.0160308$, $y(0) = 0.0001603$, $z(0) = 0$, $x'(0) = 0$, $y'(0) = 0$, $z'(0) = 0.4896355662686994799$ and the same coefficients $a = 1$, $b = 1$, $c = 1$, $n = 0.001$ and $e = 0.1$ we obtain

t	$\tilde{x}(t)$	$\tilde{y}(t)$	$\tilde{z}(t)$
0.0	0.016030800000	0.000160300000	0.000000000000
0.5	0.014188907474	0.000141882054	0.234772579726
1.0	0.010353260108	0.000103528279	0.412205882603
1.5	0.008043419692	0.000080433632	0.488901069945
2.0	0.009378323944	0.000093786570	0.446069255893
2.5	0.013131021625	0.000131316627	0.294205994254
3.0	0.015855705194	0.000158564972	0.070475284080
3.5	0.015051130099	0.000150518792	-0.170463104295
4.0	0.011455858557	0.000114563812	-0.369774646824
4.5	0.008370609075	0.000083707774	-0.478734174141
5.0	0.008628875328	0.000086286480	-0.470658621672
5.5	0.011995098161	0.000119945309	-0.347527105133
6.0	0.015380602114	0.000153798338	-0.139487177037

Finally the systems (Σ_1) and (Σ_2) are solved in $[0,2]$ with coefficients $a = 1, b = 2, c = 3, n = 0.001, e = 0.001$ and initial values $x(0) = y(0) = z(0) = x'(0) = y'(0) = z'(0) = 1$. Evaluating our approximations for $t = 2$ with N subintervals each time, we obtain

(Σ_1)

	N	Error	approximation	num. convergence order
$\tilde{x}$	20	0.001045260386	0.493095924885	2.434537
$\tilde{y}$	20	0.000206985079	-0.731784939969	2.432719
$\tilde{x}$	40	0.000193355245	0.494141185271	2.589748
$\tilde{y}$	40	0.000038336970	-0.731577954890	2.589093
$\tilde{x}$	60	0.000067658739	0.494334540516	2.678041
$\tilde{y}$	60	0.000013418413	-0.731539617920	2.677691
$\tilde{x}$	80	0.000031313591	0.494402199255	
$\tilde{y}$	80	0.000006210890	-0.731526199507	

(Σ_2)

	N	Error	approximation	num. convergence order
$\tilde{x}$	20	0.001044503830	0.492800011695	2.434541
$\tilde{y}$	20	0.000203942718	-0.733194200930	2.432761
$\tilde{z}$	20	0.000136848932	-1.130291116896	2.434579
$\tilde{x}$	40	0.000193214713	0.493844515525	2.589750
$\tilde{y}$	40	0.000037772364	-0.732990258211	2.589108
$\tilde{z}$	40	0.000025313965	-1.130427965828	2.589767
$\tilde{x}$	60	0.000067609522	0.494037730238	2.678042
$\tilde{y}$	60	0.000013220712	-0.732952485847	2.677699
$\tilde{z}$	60	0.000008857778	-1.130453279793	2.678051
$\tilde{x}$	80	0.000031290805	0.494105339760	
$\tilde{y}$	80	0.000006119367	-0.732939265136	
$\tilde{z}$	80	0.000004099515	-1.130462137571	

where for the computation of the numerical convergence order above the formula $\ln\left(\frac{|e_k|}{|e_m|}\right) / \ln\left(\frac{m}{k}\right)$ is used.

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