Applied Mathematics E-Notes, 5(2005), 52-59 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

ON THE VELOCITY PROJECTION MAP FOR POLYHEDRAL SKOROKHOD PROBLEMS *†

Martin V. Day ‡

Received 24 June 2004

Abstract

We consider the Skorokhod problem with oblique reflection on a convex polyhedral domain. Under Assumption 2.1 of Dupuis and Ishii, insuring the Lipschitz continuity of the Skorokhod map, the associated velocity projection map is identified with a collection of complementarity problems. The solvability of the complementarity problems is shown to be equivalent to the existence of the discrete projection map.

1 Introduction

Consider the closed convex polyhedron G consisting of those $x \in \mathbb{R}^n$ satisfying a collection of N linear inequalities

$$x \cdot n_i \geq c_i. \tag{1}$$

The n_i are unit vectors and c_i are scalars. For $x \in G$ the set of indices for which the constraints are active will be denoted

$$I(x) = \{i : \ x \cdot n_i = c_i\}.$$

Thus ∂G consists of those $x \in G$ for which $I(x)$ is nonempty.

A "restoration vector" d_i is assigned to each constraint (1), normalized by $n_i \cdot d_i = 1$. At $x \in \partial G$ we define the set $d(x)$ of (normalized) convex combinations of the d_i associated with active constraints:

$$d(x) = \{\gamma = \sum_{i \in I(x)} \alpha_i d_i : \ \alpha_i \geq 0, \ |\gamma| = 1\}.$$

For completeness, take $d(x) = \{0\}$ if x is interior to G. Given a function $\psi : [0, T] \to \mathbb{R}^n$ (of some appropriate class) with $\psi(0) \in G$, the *Skorokhod Problem* is to find a function $\phi : [0, T] \to G$ such that

$$\phi(t) = \psi(t) + \int_{(0,t]} \gamma(s)\, d\ell,$$

*Mathematics Subject Classifications: 90C33, 93A30, 60K25.

†Supported in part by NSF grant DMS-0102266.

‡Department of Mathematics, Virginia Tech, Blacksburg, Virginia, USA 24061-0123

where ℓ is a finite measure on $[0, T]$, supported on $\{t : \phi(t) \in \partial G\}$, and $\gamma(t)$ is a measurable function with $\gamma(t) \in d(\phi(t))$ for all t. The idea is that $\phi(t)$ is the result of pushing $\psi(t)$ in the direction d_i when necessary to prevent violation of constraint i, producing a function $\phi(t)$ that remains in G. A careful treatment of this problem was given by Dupuis and Ishii [6], with additional work in [7]. The map $\Gamma : \psi(\cdot) \mapsto \phi(\cdot)$ is called the *Skorokhod map*, to whatever extent it is well-defined.

Skorokhod problems are particularly useful in the study of queueing systems. See the many references in [7], and also [4] where Γ is called the "oblique reflection mapping". Several recent papers (such as [1] and [5]) consider differential games associated with queueing systems. Only absolutely continuous functions are relevant there, in which case the Skorokhod problem can be expressed as a differential equation:

$$\dot{\phi}(t) = \pi(x(t), \dot{\psi}(t))$$

where $\pi(x, v)$ is the *velocity projection map*, defined by (7) below. (See [6].)

For practical purposes it is useful to identify $\pi(x, v)$ directly in terms of the constraints and associated d_i. This takes the form of complementarity problems (3). These are also intimately connected with the discrete projection map $\Pi(\cdot)$ appearing in (7) below. Observations and simple results connecting these objects have appeared piecemeal in the literature, often under the slightly more restrictive assumption that G is a convex cone. For instance [2] (Remark 1, page 158) points out the connection between the complementarity problem and the Skorokhod map for linear functions, $\psi(t) = tv$, $\phi(t) = tw$ in the case $G = \mathbb{R}^n$. In [3], where G is a convex cone with vertex at 0, $\Pi(\cdot)$ has a simple scaling property which implies $\pi(0, v) = \Pi(v) = w$ for the same linear functions. Putting these together we have the connection between $\pi(0, v)$ and the complementarity problem, at least at $x = 0$ for $G = \mathbb{R}^n$.

Our goal is to present the connections among $\pi(x, v)$, $\Pi(y)$ and the complementarity problems in a more organized way, and for a general convex polyhedron G. Assumption 2.1 below will be assumed throughout. Our main results are that $\pi(x, v)$ is precisely the solution map of the complementarity problem (Theorem 1), and that the solvability of the complementarity problems is equivalent to the existence of the discrete projection (Theorem 2). Theorem 2 also provides a slight strengthening of the result from [6] that the existence of Π in a neighborhood of G is sufficient for its existence globally.

2 Preliminaries

The n_i, d_i must satisfy some conditions for the Skorokhod problem to be well-posed and have decent continuity properties. Dupuis and Ishii [6] present separate sufficient conditions for continuity and existence. Their sufficient condition for continuity is the following (we retain the same title as in [6] and [7]):

ASSUMPTION 2.1. There exists a compact, convex set $B \subseteq \mathbb{R}^n$ with $0 \in B^\circ$, such that for each $i = 1, \ldots N$ and $z \in \partial B$, and any inward normal v to B at z,

$$|z \cdot n_i| < 1 \quad \text{implies} \quad v \cdot d_i = 0.$$

We refer the reader to [7] and [6] for details, properties of B, and techniques to verify its existence. Assumption 2.1 will be a hypothesis for all our results below.

The sufficient condition for existence in [6] is Assumption 3.1. This postulates the existence of a *discrete projection map* $\Pi : \mathbb{R}^n \to G$ such that $y = \Pi(x)$ satisfies the following properties

$$\begin{aligned} &y = x \text{ if } x \in G, \\ &y \in \partial G \text{ and } r(y - x) \in d(y) \text{ for some } r > 0 \text{ if } x \notin G. \end{aligned} \tag{2}$$

We do *not* assume the existence of $\Pi(y)$ here. Rather Theorem 2 below will establish an equivalence between the existence of $\Pi(y)$ and the existence of solutions to the following family of complementarity problems: given $x \in \partial G$ and $v \in \mathbb{R}^n$, find $w \in \mathbb{R}^n$ such that

$$\begin{aligned} &w = v + \sum_{i \in I(x)} \beta_i d_i, \quad \text{where } \beta_i \text{ are scalars satisfying} \\ &\beta_i \geq 0,\ n_i \cdot w \geq 0, \text{ and } \beta_i(n_i \cdot w) = 0 \text{ for each } i \in I(x). \end{aligned} \tag{3}$$

Specifically we will call this the *complementarity problem for v at x*.

The major implication of Assumption 2.1 is Lipschitz continuity of the Skorokhod map on functions in $D([0,T], R^n)$ (right continuous functions having left limits). The discrete projection $\Pi(\cdot)$ is associated with solutions of the Skorokhod problem for piecewise constant functions. As a consequence, Assumption 2.1 implies that $\Pi(\cdot)$ is Lipschitz, on whatever domain it is defined: if $\Pi(x_i)$ exists for two x_i values, $i = 1, 2$, then

$$|\Pi(x_2) - \Pi(x_1)| \leq K|x_2 - x_1|. \tag{4}$$

(See the fourth bullet, page 1690 of [3].) The value K will appear in several results below. We will see as a consequence of Lemma 1 below that Assumption 2.1 also implies that solutions of (3) are unique.

Another key quantity for our arguments is the slack in inactive constraints for $x \in \partial G$:

$$\delta_x = \min_{j \in I(x)^c} (x \cdot n_j - c_j). \tag{5}$$

With the convention that $\delta_x = +\infty$ if $I(x)^c = \{1, \ldots, N\} \setminus I(x)$ is empty, we see that δ_x is always positive, possibly infinite. Consider any y with $|y - x| < \delta_x$. Since the n_i are all unit vectors, it follows that $y \cdot n_j > c_j$ for $j \in I(x)^c$. Thus $y \in G$ iff $y \cdot n_i \geq c_i$ for $i \in I(x)$. Moreover $I(y) \subseteq I(x)$ for such y.

3 Local Equivalence

Our first result connects the complementarity problem at $x \in \partial G$ to the existence of $\Pi(\cdot)$ in a neighborhood of x.

LEMMA 1. Suppose Assumption 2.1 holds, $x \in \partial G$ and $\epsilon > 0$.

a) If $\epsilon|v| < \delta_x/K$ and $y = \Pi(x + \epsilon v)$ exists, then $w = \frac{1}{\epsilon}(y - x)$ solves the complementarity problem for v at x.

b) If $\epsilon|w| < \delta_x$ and w solves the complementarity problem for v at x, then $\Pi(x+\epsilon v) = x + \epsilon w$.

PROOF. For part a), since $\Pi(x) = x$ it follows from (4) and $\epsilon|v| < \delta_x/K$ that

$$|y - x| = |\epsilon w| = |\Pi(x + \epsilon v) - x| \leq K|\epsilon v| < \delta_x.$$

By definition of δ_x this means $y \cdot n_i > c_i$ for any $i \notin I(x)$. For any $i \in I(x)$ we have

$$w \cdot n_i = \frac{1}{\epsilon}(y \cdot n_i - x \cdot n_i) = \frac{1}{\epsilon}(y \cdot n_i - c_i) \geq 0.$$

Thus

$$I(y) = \{i \in I(x) : \; w \cdot n_i = 0\}. \tag{6}$$

We also know from the definition of Π that there exist $\alpha_i \geq 0$, $i \in I(y)$,

$$\epsilon(w - v) = \Pi(x + \epsilon v) - (x + \epsilon v) = \sum_{i \in I(y)} \alpha_i d_i.$$

Thus with $\beta_i = \alpha_i/\epsilon$ for $i \in I(y)$ and $\beta_i = 0$ for $i \in I(x) \setminus I(y)$ we have $w = v + \sum_{i \in I(x)} \beta_i d_i$ and

$$\beta_i \geq 0; \quad w \cdot n_i \geq 0; \quad \text{and } \beta_i\,(w \cdot n_i) = 0,$$

the last equality following from (6). In other words, w is a solution to the complementarity problem (3) for v at x.

For b), the hypotheses imply $|(x + \epsilon w) - x| < \delta_x$. Therefore for $x + \epsilon w \in G$ it is sufficient that $w \cdot n_i \geq 0$ for $i \in I(x)$, which is true from the features of the complementarity problem. If $w = v$ then $x + \epsilon v = x + \epsilon w \in G$ so that $\Pi(x + \epsilon v) = x + \epsilon w$, establishing our claim. Suppose instead that $w \neq v$. Then some $\beta_j > 0$ and consequently $w \cdot n_j = 0$, which implies $x + \epsilon w \in \partial G$. We know from $\epsilon|w| < \delta_x$ that $I(x + \epsilon w) = \{i \in I(x) : w \cdot n_i = 0\}$. Let $F = \{i \in I(x) : \; \beta_i > 0\}$. The complementarity conditions imply that $F \subseteq I(x + \epsilon w)$ and therefore $(x + \epsilon w) - (x + \epsilon v) = \sum_{i \in F} \epsilon \beta_i d_i$. Since this is nonzero with nonnegative coefficients, when scaled to norm 1 it belongs to $d(x + \epsilon w)$. Hence $\Pi(x + \epsilon v) = x + \epsilon w$, as claimed.

Lemma 1 has several consequences. First it implies that the solutions of the complementarity problem at a fixed $x \in \partial G$ are unique (if they exist). Indeed if solutions w_1, w_2 exist for v_1, v_2 respectively, then for sufficiently small $\epsilon > 0$ the lemma implies $\Pi(x + \epsilon v_i) = x + \epsilon w_i$. It follows from (4) that

$$|w_2 - w_1| \leq K|v_2 - v_1|.$$

Thus the solution map $v \mapsto w$ is Lipschitz on the set of v for which it is defined. Moreover (taking $v_1 = w_1 = 0$) we see that solutions of the complementarity problem satisfy $|w| \leq K|v|$.

A second consequence is that solutions of the complementarity problem are given by directional derivatives of Π. Indeed if w solves the complementarity problem for v at x, then $\Pi(x+\epsilon v) = x+\epsilon w$ for all $\epsilon > 0$ sufficiently small, and consequently

$$\frac{\Pi(x+\epsilon v)-x}{\epsilon} = w.$$

The velocity projection map is defined in [6] by

$$\pi(x,v) = \lim_{\Delta\downarrow 0} \frac{\Pi(x+\Delta v)-x}{\Delta}. \tag{7}$$

So we see that if the complementarity problem for v at x has a solution, then $\pi(x,v)$ exists and agrees with the solution. Conversely, if $\pi(x,v)$ exists, then $\Pi(x+\epsilon v)$ exists for all sufficiently small $\epsilon > 0$, so by part a) of the lemma the complementarity problem for v has a solution, which must then agree with $\pi(x,v)$.

THEOREM 1. Suppose Assumption 2.1 holds and $x \in \partial G$. $w = \pi(x,v)$ as defined by (7) exists iff w solves the complementarity problem for v at x. $\Pi(y)$ exists for all y in some neighborhood of x iff the complementarity problem (3) has a solution at x for every $v \in \mathbb{R}^n$.

PROOF. Our discussion above establishes all but the “if” assertion of the last sentence. Suppose the complementarity problem (3) has a solution at x for every $v \in \mathbb{R}^n$. Consider $|y-x| < \delta_x/K$. Let w solve the complementarity problem for $v = y-x$ at x. Then $|w| \leq K|v| < \delta_x$, so by part b) of the lemma, $\Pi(x+v) = \Pi(y)$ does exist for all y in the δ_x/K neighborhood of x.

4 Global Existence of the Discrete Projection

Theorem 4.4 of [7] says that $\Pi(\cdot)$ exists globally if it exists in some δ-neighborhood of G: $\{y :\ |y-x| < \delta \text{ for some } x \in G\}$. The following lemma will allow us to prove global existence from solvability of the complementarity problems. We save its proof until after that of the theorem.

LEMMA 2. There exists a $\delta > 0$ with the property that for every $y \notin G$ with $\mathrm{d}(y,G) < \delta$ there exists $x \in \partial G$ with $|y-x| < \delta_x/K$.

THEOREM 2. Under Assumption 2.1 the following are equivalent:

a) For every $x \in \partial G$ and $v \in \mathbb{R}^n$ there exists a solution w of the complementarity problem (3).

b) $\Pi(y)$ exists for all y in some open set containing G.

c) $\Pi(y)$ is defined for all $y \in \mathbb{R}^n$.

PROOF (Theorem 2). We only need to prove that a) implies c), since the rest of the implications are provided by Theorem 1. Let $\delta > 0$ be as in Lemma 2. Consider any y in the δ-neighborhood of G. If $y \notin G$ there exists there exists $x \in \partial G$ with

$|y-x| < \delta_x/K$. The proof of Theorem 1 shows that $\Pi(y)$ exists. Thus $\Pi(\cdot)$ is defined on the δ neighborhood of G. Theorem 4.4 of [7] now implies that $\Pi(\cdot)$ is defined globally.

That b) $\Rightarrow$ c) is a slight extension of Theorem 4.4 of [7] in the case of unbounded G, because then not all neighborhoods of G contain a δ-neighborhood. Lemma 2 is elementary in the bounded case. We need to work harder for the general case.

PROOF (Lemma 2). For each $I \subseteq \{1, \ldots, N\}$ define the face F_I of ∂G for which the constraints $i \in I$ are active:

$$F_I = \{x : \; x \cdot n_i = c_i \text{ for } i \in I, \text{ and } x \cdot n_j > c_j \text{ for } j \in I^c\}.$$

Let L_I be the linear subspace of $\mathbb{R}^n$ defined by $v \cdot n_i = 0$ for all $i \in I$. Clearly any two points of F_I differ by an element of L_I.

We will show that for each nonempty face F_I there exists $\delta_I > 0$ with the property that

$$d(y, F_I) < \delta_I \text{ implies that there exists } x \in \partial G \text{ with } |y-x| < \delta_x/K. \tag{8}$$

To see this, first consider a nonempty face F_I of maximal order, i.e. F_J is empty for all $I \subsetneq J$. The $x \cdot n_j$ for $j \notin I$ must then be constant over F_I. (Otherwise we could add some vector from L_I to x obtain a point in G with at least one more active constraint, which is contrary to the maximality.) Thus δ_x/K is constant over F_I and serves as δ_I.

Now suppose there were some nonempty F_I for which no $\delta_I > 0$ exists. We can assume I is largest possible with this property. Then from the preceding paragraph there are nonempty F_J with $I \subsetneq J$, and $\delta_J > 0$ does exist for all such J. We will produce a contradiction by constructing a suitable $\delta_I > 0$.

Define

$$\epsilon = \frac{1}{2} \min_{I \subsetneq J} \delta_J > 0,$$

and let F_I^ϵ be the set of those y in the affine hyperplane defined by the active constraints $y \cdot n_i = c_i$, $i \in I$, which are at least ϵ away from any point where some other constraint ($j \in I^c$) is active:

$$F_I^\epsilon = \{x \in F_I : \; y \in F_I \text{ whenever } y - x \in L_I \text{ and } |y-x| < \epsilon\}.$$

We claim that $0 < \inf_{x \in F_I^\epsilon} \delta_x$. To see this consider any $j \in I^c$. It suffices to show that $x \cdot n_j - c_j$ has a positive lower bound over F_I^ϵ. There are two cases to consider. If $L_I \perp n_j$ then $x \cdot n_j - c_j$ is a positive constant over F_I^ϵ. If $L_I \perp n_j$ fails, there exists a unit vector $u \in L_I$ with $u \cdot n_j > 0$. Consider any $x \in F_I^\epsilon$ and $0 < \epsilon' < \epsilon$. Let $y = x - \epsilon' u$. Since $|y - x| < \epsilon$ and $x \cdot n_i = y \cdot n_i$ all $i \in I$, the definition of F_I^ϵ implies that $y \cdot n_j - c_j > 0$. Therefore

$$x \cdot n_j = (y + \epsilon' u) \cdot n_j > c_j + \epsilon' u \cdot n_j.$$

Taking the supremum over $\epsilon' < \epsilon$ implies that $x \cdot n_j - c_j \geq \epsilon u \cdot n_j$ for all $x \in F_I^\epsilon$, establishing $\epsilon u \cdot n_j$ as the desired lower bound.

Now consider

$$\delta_I = \min\left(\epsilon, \inf_{x\in F_I^\epsilon} \delta_x/K\right). \tag{9}$$

It follows that $\delta_I > 0$. Suppose that $d(y, F_I) < \delta_I$. Then there is an $x \in F_I$ with $|y-x| < \delta_I$. If $x \in F_I^\epsilon$ then $x \in \partial G$ and

$$|y-x| < \delta_I \le \delta_x/K.$$

Otherwise, $x \in F_I \setminus F_I^\epsilon$. In that case, by definition of F_I^ϵ there exists y' with $|y'-x| < \epsilon$, $y' \cdot n_i = c_i$ for all $i \in I$ but $y' \cdot n_j \le c_j$ for some $j \in I^c$. Then along the line segment from x to y' there exists a z with $z \cdot n_i = c_i$ for all $i \in I \cup \{k\}$, some $k \notin I$, and $z \cdot n_j \ge c_j$ for all remaining j. Then $z \in G$ and $I \cup \{k\} \subseteq I(z)$. Therefore $z \in F_J$ where $I \subsetneq J = I(z)$. Since z is on the line segment between x and y', $|z-x| \le |y'-x| < \epsilon$. Therefore $d(y, F_J) \le |z-x| + |x-y| < 2\epsilon \le \delta_J$. By (8) for J, there exists $x' \in \partial G$ with $|y-x'| < \delta_{x'}/K$. Thus δ_I defined by (9) does satisfy (8).

By contradiction $\delta_I > 0$ satisfying (8) does exist for all nonempty faces F_I. The lemma follows.

5 Coercivity as a Sufficient Condition

The complementarity problem (3) can be expressed in standard form: given $q = (n_i \cdot v)$ find $\beta = (\beta_i)$ such that

$$\begin{aligned} \beta &\ge 0; \quad \text{(componentwise)} \\ z &\ge 0 \quad \text{where } z = M\beta + q; \text{ and} \\ z \cdot \beta &= 0, \end{aligned} \tag{10}$$

where M is the matrix with entries $n_i \cdot d_j$, $i, j \in I(x)$, which we will denote $M = N_I^T D_I$, $I = I(x)$. While several sufficient conditions for solvability of (10) are known, a particularly simple one is that M be coercive (i.e. $u \cdot Mu > 0$ for all $u \ne 0$). See for instance Chapter 1 of [8], Corollary 4.3 and Theorem 5.5. Thus, under Assumption 2.1, a sufficient condition for the global existence of $\Pi(x)$ is that $M = N_I^T D_I$ be coercive for all I with F_I nonempty.

Coercivity of $M = N_I^T D_I$ is equivalent to positive definiteness of

$$\frac{1}{2}(N_I^T D_I + D_I^T N_I),$$

making it rather easy to check. On the other hand coercivity requires that $\{n_i : i \in I\}$ is linearly independent. Thus in $\mathbb{R}^n$ we can only hope to appeal to coercivity if no more than n of the constraints (1) are active at each $x \in G$. Otherwise different conditions would need to be used.

References

[1] R. Atar and P. Dupuis, A differential game with constrained dynamics and viscosity solutions of a related HJB equation, Nonlinear Analysis, 51(2002), 1105–1130.

[2] A. Berhard and A. El Kharroubi, Régularizations déterministes et stochastiques dans le premier “orthant” de $\mathbb{R}^n$, Stochastics and Stochastics Reports, 34(1991), 149–167.

[3] A. Budhiraja and P. Dupuis, Simple necessary and sufficient conditions for the stability of constrained processes, SIAM J. Appl. Math., 59(1999), 1686–1700.

[4] H. Chen and D. D. Yao, Fundamentals of Queueing Networks, Springer-Verlag, New York, 2001.

[5] M. V. Day, J. Hall, J. Menendez, D. Potter and I. Rothstein, Robust optimal service analysis of single-server re-entrant queues, Computational Optimization and Applications, 22(2002), 261–302.

[6] P. Dupuis and H. Ishii, On Lipschitz continuity of the solution mapping of the Skorokhod problem, with applications, Stochastics and Stochastics Reports, 35(1991), 31–62.

[7] P. Dupuis and K. Ramanan, Convex duality and the Skorokhod problem, I and II, Prob. Theor. and Rel. Fields, 115(1999), 153–195, 197–236.

[8] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and their Applications, Academic Press, New York, 1980.