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GEODESIC FLOWS ON THE BOTT-VIRASORO GROUP WITH DUBINSKII NORM *

Partha Guha[†]

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Abstract

It is known that the Korteweg-De Vries equation, the Camassa-Holm equation and the Harry Dym (or Hunter-Saxton) equation are geodesic flows on the Bott-Virasoro group with respect to L^2 and H^1 right invariant metrics. In this Note we study geodesic flow on the Bott-Virasoro group with respect to the Sobolev metric of exponential type or Dubinskii metric. It is shown that the Harry Dym equation follows from linearization of this geodesic flow.

1 Introduction

The connection between the geodesic equation on the Bott-Virasoro group and the periodic Korteweg-de Vries (KdV) equation follows from the work of Ovsienko and Khesin [14]. It has been discussed in various places [4,8,15,16].

R. Camassa and D. Holm [2] derived a new completely integrable dispersive shallow water equation using an asymptotic expansion in the Hamiltonian of the Euler equation of hydrodynamics. It is given by

$$u_t + 2Ku_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \tag{1}$$

where K is a constant and has units of speed. The linear dispersion term in the Camassa-Holm equation vanishes and its remaining nonlinear dynamics allows the superposition of N-solitons. The N-soliton solutions of Camassa-Holm equation is called "peakons". Misiolek [13] showed that the Camassa-Holm equation is the Euler-Poincaré equation for the geodesic flow on the Bott-Virasoro group with respect to the right invariant Sobolev H^1 metric.

The Harry-Dym equation is connected with the geodesic flow of the weighted H^1 metric on the Bott-Virasoro group [6]. Recently, Khesin and Misiolek [9] showed that the KdV equation, the Camassa-Holm equation and the Harry Dym or Hunter-Saxton equation have the same symmetry group and similar bihamiltonian structure.

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†S.N. Bose National Centre for Basic Sciences, JD Block, Sector-3, Salt Lake, Calcutta-700098, India

In fluid dynamics the Harry Dym equation follows from the nonlinear monochromatic short surface waves equation in ideal fluid

$$u_{xxt} + u_x u_{xx} + \frac{1}{2} u u_{xxx} = 0. \tag{2}$$

These monochromatic short surface waves in fluids are shown to emerge as a result of superposition of two surface motions, an oscillatory flow and a laminar flow. The oscillatory flow corresponds to mechanical perturbations which propagate like a wave. The laminar flow may be created in various ways, e.g. by the action of an external wind, or by an external electric field acting on a charged surfaces etc. [10].

In this Note we will study the geodesic flow with respect to Dubinskii metric. The Sobolev spaces of infinite order

$$W^{\infty}\{a_n, p, r\}_{\Omega} = \{u(x) \in C^{\infty}(\Omega) : p(u) = \sum_{n=0}^{\infty} a_n ||\Delta^{\alpha} u(x)||_r^p < \infty\}$$

be defined by a rapidly decreasing sequence $\{a_n\}$. This metric was derived by Yu. A. Dubinskii [3] while continuation of research on Banach spaces of infinitely differentiable function.

Thus the Dubinskii metric must be the square root of either $||e^{a\Delta} f||_2^2$ or $||e^{a\sqrt{\Delta}} f||_2^2$, where a is some parameter and Δ be the corresponding Laplace-Beltrami operator associated to the manifold M. Let us define

$$||e^{\Delta} f||_2^2 = \langle e^{\Delta} f, e^{\Delta} f \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} ||\nabla^{(n)} f||_2^2. \tag{3}$$

In fact other possible definitions could be an infinite series with positive coefficients and positive radius of convergence.

Let us stick to equation (3), and $M = S^1$. It is also suitable to define some kind of averaged Euler equation.

We now state our main results:

THEOREM 1.1. The Euler-Poincaré equation on the coadjoint orbits on the dual of the Virasoro algebra describing the geodesic flow on the Bott-Virasoro group with respect to the W^{∞} metric, defined by

$$||e^{\partial} f||_2^2 = \sum_{0}^{\infty} \frac{1}{n!} ||\partial^{(n)} f||_2^2, \tag{4}$$

yields the following equation

$$e^{-a^2 \partial_x^2} u_t = -\lambda u_{xxx} + 3 e^{-a^2 \partial_x^2} u_x u.$$

The Harry Dym equation appears as a linearize flow of this equation.

2 Background

Ovsienko and Khesin showed that the KdV equation is the Euler-Poincaré flow on the central extension of the group $Diff(S^1)$, group of diffeomorphism of the circle parametrized by $x : 0 \leq x \leq 2\pi$. For all practical purposes we restrict ourselves to the space of orientation preserving C^∞ diffeomorphism of S^1, denoted by $Diff_+(S^1)$.

It is natural to consider the algebra $Vect(S^1)$ of vector fields on S^1 as its algebra. The central extension of the algebra of vector fields $Vect(S^1)$ is also called the Virasoro algebra [7,8].

The $Vect(S^1)$ has a unique non-trivial central extension by means of $\mathbf{R}$

$$0 \longrightarrow \mathbf{R} \longrightarrow Vir \longrightarrow Vect(S^1) \longrightarrow 0$$

described by the Gelfand-Fuks cocycle [7,8]

$$\omega_1 \left(f\frac{d}{dx}, g\frac{d}{dx} \right) = \int_{S^1} f'g''dx.$$

The elements of Vir can be identified with the pairs (2π periodic function , real number). The commutator in Vir takes the form

$$\left[\left(f(x)\frac{d}{dx}, a \right), \left(g(x)\frac{d}{dx}, b \right) \right] = \left(fg' - gf', \int_{S^1} f'g'' \right). \tag{5}$$

The dual space Vir^* can be identified to the set

$$\{(\lambda, u) | \lambda \in \mathbf{R} \text{ and } u \text{ is a quadratic differential } \}.$$

Let G be a Lie group and $\mathfrak{g}$ be its corresponding Lie algebra and its dual is denoted by $\mathfrak{g}^*$. The dual space $\mathfrak{g}^*$ to any Lie algebra $\mathfrak{g}$ carries a natural Lie-Poisson structure [11,12]:

$$\{f, g\}_{LP}(\mu) := \langle [df, dg], \mu \rangle$$

for any $\mu \in \mathfrak{g}^*$ and $f, g \in C^\infty(S^1)$.

LEMMA 2.1. The Hamiltonian vector field on $\mathfrak{g}^*$ corresponding to a Hamiltonian function f, computed with respect to the Lie-Poisson structure is given by

$$\frac{d\mu}{dt} = ad^*_{df}\mu \tag{6}$$

PROOF. It follows from the following identities

$$i_{X_f} dg|_\mu = L_{X_f} g|_\mu = \{f, g\}_{LP}(\mu) = \langle [dg, df], \mu \rangle = \left\langle dg, ad^*_{df}\mu \right\rangle.$$

This implies that $X_f = ad^*_{df}\mu$. Thus the Hamiltonian equation $\frac{d\mu}{dt} = X_f$ yields our result.

Let I be an inertia operator

$$I : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$

and then $\mu \in \mathfrak{g}^*$ evolve by

$$\frac{d\mu}{dt} = (I^{-1}\mu) \cdot \mu, \tag{7}$$

where right hand side denote the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. This equation is called the Euler-Poincaré equation.

DEFINITION 2.2. The Euler-Poincaré equation on $\mathfrak{g}^*$ corresponding to the Hamiltonian $H(\mu) = \frac{1}{2} < I^{-1}\mu, \mu >$ is given by the following

$$\frac{d\mu}{dt} = -ad^*_{I^{-1}\mu}\mu,$$

evolution of a point $\mu \in \mathfrak{g}^*$.

The Euler-Poincaré equation is the Hamiltonian flow on the coadjoint orbits on the dual of Bott-Virasoro algebra generated by the Hamiltonian

$$H\left(u\frac{d}{dx}, a\right) = \frac{1}{2}\int_{S^1} u^2 dx + a^2,$$

where a is just a constant.

3 W^∞-Dubinskii Norm and Geodesic Flow

Let us define the W^∞ metric on the space of Virasoro algebra:

DEFINITION 3.1.

$$\left\langle (f(x)\frac{d}{dx}, b), (g(x)\frac{d}{dx}, c)\right\rangle_{W^\infty} = \left\langle exp(a\frac{d}{dx})f, exp(a\frac{d}{dx})g\right\rangle_{L^2} + bc. \tag{8}$$

Let us define the coadjoint action of the Virasoro algebra on its dual

$$\left\langle ad^*_{\hat{f}}\hat{u}, \hat{g}\right\rangle_{W^\infty} = \left\langle \hat{u}, [\hat{f}, \hat{g}]\right\rangle_{W^\infty}. \tag{9}$$

LEMMA 3.2. The following is valid:

$$ad^*_{\hat{f}}\hat{u} = (e^{-a^2\partial^2})^{-1}\left[fe^{-a^2\partial^2}u_x + 2f'e^{-a^2\partial^2}u + \lambda f'''\right]. \tag{10}$$

PROOF. We know

$$\left\langle ad^*_{\hat{f}}\hat{u}, \hat{g}\right\rangle_{W^\infty} = \left\langle \hat{u}, -(fg' - fg')\frac{d}{dx}, \int_{S^1} f'g''\right\rangle_{W^\infty}.$$

But

$$\begin{aligned} R.H.S. &= -\langle u, fg' - fg'\rangle_{H^\infty} + \lambda\int_{S^1} f'g''dx \\ &= \left\langle e^{a\frac{d}{dx}}u, e^{a\frac{d}{dx}}(fg' - f'g)\right\rangle_{L^2} + \lambda\int_{S^1} f'''gdx \\ &= \int_{S^1}\left[f(e^{-a^2\partial^2})u_x + 2f'(e^{-a^2\partial^2})u + \lambda f'''\right]gdx, \end{aligned}$$

and

$$L.H.S. = \int_{S^1} \left[e^{a\frac{d}{dx}} (ad^*_{\hat{f}} \hat{u}) e^{a\frac{d}{dx}} g \right] dx = \int_{S^1} \left[e^{-a^2\partial^2}) ad^*_{\hat{f}} \hat{u} \right] g dx.$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.

The Hamiltonian operator is given by

$$\begin{aligned} \mathcal{O}_\lambda &= (e^{-a^2\partial^2})^{-1} \left[-\lambda \partial^3 + (e^{-a^2\partial^2}) u_x + 2(e^{-a^2\partial^2}) u \partial_x \right] \\ &= (e^{-a^2\partial^2})^{-1} \left[-\lambda \partial^3 \oplus e^{-a^2\partial^2} (\partial u + u \partial) \right] \\ &= (e^{-a^2\partial^2})^{-1} [\lambda \mathcal{O}_1 + \mathcal{O}_2], \end{aligned}$$

where $(e^{-a^2\partial^2})^{-1}$ is a formal inverse of a bounded operator.

If we substitute this into the Euler-Poincaré equation

$$u_t = \mathcal{O}_\lambda \frac{\delta H}{\delta u} = (e^{-a^2\partial^2})^{-1} [\lambda \mathcal{O}_1 + \mathcal{O}_2] \frac{\delta H}{\delta u}.$$

Thus we obtain

$$(e^{-a^2\partial_x^2}) u_t = \left[-\lambda u_{xxx} + 3(e^{-a^2\partial_x^2}) u_x u \right]. \tag{11}$$

REMARK. We *cannot* express $\mathcal{O}_\lambda = -\lambda e^{a^2\partial^2} \partial_x^3 + u_x$. The operator $e^{a^2\partial^2}$ is an unbounded operator. Moreover this would lead to different class of system

$$u_t = -\lambda e^{a^2\partial^2} u_{xxx} + 3 u_x u. \tag{12}$$

The first expression of the R.H.S. denotes an unbounded operator acting on an unknown object. Of course this new equation yields the KdV equation with an additional higher-order dispersion term, and this would lead to infinitely many higher order cocycle terms for Virasoro algebra. Thus we come to a contradiction. In fact, we can not go back to the Camassa-Holm type systems from equation (12). If we retain upto the second term of the exponential derivative then equation (12) boils down to the quintic KdV - the KdV equation with an additional fifth-order dispersion term

$$u_t + 6 u u_x + \gamma u_{xxx} + \beta u_{xxxxx} = 0.$$

This equation has been used as a model for gravity-capillary waves on a shallow layer. It has been shown [1] that this equation possesses infinitely many multi-pulsed stationary solitary wave solutions. Thus the solutions of this equation are not peakons but compactons.

COROLLARY 3.3. (i) When $a \longrightarrow 0$, then the equation (12) boils down to the KdV equation for $\lambda = 1$. (ii) For a small value of a this becomes the Camassa-Holm equation

$$u_t - a^2 u_{xxt} = -\lambda u_{xxx} + 3(1 - a^2\partial^2) u_x u$$

COROLLARY 3.4. The linearization of the geodesic flow of W^∞ Dubinskii metric yields the Harry-Dym equation.

Indeed,

$$\frac{d}{db}(e^{-b\partial_x^2}u_t)|_{b=0} = \frac{d}{db}[-\lambda u_{xxx} + +3(e^{a^2\partial_x^2})u_x u]|_{b=0}.$$

4 Conclusion and Outlook

In this modest Note we have considered the geodesic flow on the Bott-Virasoro group with respect to W^∞ metric. This flow is highly nontrivial and only linearize flow yields the Harry Dym equation. It would be interesting to study the analysis of this flow.

We can suggest two possible immediate generalization of the present construction. Firstly, one can study the geodesic flow with respect to Dubinskii metric on the space of $Di\hat{ff}(S^1) \ltimes C^\infty(S^1)$ [5]. This will give rise to $2+1$ dimensional generalization of our equation. Next one will be to study the supersymmetric generalization. The Neveu-Schwarz superalgebra, which contains the Virasoro algebra as its even part, must play a central role. Thus, one must study the Euler-Poincaré flow on the dual space of Neveu-Schwarz algebra. This would yield the geodesic equations on the superconformal group with respect to W^∞.

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