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L^∞-ERROR ESTIMATE FOR A NONCOERCIVE SYSTEM OF ELLIPTIC QUASI-VARIATIONAL INEQUALITIES: A SIMPLE PROOF *

Messaoud Boulbrachene†

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Abstract

In this paper we provide a simple proof to derive L^∞-error estimate for a noncoercive system of quasi-variational inequalities related to the management of energy production. The key idea is a discrete L^∞-stability property owned by the corresponding coercive problem.

1 Introduction

We are interested in the piecewise linear finite element approximation of the solution of the following system of quasi-variational inequalities (QVIs) : find $U = (u^1, ..., u^J) \in (H_0^1(\Omega))^J$ satisfying

$$\begin{cases} a^i(u^i, v - u^i) \geq (f^i, v - u^i), \ \forall v \in H_0^1(\Omega) \\ u^i \leq Mu^i; \ u^i \geq 0; \ v \leq Mu^i \end{cases} \tag{1}$$

in which Ω is a bounded smooth domain of $\Re^N$ where $N \geq 1$, each $a^i(.,.)$ is a continuous elliptic bilinear form assumed to be noncoercive, $(.,.)$ is the inner product in $L^2(\Omega)$ and each f^i is a regular function.

This system arises in the management of energy production problems, where J power generation machines are involved, see e.g. [1], [2] and the references therein. In the case studied here Mu^i represents a (cost function) and the prototype encountered is

$$Mu^i = k + \inf_{\mu \neq i} u^\mu. \tag{2}$$

In (2), k represents the switching cost. It is positive when the machine is "turn on" and equal to zero when the machine is "turn off". Note also that operator M provides the coupling between the unknowns $u^1, ..., u^J$.

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†Department of Mathematics, College of Science, Sultan Qaboos University, P.O. Box 36 Muscat 123, Sultanate of Oman

In [3], we established an L^∞-error estimate for the coercive problem. This result was then extended to the noncoercive case where a subsolution approach was employed (see [4]).

In this paper we propose a new proof to derive the same L^∞ convergence order for the noncoercive problem. This proof is much simpler than the one introduced in [4] as it rests on the sole discrete L^∞ stability property with respect to the right hand side of the solution of the corresponding coercive problem.

The paper is organized as follows. In section 2 we state both the continuous and discrete problems. In section 3 we prove a discrete L^∞ stability property for the corresponding coercive problem and give the main result of the paper.

2 Statement of the Problems

2.1 Preliminaries

We are given functions $a^i_{jk}(x)$ in $C^{1,\alpha}(\bar{\Omega})$, $a^i_k(x)$, $a^i_0(x)$ in $C^{0,\alpha}(\bar{\Omega})$ such that

$$\sum_{1\leq j,k\leq N} a^i_{jk}(x)\xi_j\xi_k \geq \alpha \mid \zeta \mid^2;\ \zeta \in \Re^N;\ \alpha > 0, \tag{3}$$

$$a^i_0(x) \geq \beta > 0,\ x \in \Omega. \tag{4}$$

We define the bilinear form

$$a^i(u,v) = \int_\Omega \left(\sum_{1\leq j,k\leq N} a^i_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} a^i_k(x)\frac{\partial u}{\partial x_k}v + a^i_0(x)uv \right) dx. \tag{5}$$

We are also given right hand side f^i , $1 \leq i \leq J$, such that

$$f^i \in C^{0,\alpha}(\bar{\Omega});\ f^i \geq f^0 > 0. \tag{6}$$

Finally for $W = (w^1, ..., w^J) \in (L^\infty(\Omega))^J$ we introduce the norm

$$\| W \|_\infty = \max_{1\leq i\leq J} \left\| w^i \right\|_{L^\infty(\Omega)}. \tag{7}$$

2.2 The Continuous Problem

To solve the noncoercive problem, we transform (1) into the following auxiliary system: find $U = (u^1, ..., u^J) \in (H^1_0(\Omega))^J$ such that:

$$\begin{cases} b^i(u^i, v-u^i) \geq (f^i + \lambda u^i, v-u^i),\ \forall v \in H^1_0(\Omega) \\ u^i \leq Mu^i;\ u^i \geq 0;\ v \leq Mu^i \end{cases} \tag{8}$$

where

$$b^i(u,\, v) = a^i(u,\, v) + \lambda(v, v) \tag{9}$$

and $\lambda > 0$ is large enough such that

$$b^i(v,\, v) \geq \gamma \, \|v\|^2_{H^1(\Omega)}\,, \;\; \gamma \; > 0; \; v \in H^1(\Omega). \tag{10}$$

THEOREM 1 (cf. [2]). Under the preceding assumptions, there exists a unique solution $U = (u^1, ..., u^J)$ to system (1). Furthermore, this solution belongs to $(W^{2,p}(\Omega))^J$ for $2 \leq p < \infty$.

NOTATION 1. Throughout the paper $U = \partial(F + \lambda U,\, MU)$ will denote the solution of (8) where $F = (f^1, ..., f^J)$ and $MU = (Mu^1, ..., Mu^J)$.

2.3 The Discrete Problem

Let Ω be decomposed into triangles and let τ_h denote the set of all those elements; $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform.

Let V_h denote the standard piecewise linear finite element space and $B^i,\ 1 \leq i \leq J$, be the matrices with generic coefficient

$$B^i_{ls} = b^i(\varphi_l, \varphi_s),\ 1 \leq l, s \leq m(h), \tag{11}$$

where $\varphi_s,\ s = 1, 2, ..., m(h)$ are the nodal basis functions of the space V_h.

Also let r_h denote the usual interpolation operator.

DEFINITION. A real $n \times n$ matrix $A = [a_{ij}]$ with $a_{ij} \leq 0$ for all $i \neq j$ is an M-matrix if A is nonsingular and $A^{-1} \geq 0$.

The discrete maximum principle assumption (d.m.p): We assume that the matrices B^i, are M-matrices (cf. [5]). The discrete counterpart of system (1) then reads as follows: find $U_h = (u^1_h, ..., u^J_h) \in (V_h)^J$ such that

$$\left\{ \begin{array}{l} a^i(u^i_h, v - u^i_h) \geq (f^i, v - u^i_h),\ \forall v \in V_h \\ u^i_h \leq r_h M u^i_h;\ u^i_h \geq 0;\ v \leq r_h M u^i_h \end{array} \right. \tag{12}$$

or equivalently

$$\left\{ \begin{array}{l} b^i(u^i_h, v - u^i_h) \geq (f^i + \lambda u^i_h, v - u^i_h),\ \forall v \in V_h \\ u^i_h \leq r_h M u^i_h;\ u^i_h \geq 0;\ v \leq r_h M u^i_h \end{array} \right. \tag{13}$$

THEOREM 2 (cf. [4]). Let the **d.m.p** hold. Then, system (12) or (13) admits a unique solution.

NOTATION 2. Let $U_h = \partial_h(F + \lambda U_h,\, MU_h)$ where $MU_h = (Mu^1_h, ..., Mu^J_h)$.

3 L^∞-Error Analysis

The following is a monotonicity property for the coercive problem corresponding to system (12). This result will play a crucial role in proving the discrete L^∞- stability property.

3.1 A Discrete Monotonicity Property

Let $F = (F^1, ..., F^J) \in (L^\infty(\Omega))^J$ and $Z_h = (z_h^1, ..., z_h^J)$ be the solution of the coercive system of QVIs:

$$\begin{cases} b^i(z_h^i, v - z^i) \geq (F^i, v - z_h^i), \ v \in V_h \\ z_h^i \leq r_h M z_h^i; \ z_h^i \geq 0; \ v \leq r_h M z_h^i \end{cases} . \tag{14}$$

Let $Z_h = \partial_h(F, MZ_h)$ denote that solution with $z_h^i = \sigma_h(F^i,\ Mz_h^i)$. Following [3], existence of a unique solution can be obtained by introducing two monotone sequences: **a decreasing sequence**: $\bar{Z}_h^n = (\bar{z}_h^{1,n}, ..., \bar{z}_h^{J,n})$ such that

$$\bar{z}_h^{i,n+1} = \sigma_h(F^i,\ M\bar{z}_h^{i,n})$$

where $\bar{z}_h^{i,0}$ is the unique solution of $b(\bar{z}_h^{i,0}, v) = (F^i, v)$ for all $v \in H_0^1(\Omega)$, and **an increasing sequence:** $\underline{Z}_h^n = (\underline{z}_h^{1,n}, ..., \underline{z}_h^{J,n})$ such that

$$\underline{z}_h^{i,n+1} = \sigma_h(F^i,\ M\underline{z}_h^{i,n})$$

with $\underline{z}^{i,0} = 0$.

THEOREM 3 (cf. [3]). The sequences $(\bar{Z}_h^n)$ and $(\underline{Z}_h^n)$ converge respectively from above and below to the unique solution of system (14).

Let $F, \tilde{F}$ in $(L^\infty(\Omega))^J$ and $Z_h = \partial_h(F, MZ_h)$, $\tilde{Z}_h = \partial_h(\tilde{F}, M\tilde{Z}_h)$ be the corresponding solutions to system (14). Then, we have the following monotonicity principle.

PROPOSITION 1. Under the **d.m.p**, if $F \geq \tilde{F}$, then $\partial_h(F, MZ_h) \geq \partial_h(\tilde{F},\ M\tilde{Z}_h)$.

PROOF. First, let $\bar{Z}_h^0 = (\bar{z}_h^{1,0}, ..., \bar{z}_h^{J,0})$ and $\overline{\tilde{Z}}_h^0 = (\overline{\tilde{z}}_h^{1,0}, ..., \overline{\tilde{z}}_h^{J,0})$ be such that $\bar{z}_h^{i,0}$ and $\overline{\tilde{z}}_h^{i,0}$ are solutions to equations $b(\bar{z}_h^{i,0}, v) = (F^i, v)$ for all $v \in H_0^1(\Omega)$ and $b(\overline{\tilde{z}}_h^{i,0}, v) = (\tilde{F}^i, v)$ for all $v \in H_0^1(\Omega)$, respectively. Then, the corresponding decreasing sequences

$$\bar{Z}_h^n = (\bar{z}_h^{1,n}, ..., \bar{z}_h^{J,n}) \text{ and } \overline{\tilde{Z}}_h^n = (\overline{\tilde{z}}_h^{1,n}, ..., \overline{\tilde{z}}_h^{J,n})$$

satisfy the following monotonicity principle

$$F^i \geq \tilde{F}^i \Rightarrow \bar{z}_h^{i,n} \geq \overline{\tilde{z}}_h^{i,n},\ i = 1, ..., J.$$

Indeed, since

$$\bar{z}_h^{i,n+1} = \sigma(F^i, M\bar{z}_h^{i,n})$$

and

$$\overline{\tilde{z}}_h^{i,n+1} = \sigma(\tilde{F}^i, M\overline{\tilde{z}}_h^{i,n})$$

then, due to the **d.m.p**, $F^i \geq \tilde{F}^i$ implies $\bar{z}_h^{i,0} \geq \overline{\tilde{z}}_h^{i,0}$, $i = 1, 2, ..., J$. So, $M\bar{z}_h^{i,0} \geq M\overline{\tilde{z}}_h^{i,0}$, and thus, from the **d.m.p** and standard comparison results in discrete coercive variational inequalities, it follows that

$$\bar{z}_h^{i,1} \geq \overline{\tilde{z}}_h^{i,1}.$$

Now assume that $\bar{z}_h^{i,n-1} \geq \bar{\tilde{z}}_h^{i,n-1}$. Then, as $F^i \geq \tilde{F}^i$, applying the same comparison argument as before, we get

$$\bar{z}_h^{i,n} \geq \bar{\tilde{z}}_h^{i,n}.$$

Finally, by Theorem 3, taking the the limit as $n \to \infty$, we get $Z_h \geq \tilde{Z}_h$.

3.2 A Discrete L^∞-Stability Property

We have the following result.

THEOREM 4. Under conditions of Proposition 1, we have

$$\left\| \partial_h(F, MZ_h) - \partial_h(\tilde{F}, M\tilde{Z}_h) \right\|_\infty \leq \frac{1}{\lambda + \beta} \left\| F - \tilde{F} \right\|_\infty .$$

PROOF. Let us first set

$$\Phi = \frac{1}{\lambda + \beta} \left\| F - \tilde{F} \right\|_\infty ; \; \Phi^i = \frac{1}{\lambda + \beta} \left\| F^i - \tilde{F}^i \right\|_\infty .$$

Then, we have

$$F^i \leq \tilde{F}^i + \left\| F^i - \tilde{F}^i \right\|_\infty \leq \tilde{F}^i + \frac{a_0(x) + \lambda}{\lambda + \beta} \left\| F - \tilde{F}^i \right\|_\infty .$$

So, due to (4),

$$F^i \leq \tilde{F}^i + (a_0(x) + \lambda)\Phi.$$

Now making use of Proposition 1, we get

$$z_h^i \leq \tilde{z}_h^i + \Phi^i.$$

Similarly, interchanging the roles of F and $\tilde{F}$, we obtain

$$\tilde{z}_h^i \leq z_h^i + \Phi^i.$$

Thus

$$\left\| z_h^i - \tilde{z}_h^i \right\|_{L^\infty(\Omega)} \leq \; \Phi^i, \; i = 1, ..., J,$$

which completes the proof.

3.3 L^∞-Error Estimate

In what follows we prove the main result of this paper. For this purpose, we introduce the following auxiliary coercive discrete system of QVIs: find $\bar{Z}_h = (\bar{z}_h^1, ..., \bar{z}_h^J)$ solution to

$$\left\{ \begin{array}{l} b^i(\bar{z}_h^i, v - \bar{z}^i) \geq (f^i + \lambda u^i, \; v - \bar{z}_h^i), \; \forall v \in V_h \\ \bar{z}_h^i \leq r_h M \bar{z}_h^i ; \; \bar{z}_h^i \geq 0; \; v \leq r_h M \bar{z}_h^i \end{array} \right. \tag{15}$$

where the right hand side $f^i + \lambda u^i$ is the same as that of the continuous system (8). So, the solution of system (15) is nothing else than the discrete counterpart of that of (1) or (8). Consequently, thanks to [3], we have the following error estimate.

LEMMA 1 (cf. [3]). There exists a constant C independent of h such that:

$$\left\|U - \bar{Z}_h\right\|_\infty \leq Ch^2 \left|\log h\right|^3 .$$

THEOREM 5. Under conditions of Theorem 4 and Lemma 1, we have

$$\left\|U - U_h\right\|_\infty \leq Ch^2 \left|\log h\right|^3 .$$

PROOF. Indeed, since $\bar{Z}_h = \partial_h(F + \lambda U,\, M\bar{Z}_h)$, then

$$\begin{aligned} \left\|U - U_h\right\|_\infty &\leq \left\|U - \bar{Z}_h\right\|_\infty + \left\|\bar{Z}_h - U_h\right\|_\infty \\ &\leq \left\|U - \bar{Z}_h\right\|_\infty + \left\|\partial_h(F + \lambda U, M\bar{Z}_h) - \partial_h(F + \lambda U_h, MU_h\right\|_\infty \\ &\leq Ch^2 \left|\log h\right|^3 + \frac{\lambda}{\lambda + \beta} \left\|U - U_h\right\|_\infty \end{aligned}$$

where we have used Lemma 1 and Theorem 4. Thus the desired error estimate follows.

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