Applied Mathematics E-Notes, 5(2005), 44-51 © ISSN 1607-2510
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DECOMPOSITION PRINCIPLE FOR THE RECIPROCAL OF THE FACTORIAL*

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Received June 6, 2004

Abstract

This note reports on a functional decomposition result for $1/k!$ with a potential for novel computational and number-theoretic applications.

1 Main Result

Structure decomposition is a fundamental principle in practically all branches of mathematics. It states that "*an unordered structure should always be decomposable into two more ordered structures*". In linear algebra, e.g., the principle reveals itself in the fact that any (unordered) square matrix is decomposable into a sum of a symmetric (more ordered) matrix and a skew symmetric (more ordered) matrix . In functional analysis, the principle stipulates that an arbitrary real function is decomposable into a sum of associated odd and even functions. Furthermore, in combinatorics, numerics, special functions and in several other fields the reciprocal of the factorial $1/k!$ often recurs and a number of pertaining mathematical manipulations can be anticipated to significantly simplify if it were possible to establish a decomposition principle for $1/k!$ in terms of $1/(2k)!$ and $1/(2k+1)!$. This note is in fact a report on a new result in this field invoking the perforated factorial

$$(2k+1 \mid *)! = \prod_{j=0}^{k} (2j+1) = (2k+1)!/2^k k!$$

and the summation functionals $\Im$ and $\aleph$ defined by

$$\Im\left[\frac{1}{(2k)!}\right] = -\frac{1}{4}\sum_{n=0}^{k} \frac{2^n}{(2n \mid_{2}^{1})!},$$

$$\aleph\left[\frac{1}{(2k+1)!}\right] = -\frac{1}{4}\sum_{n=0}^{k} \frac{2^n}{(2n+1 \mid_{2}^{1})!},$$

*Mathematics Subject Classifications: 33B10, 33F05.

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in terms of the perturbed factorial

$$\left(m \,|^{1}_{2}\right)! = \frac{m!}{(m-2)\,(m+1)}$$

where m is a non-negative integer different from 2.

THEOREM 1. The following holds:

$$1/k! = (2k-1 \mid *)!\,\Im\left[\frac{1}{(2k)!}\right] + (2k+1 \mid *)!\,\aleph\left[\frac{1}{(2k+1)!}\right].$$

PROOF. Clearly when $m = 2$, $\left(2 \,|^{1}_{2}\right)! = \infty$, and $\frac{1}{(2|^{1}_{2})!} = 0$ while $\left(0 \,|^{1}_{2}\right)! = \left(1 \,|^{1}_{2}\right)! = -\frac{1}{2}$. Moreover, it is demonstrated in the Appendix that

$$\sum_{n=0}^{\infty} \frac{2^n}{(2n \,|^{1}_{2})!} = 0.$$

The proof turns out to be an indirect consequence of the 'rational triplic' form for the exponential

$$e^x = \frac{2}{x^2-2}\left(\sum_{n=3}^{\infty}\sum_{m=0}^{\infty}\frac{(m+2)}{(m+n)!}x^{m+n} - x - 1\right) \tag{1}$$

recently reported by Haidar [1]. It starts with the following two identities

$$\frac{1}{(2k)!} = -\sum_{n=0}^{k}\frac{1}{(2n \,|^{1}_{2})!2^{k-n+1}} \tag{2}$$

and

$$\frac{1}{(2k+1)!} = -\sum_{n=0}^{k}\frac{1}{(2n+1 \,|^{1}_{2})!2^{k-n+1}} \tag{3}$$

which can be demonstrated by means of the examples

$$\frac{1}{8!} = -\frac{1}{(0 \,|^{1}_{2})!2^5} - \frac{1}{(2 \,|^{1}_{2})!2^4} - \frac{1}{(4 \,|^{1}_{2})!2^3} - \frac{1}{(6 \,|^{1}_{2})!2^2} - \frac{1}{(8 \,|^{1}_{2})!2},$$

and

$$\frac{1}{9!} = -\frac{1}{(1 \,|^{1}_{2})!2^5} - \frac{1}{(3 \,|^{1}_{2})!2^4} - \frac{1}{(5 \,|^{1}_{2})!2^3} - \frac{1}{(7 \,|^{1}_{2})!2^2} - \frac{1}{(9 \,|^{1}_{2})!2}.$$

The two identities can then be rewritten respectively in the following way:

$$\frac{1}{(2k)!} = \frac{1}{2^k} - \sum_{n=2}^{k}\frac{(n-1)\,(2n+1)}{(2n)!2^{k-n}},$$

and

$$\frac{1}{(2k+1)!} = \frac{1}{2^k} - \sum_{n=2}^{k} \frac{(n+1)(2n-1)}{(2n+1)!2^{k-n}}.$$

Conceive equation (1) as a Padé-like form for e^x:

$$e^x = \frac{1 + x - \frac{2}{3!}x^3 - 5\frac{x^4}{4!} - 9\frac{x^5}{5!}}{1 - \frac{x^2}{2}} = \frac{1 + x - \sum_{n=3}^{\infty} \frac{(n+1)(n-2)}{2}\frac{x^n}{n!}}{1 - \frac{x^2}{2}}$$

which is obviously the same as

$$e^x = \left[1 + x - \sum_{n=3}^{\infty} \frac{(n+1)(n-2)}{2}\frac{x^n}{n!}\right]\left[\sum_{m=0}^{\infty} \frac{x^{2m}}{2^m}\right]$$

and

$$e^x = \sum_{m=0}^{\infty} \frac{x^{2m}}{2^m} + \sum_{m=0}^{\infty} \frac{x^{2m+1}}{2^m} - \left(\sum_{n=3}^{\infty} \frac{(n+1)(n-2)}{2}\frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{x^{2m}}{2^m}\right). \tag{4}$$

Consider now the product

$$\left(\sum_{n=3}^{\infty} \frac{(n+1)(n-2)}{2}\frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{x^{2m}}{2^m}\right) = \left(\frac{1}{2}\sum_{n=3}^{\infty} \frac{(n+1)(n-2)}{n!}x^n\right)\left(\sum_{m=0}^{\infty} \frac{x^{2m}}{2^m}\right),$$

and this is equal to

$$\left(\sum_{n=3}^{\infty} \frac{(n+1)(n-2)}{n!}x^n\right)\left(\sum_{m=0}^{\infty} \frac{x^{2m}}{2^{m+1}}\right).$$

Note that the least power of x in this product is 3. The coefficient of x^{2r} $(r \geq 2)$ is

$$\sum_{m=0}^{r-2} \frac{(2r-2m+1)(r-m-1)}{(2(r-m))!2^m}$$

while the coefficient of x^{2r+1} $(r \geq 1)$ is

$$\sum_{m=0}^{r-1} \frac{(2r-2m-1)(r-m+1)}{(2(r-m)+1)!2^m}$$

Equation (2) therefore becomes

$$\begin{aligned} e^x &= \sum_{m=0}^{\infty} \frac{x^{2m}}{2^m} + \sum_{m=0}^{\infty} \frac{x^{2m+1}}{2^m} - \\ & \left(\sum_{n=3}^{\infty} \frac{(n+1)(n-2)}{2} \frac{x^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{x^{2m}}{2^m} \right) \\ &= 1 + \frac{x^2}{2} + \sum_{r=2}^{\infty} \left(\frac{1}{2^r} - \sum_{m=0}^{r-2} \frac{(2r-2m+1)(r-m-1)}{(2(r-m))!2^m} \right) x^{2r} + \\ & x + \sum_{r=1}^{\infty} \left(\frac{1}{2^r} - \sum_{m=0}^{r-1} \frac{(2r-2m-1)(r-m+1)}{(2(r-m)+1)!2^m} \right) x^{2r+1}. \end{aligned}$$

Comparison with the Maclaurin's series representation of e^x will lead to the proof.

2 Applications

The present decomposition is clearly redundant for any harmonic analysis of periodic functions such as

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \text{ or } \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

It has, however, much more promise in a modal analysis of severely aperiodic functions like e^x and mildly aperiodic, or recurrent functions like Bessel functions of the first kind $J_m(x)$ [2].

(i) Indeed, the decomposition

$$e^x = \sum_{k=0}^{\infty} \left\{ (2k-1 \mid *)! \, \Im \left[\frac{1}{(2k)!} \right] + (2k+1 \mid *)! \, \aleph \left[\frac{1}{(2k+1)!} \right] \right\} x^k \tag{5}$$

is expected to further the accelerating convergence effect of the pertaining intrinsic Haidar's rational triplic form (1) for the exponential.

(ii) Decomposition of the series representation of Bessel functions of the first kind into four power series

$$\begin{aligned}
& J_m(x) \\
= & \sum_{m=0}^{\infty} \frac{(-1)^r}{r!\,(m+r)!}\left(\frac{x}{2}\right)^{m+2r} \\
= & \sum_{r=0}^{\infty}(2r-1|*)!\,(2m+2r-1|*)!\Im\left[\frac{1}{(2r)!}\right]\Im\left[\frac{1}{(2m+2r)!}\right](-1)^r\left(\frac{x}{2}\right)^{m+2r} \\
& +\sum_{r=0}^{\infty}(2r+1|*)!\,(2m+2r-1|*)!\aleph\left[\frac{1}{(2r+1)!}\right]\Im\left[\frac{1}{(2m+2r)!}\right](-1)^r\left(\frac{x}{2}\right)^{m+2r} \\
& +\sum_{r=0}^{\infty}(2r-1|*)!\,(2m+2r+1|*)!\Im\left[\frac{1}{(2r)!}\right]\aleph\left[\frac{1}{2m+2r+1)!}\right](-1)^r\left(\frac{x}{2}\right)^{m+2r} \\
& +\sum_{r=0}^{\infty}\left\{(2r+1|*)!\,(2m+2r+1|*)!\aleph\left[\frac{1}{(2r+1)!}\right]\right. \\
& \left.\times\aleph\left[\frac{1}{(2m+2r+1)!}\right](-1)^r\left(\frac{x}{2}\right)^{m+2r}\right\}.
\end{aligned}$$

Such nonstandard harmonic (modal) analysis can be employed in sifting possible almost-periods that are attributable to $J_m(x)$.

(iii) In number theory one can state the following novel results.

COROLLARY 1. If n is a natural number > 2, then there exists at least one prime number p satisfying

$$2r < p < \frac{1}{2}\left\{(4r-1|*)!\,(4r+1|*)!\Im\left[\frac{1}{(4r)!}\right]\aleph\left[\frac{1}{(4r+1)!}\right]\right\}^{-1/2}$$

for $n = 2r$ and $r > 1$, and

$$2r+1 < p < \frac{1}{2}\left\{(4r+1|*)!\,(4r+3|*)!\Im\left[\frac{1}{(4r+2)!}\right]\aleph\left[\frac{1}{(4r+3)!}\right]\right\}^{-1/2}$$

for $n = 2r+1$ and $r > 1$.

PROOF. Invoke the well known result [3] which states that if n is a natural number > 2, then there exists at least one prime number p satisfying

$$n < p < n!, \tag{6}$$

i.e.

$$\frac{1}{n} > \frac{1}{p} > \frac{1}{n!}.$$

Decompose $1/n!$ according to the Theorem 1, using the $\Im$ and $\aleph$ functional notation. The proof is then completed by inversion after invoking the Arithmetic-Geometric inequality.

Result (6) was, however, proved by Tchebyshev in 1850 to be replaceable by the following sharper fact.

THEOREM 2 (Tchebyshev). If $n > 5$, then between n and $2n$, there exist at least two primes p and q satisfying

$$n < p < q < 2n. \tag{7}$$

Therefore, motivated by (7) and making use of the definition

$$\rho(n) = 4n \left\{ (2k-1|*)!\,(2k+1|*)! \Im \left[\frac{1}{(2k)!} \right] \aleph \left[\frac{1}{(2k+1)!} \right] \right\}^{1/2},$$

we can show that $\lim_{n\to\infty} \rho(n) > 1$.

To provide a proof for this result, note that

$$\rho(n) = n \prod_{j=1}^{n} (2j-1) \left[(2n+1) \sum_{k=0}^{n} \frac{2^k}{(2k|\frac{1}{2})!} \sum_{k=0}^{n} \frac{2^k}{(2k+1|\frac{1}{2})!} \right].$$

But $\sum\limits_{k=0}^{\infty} a_k \sum\limits_{k=0}^{\infty} b_k = \sum\limits_{k=0}^{\infty} c_k$ where $c_k = \sum\limits_{r=0}^{k} a_r b_{k-r}$. Therefore

$$\begin{aligned}
\lim_{n\to\infty} \rho(n) &= \lim_{n\to\infty} n \prod_{j=1}^{n} (2j-1) \left[(2n+1) \sum_{k=0}^{n} \sum_{r=0}^{k} \frac{2^r}{(2r|\frac{1}{2})!} \frac{2^{k-r}}{(2k-2r+1|\frac{1}{2})!} \right]^{1/2} \\
&= \lim_{n\to\infty} n \prod_{j=1}^{n} (2j-1) \\
&\quad \times \left[(2n+1) \sum_{k=0}^{n} \sum_{r=0}^{k} \frac{2^r (2r-2)(2r+1)(2k-2r-1)(2k-2r+2)}{(2r)!\,(2k-2r+1)!} \right]^{1/2}.
\end{aligned}$$

It is straightforward to show that $\rho(6) > 1$, then the proof can be carried out by induction. Suppose $\lim\limits_{n\to\infty} \rho(n) > 1$ is true for $n = m$, then it should also be true for $n = m+1$.

So if

$$\begin{aligned}
& \lim_{m\to\infty} \rho(m) \\
= & \lim_{m\to\infty} m \prod_{j=1}^{m} (2j-1) \\
& \times \left[(2m+1) \sum_{k=0}^{m} \sum_{r=0}^{k} \frac{2^r (2r-2)(2r+1)(2k-2r-1)(2k-2r+2)}{(2r)!\,(2k-2r+1)!} \right]^{1/2} \\
> & \ 1,
\end{aligned}$$

we have

$$\begin{aligned}
& \lim_{m\to\infty} \rho\,(m+1) \\
= & \lim_{m\to\infty} (m+1) \prod_{j=1}^{m} (2j-1) \\
& \times \left[(2m+3) \sum_{k=0}^{m+1} \sum_{r=0}^{k} \frac{2^r\,(2r-2)\,(2r+1)\,(2k-2r-1)\,(2k-2r+2)}{(2r)!\,(2k-2r+1)!} \right]^{1/2} \\
= & \lim_{m\to\infty} (m+1) \prod_{j=1}^{m} (2j-1) \\
& \times \left[(2m+3) \sum_{k=0}^{m} \sum_{r=0}^{k} \frac{2^r\,(2r-2)\,(2r+1)\,(2k-2r-1)\,(2k-2r+2)}{(2r)!\,(2k-2r+1)!} \right. \\
& \left. + (2m+3) \sum_{r=0}^{m+1} \frac{2^r\,(2r-2)\,(2r+1)\,(2k-2r-1)\,(2k-2r+2)}{(2r)!\,(2k-2r+1)!}^{1/2} \right]
\end{aligned}$$

and this is clearly $> \lim_{m\to\infty} \rho\,(m) > 1$. Here the proof ends.

The previous corollary and result indicate a possibility for different search domains for large prime numbers according to the parity of n. The practical value of this remark requires, perhaps, a further multi-disciplinary investigation.

3 Appendix

The identities (2-3) could be verified directly by induction: for example, one can verify that

$$\frac{1}{(2k)!} = -\sum_{n=0}^{k} \frac{1}{(2n|\frac{1}{2})!2^{k-n+1}}$$

is true for $k=0$. The inductive step is as follows:

$$\begin{aligned}
\frac{1}{(2k+2)!} - \frac{1}{2}\frac{1}{(2k)!} &= -\frac{k\,(2k+3)}{(2k+2)!} - \sum_{n=0}^{k+1} \frac{1}{(2n|\frac{1}{2})!2^{k-n+2}} + \frac{1}{2}\sum_{n=0}^{k} \frac{1}{(2n|\frac{1}{2})!2^{k-n+1}} \\
&= -\sum_{n=0}^{k+1} \frac{1}{(2n|\frac{1}{2})!2^{k-n+2}} + \sum_{n=0}^{k} \frac{1}{(2n|\frac{1}{2})!2^{k-n+2}} \\
&= -\frac{1}{(2k+2|\frac{1}{2})!2} \\
&= -\frac{(2k)\,(2k+3)}{(2k+2)!2} \\
&= -\frac{(k)\,(2k+3)}{(2k+2)!}.
\end{aligned}$$

Notice also that

$$\frac{2^k}{(2k)!} = \frac{1}{k!} \frac{1}{(1)\,(3)\cdots(2k-1)}$$

For m an integer, define the odd factorial $(m|*)!$ to be the product of all odd integers less or equal to m. Factoring out 2^k, one gets

$$\frac{2^k}{(2k)!} = -\sum_{n=0}^{k} \frac{1}{(2n|\frac{1}{2})!2^{-n+1}}$$

and

$$\begin{aligned}\frac{1}{k!}\frac{1}{(2k-1|*)!} &= -\sum_{n=0}^{k} \frac{2^{n-1}}{(2n|\frac{1}{2})!} \\ &= -\frac{1}{(0|\frac{1}{2})!2} - \frac{1}{(2|\frac{1}{2})!} - \frac{2}{(4|\frac{1}{2})!} - \cdots - \frac{2^{k-1}}{(2k|\frac{1}{2})!}\end{aligned}$$

Acknowledgment. This work is partially supported by a grant from the Lebanese National Council for Scientific Research, Beirut, Lebanon.

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