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THE GENERALIZED HERON MEAN AND ITS DUAL FORM *

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Abstract

In this paper, we define the generalized Heron mean $H_r(a,b;k)$ and its dual form $h_r(a,b;k)$, and obtain some propositions for these means.

1 Introduction

For positive numbers a, b, let $A = A(a,b) = \frac{a+b}{2}$, $G = G(a,b) = \sqrt{ab}$, $H = H(a,b) = \frac{a+\sqrt{ab}+b}{3}$, and

$$L = L(a,b) = \begin{cases} \frac{a-b}{\ln a - \ln b} & a \neq b \\ a & a = b \end{cases}.$$

These are respectively called the arithmetic, geometric, Heron, and logarithmic means. Let r be a real number, the r-order power mean (see [1]) is defined by

$$M_r = M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{1/r} & r \neq 0 \\ \sqrt{ab} & r = 0 \end{cases}. \tag{1}$$

The well-known Lin inequality (see also [1]) is stated as $G \leqslant L \leqslant M_{\frac{1}{3}}$.

In 1993, the following interpolation inequalities are summarized and stated by Kuang in [1]:

$$G \leqslant L \leqslant M_{\frac{1}{3}} \leqslant M_{\frac{1}{2}} \leqslant H \leqslant M_{\frac{2}{3}} \leqslant A. \tag{2}$$

In [2], Jia and Cao studied the power-type generalization of Heron mean

$$H_r = H_p(a,b) = \begin{cases} \left(\frac{a^r+(ab)^{r/2}+b^r}{3}\right)^{1/r} & r \neq 0 \\ \sqrt{ab} & r = 0 \end{cases} \tag{3}$$

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and obtained inequalities $L \leqslant H_p \leqslant M_q$, where $p \geqslant \frac{1}{2}, q \geqslant \frac{2}{3}p$. Furthermore, $p = \frac{1}{2}, q = \frac{1}{3}$ are the best constants.

In 2003, Xiao and Zhang [3] gave another generalization of Heron mean and its dual form respectively as follows

$$H(a,b;k) = \frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{k-i}{k}} b^{\frac{i}{k}}, \tag{4}$$

and

$$h(a,b;k) = \frac{1}{k}\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}} b^{\frac{i}{k+1}}, \tag{5}$$

where k is a natural number. They proved that $H(a,b;k)$ is a monotone decreasing function and $h(a,b;k)$ is a monotone increasing function in k, and $\lim_{k\to+\infty} H(a,b;k) = \lim_{k\to+\infty} h(a,b;k) = L(a,b)$.

Combining (3)-(5), two classes of new means for two variables will be defined.

DEFINITION 1. Suppose $a > 0$, $b > 0$, k is a natural number and r is a real number. Then the generalized power-type Heron mean and its dual form are defined as follows

$$H_r(a,b;k) = \begin{cases} \left(\frac{1}{k+1}\sum_{i=0}^{k} a^{(k-i)r/k} b^{ir/k}\right)^{1/r}, & r \neq 0; \\ \sqrt{ab}, & r = 0; \end{cases} \tag{6}$$

and

$$h_r(a,b;k) = \begin{cases} \left(\frac{1}{k}\sum_{i=0}^{k} a^{(k+1-i)r/(k+1)} b^{ir/(k+1)}\right)^{1/r}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases} \tag{7}$$

According to Definition 1, we easily find the following characteristic properties and two remarks for $H_r(a,b;k)$ and $h(a,b;k)$.

PROPOSITION 1. If k is a natural number, and r is a real number, then

(*a*) $H_r(a,b;k) = H_r(b,a;k)$ and $h_r(a,b;k) = h_r(b,a;k)$;

(*b*) $\lim\limits_{r\to 0} H_r(a,b;k) = \lim\limits_{r\to 0} h_r(a,b;k) = \sqrt{ab}$;

(*c*) $H_r(a,b;1) = M_r(a,b)$, $H_r(a,b;2) = H_r(a,b)$ and $h_r(a,b;1) = \sqrt{ab}$;

(*d*) $\lim\limits_{k\to+\infty} H_r(a,b;k) = \lim\limits_{k\to+\infty} h_r(a,b;k) = [L(a^r,b^r)]^{\frac{1}{r}}$;

(*e*) $a \leqslant H_r(a,b;k) \leqslant b$ and $a \leqslant h_r(a,b;k) \leqslant b$ if $0 < a < b$;

(*f*) $H_r(a,b;k) = h_r(a,b;k) = a$ if, and only if, $a = b$;

(*g*) $H_r(ta,tb;k) = tH_r(a,b;k)$ and $h_r(ta,tb;k) = th_r(a,b;k)$ if $t > 0$.

REMARK 1. Suppose $a>0, b>0,\ k$ is a natural number and r is a real number. Then the generalized power-type Heron mean $H_r(a,b;k)$ and its dual form $h_r(a,b;k)$ can be written as

$$H_r(a,b;k)=\begin{cases}\left[\dfrac{a^{\frac{(k+1)r}{k}}-b^{\frac{(k+1)r}{k}}}{(k+1)(a^{\frac{r}{k}}-b^{\frac{r}{k}})}\right]^{\frac{1}{r}}, & r\neq 0, a\neq b;\\ \sqrt{ab}, & r=0, a\neq b;\\ a, & r\in R, a=b;\end{cases} \tag{8}$$

and

$$h_r(a,b;k)=\begin{cases}\left[\dfrac{a^{\frac{kr}{k+1}}-b^{\frac{kr}{k+1}}}{-k(a^{-\frac{r}{k+1}}-b^{-\frac{r}{k+1}})}\right]^{\frac{1}{r}}, & r\neq 0, a\neq b;\\ \sqrt{ab}, & r=0, a\neq b;\\ a, & r\in R, a=b.\end{cases} \tag{9}$$

REMARK 2. Let $a>0, b>0,\ k$ is a natural number, then the following Detemple-Robertson mean $D_r(a,b)$ (see [4]) and its dual form $d_k(a,b)$ are respectively the special cases for $H_r(a,b;k)$ and $h_k(a,b;k)$:

$$D_k(a,b)=[H_k(a,b;k)]^k=\frac{1}{k+1}\sum_{i=0}^{k}a^{k-i}b^i=\begin{cases}\dfrac{a^{k+1}-b^{k+1}}{(k+1)(a-b)}, & a\neq b;\\ a^k, & a=b;\end{cases} \tag{10}$$

and

$$d_k(a,b)=[h_{k+1}(a,b;k)]^{k+1}=\frac{1}{k}\sum_{i=1}^{k}a^{k+1-i}b^i=\begin{cases}\dfrac{ab(a^k-b^k)}{k(a-b)}, & a\neq b;\\ a^{k+1}, & a=b.\end{cases} \tag{11}$$

In this paper, we obtain the monotonicity and logarithmic convexity of the generalized power-type Heron mean $H_r(a,b;k)$ and its dual form $h_r(a,b;k)$.

2 Lemmas

In order to prove the theorems of the next section, we require some lemmas in this section.

LEMMA 1 ([1]). Let $a_1,...,a_n$ be real numbers with $a_i\neq a_j$ for $i\neq j$, and

$$M_r(a)=\begin{cases}\left[\dfrac{1}{n}\sum_{i=1}^{n}a_i^r\right]^{\frac{1}{r}}, & 0<\mid r\mid<+\infty;\\ \prod_{i=1}^{n}a_i^{\frac{1}{n}}, & r=0.\end{cases} \tag{12}$$

Then $M_r(a)$ is a monotone increasing function in r, and $f(r) = [M_r(a)]^r$ is a logarithmic convex function with respect to $r > 0$.

LEMMA 2 ([5],[6]). Let p, q be arbitrary real numbers, and $a, b > 0$. Then the extended mean values

$$E_{p,q}(a,b) = \begin{cases} \left[\frac{q}{p}\frac{a^p-b^p}{a^q-b^q}\right]^{1/(p-q)}, & pq(p-q)(a-b) \neq 0; \\ \left[\frac{1}{p}\frac{a^p-b^p}{\ln a - \ln b}\right]^{1/p}, & p(a-b) \neq 0, q = 0; \\ e^{-1/p}\left(\frac{a^{a^p}}{b^{b^p}}\right)^{1/(a^p-b^p)} & p(a-b) \neq 0, p = q; \\ \sqrt{ab}, & (a-b) \neq 0, p = q = 0; \\ a, & a = b. \end{cases} \tag{13}$$

are monotone increasing with respect to both p and q, or to both a and b; and are logarithmical concave on $(0, +\infty)$ with respect to either p or q, respectively; and logarithmical convex on $(-\infty, 0)$ with respect to either p or q, respectively.

LEMMA 3 ([7]). Let p, q, u, v be arbitrary with $p \neq q, u \neq v$. Then the inequality

$$E_{p,q}(a,b) \geqslant E_{u,v}(a,b) \tag{14}$$

is satisfied for all $a, b > 0, a \neq b$ if and only if $p+q \geqslant u+v$, and $e(p,q) \geqslant e(u,v)$, where

$$e(x,y) = \begin{cases} (x-y)/\ln(x/y), & for\ xy > 0, x \neq y; \\ 0, & for\ xy = 0; \end{cases}$$

if either $0 \leqslant \min\{p,q,u,v\}$ or $\max\{p,q,u,v\} \leqslant 0$; and

$$e(x,y) = (\mid x \mid - \mid y \mid)/(x-y), \text{ for } x, y \in \mathrm{R}, x \neq y,$$

if either $\min\{p,q,u,v\} < 0 < \max\{p,q,u,v\}$.

LEMMA 4. If k is a natural number. Then

$$(k+2)^{k(k+3)} \geqslant (k+1)^{(k+1)(k+2)}, \tag{15}$$

or

$$\frac{k}{(k+2)\ln(k+1)} \geqslant \frac{k+1}{(k+3)\ln(k+2)}. \tag{16}$$

PROOF. When $k = 1, 2$, we have $(1+2)^{1\cdot(1+3)} = 81 > 64 = (1+1)^{(1+1)(1+2)}$, and $(2+2)^{2\cdot(2+3)} = 1048576 > 531441 = (2+1)^{(2+1)(2+2)}$, respectively. i.e. (15) or (16) holds.

If $k \geqslant 3$, then we have

$$\frac{k^3}{6} \geqslant \frac{k^2}{2},\ \frac{k^4}{24} \geqslant k, \tag{17}$$

and

$$k(k+3)-i \geqslant k(k+1), 1 \leqslant i \leqslant 3. \tag{18}$$

Using the binomial theorem, we obtain

$$\begin{aligned}\left(1+\frac{1}{k+1}\right)^{k(k+3)} &= 1+\frac{k(k+3)}{k+1}+\frac{k(k+3)[k(k+3)-1]}{2(k+1)^2} \\ &+ \frac{k(k+3)[k(k+3)-1][k(k+3)-2]}{6(k+1)^3} \\ &+ \frac{k(k+3)[k(k+3)-1][k(k+3)-2][k(k+3)-3]}{24(k+1)^4}+\cdots \end{aligned} \tag{19}$$

From (17)-(19), we get

$$\begin{aligned}\left(1+\frac{1}{k+1}\right)^{k(k+3)} &> 1+k+\frac{k^2}{2}+\frac{k^3}{6}+\frac{k^4}{24} \\ &\geqslant 1+k+\frac{k^2}{2}+\frac{k^2}{2}+k = 1+2k+k^2 = (k+1)^2 \end{aligned} \tag{20}$$

Rearranging (20), we immediately find (15) or (16). The proof of Lemma 4 is completed.

LEMMA 5 ([8]). Suppose $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n > 0$, $\frac{a_1}{b_1} \geqslant \frac{a_2}{b_2} \geqslant \cdots \geqslant \frac{a_n}{b_n} > 0$. Then the function

$$F_r(a,b) = \begin{cases} \left[\sum_{i=1}^{n} a_i^r / \sum_{i=1}^{n} b_i^r\right]^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod_{i=1}^{n} \frac{a_i}{b_i}\right)^{1/n}, & r = 0, \end{cases} \tag{21}$$

is monotone increasing one with respect to r.

LEMMA 6. Suppose $x \geqslant 1$, and k is a fixed natural number. Then the functions

$$f_k(x) = \left(\sum_{i=0}^{k} x^{k-i}\right)^{\frac{1}{k}} / \left(\sum_{i=0}^{k+1} x^{k+1-i}\right)^{\frac{1}{k+1}} \tag{22}$$

and

$$g_k(x) = \left(\sum_{i=1}^{k} x^{k+1-i}\right)^{\frac{1}{k+1}} / \left(\sum_{i=1}^{k+1} x^{k+2-i}\right)^{\frac{1}{k+2}} \tag{23}$$

are monotone decreasing with respect to $x \in [1,+\infty)$.

PROOF. Calculating the derivative for $f_k(x)$ and $g_k(x)$ about x, respectively, we get

$$f_k'(x) = \left[\sum_{i=1}^{k} \frac{i(i+1)}{2}(x^{i-1}-x^{2k-i})\right] / \left[k(k+1)\left(\sum_{i=0}^{k} x^{k-i}\right)^{\frac{k-1}{k}} \left(\sum_{i=0}^{k+1} x^{k+1-i}\right)^{\frac{k+2}{k+1}}\right].$$

Since $x \geqslant 1$ and k is a fixed natural number, we find that $x^{i-1} - x^{2k-i} \leqslant 0$ for $1 \leqslant i \leqslant k$, or $f_k'(x) \leqslant 0$. And we similarly obtain $g_k'(x) \leqslant 0$. It is easy to see that the functions $f_k(x)$ and $g_k(x)$ are monotone decreasing with respect to $x \in [1, +\infty)$. The proof of Lemma 6 is completed.

3 Monotonicity and Logarithmic Convexity

From Lemma 2 and Lemma 1, we may easily prove the following Theorem 1 and Theorem 2 respectively.

THEOREM 1. If k is a fixed natural number, then $H_r(a,b;k)$ and $h_r(a,b;k)$ are monotone increasing with respect to a and to b for fixed real numbers r, or with respect to r for fixed positive numbers a and b; and are logarithmical concave on $(0, +\infty)$, and logarithmical convex on $(-\infty, 0)$ with respect to r.

THEOREM 2. Assume a and b are fixed positive numbers, and k is a fixed natural number. Then $[H_r(a,b;k)]^r$ and $[h_r(a,b;k)]^r$ are logarithmic convex functions with respect to $r > 0$.

THEOREM 3 ([3]). For any $r > 0$, $H_r(a,b;k)$ is monotonic decreasing and $h_r(a,b;k)$ is monotone increasing with respect to k.

THEOREM 4. For fixed positive numbers a and b, $H_{\frac{k}{k+2}}(a,b;k)$ is monotonic decreasing and $h_{\frac{k+1}{k-1}}(a,b;k)$ is monotone increasing with resepct to k.

PROOF. The proof of the monotonicity of $H_{\frac{k}{k+2}}(a,b;k)$ is equivalent to the inequality

$$\left[\frac{a^{\frac{k+1}{k+2}} - b^{\frac{k+1}{k+2}}}{(k+1)(a^{\frac{1}{k+2}} - b^{\frac{1}{k+2}})}\right]^{\frac{k+2}{k}} \geqslant \left[\frac{a^{\frac{k+2}{k+3}} - b^{\frac{k+2}{k+3}}}{(k+2)(a^{\frac{1}{k+3}} - b^{\frac{1}{k+3}})}\right]^{\frac{k+3}{k+1}}, \tag{24}$$

where k is a natural number. Setting $p_1 = \frac{k+1}{k+2}, q_1 = \frac{1}{k+2}, u_1 = \frac{k+2}{k+3}$, and $v_1 = \frac{1}{k+3}$, then (24) becomes

$$E_{p_1,q_1}(a,b) \geqslant E_{u_1,v_1}(a,b). \tag{25}$$

It is easy to see that $\min\{p_1, q_1, u_1, v_1\} = \frac{1}{k+3} > 0$, and $p_1 + q_1 = 1 = u_1 + v_1$. From Lemma 4, we find that

$$e(p_1, q_1) = \frac{k}{(k+2)\ln(k+1)} \geqslant \frac{k+1}{(k+3)\ln(k+2)} = e(u_1, v_1), \tag{26}$$

where $e(x,y)$ is defined in Lemma 3. Using Lemma 3, we can obtain (25), and it immediately follows that expression (24) is true.

We may similarly prove that $h_{\frac{k+1}{k-1}}(a,b;k)$ is a monotone increasing function with respect to k. The proof is complete.

THEOREM 5. If $b_1 \geqslant b_2 > 0$ and $a_1/b_1 \geqslant a_2/b_2 > 0$, then $H_r(a_1,a_2;k)/H_r(b_1,b_2;k)$ and $h_r(a_1,a_2;k)/h_r(b_1,b_2;k)$ are monotone increasing with respect to r in $\mathbf{R}$.

PROOF. According to Definition 1, we have

$$\frac{H_r(a_1,a_2;k)}{H_r(b_1,b_2;k)} = \begin{cases} \left[\sum_{i=0}^{k} a_1^{\frac{(k-i)r}{k}} a_2^{\frac{ir}{k}} \Big/ \sum_{i=0}^{k} b_1^{\frac{(k-i)r}{k}} b_2^{\frac{ir}{k}}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{\frac{a_1 a_2}{b_1 b_2}}, & r = 0. \end{cases} \tag{27}$$

and

$$\frac{h_r(a_1,a_2;k)}{h_r(b_1,b_2;k)} = \begin{cases} \left[\sum_{i=1}^{k} a_1^{\frac{(k+1-i)r}{k+1}} a_2^{\frac{ir}{k+1}} \Big/ \sum_{i=1}^{k} b_1^{\frac{(k+1-i)r}{k+1}} b_2^{\frac{ir}{k+1}}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{\frac{a_1 a_2}{b_1 b_2}}, & r = 0. \end{cases} \tag{28}$$

For $b_1 \geqslant b_2 > 0$ and $a_1/b_1 \geqslant a_2/b_2 > 0$, we find

$$b_1 \geqslant b_1^{\frac{k-1}{k}} b_2^{\frac{1}{k}} \geqslant b_1^{\frac{k-2}{k}} b_2^{\frac{2}{k}} \geqslant \cdots \geqslant b_2 > 0, \tag{29}$$

and

$$\frac{a_1}{b_1} \geqslant \left(\frac{a_1}{b_1}\right)^{\frac{k-1}{k}} \left(\frac{a_2}{b_2}\right)^{\frac{1}{k}} \geqslant \left(\frac{a_1}{b_1}\right)^{\frac{k-2}{k}} \left(\frac{a_2}{b_2}\right)^{\frac{2}{k}} \geqslant \cdots \geqslant \frac{a_2}{b_2} > 0. \tag{30}$$

From Lemma 5, combining (27)-(30), the proof follows.

THEOREM 6. If $0 < a \leqslant b \leqslant 1/2$, then $H_r(a,b;k)/H_r(1-a,1-b;k)$ and $h_r(a,b;k)/h_r(1-a,1-b;k)$ are monotone increasing in r.

Indeed, from $0 < a \leqslant b \leqslant \frac{1}{2}$, we get $0 < 1-a \leqslant 1-b$ and $0 < \frac{a}{1-a} \leqslant \frac{b}{1-b}$. Using Theorem 5, we obtain Theorem 6.

THEOREM 7. If $b_1 \geqslant b_2 > 0$ and $a_1/b_1 \geqslant a_2/b_2 > 0$, then $(D_k(a_1,a_2)/D_k(b_1,b_2))^{\frac{1}{k}}$ and $(d_k(a_1,a_2)/d_k(b_1,b_2))^{\frac{1}{k+1}}$ are monotone increasing with respect to k in $\mathbf{N}$.

PROOF. To prove $(D_k(a_1,a_2)/D_k(b_1,b_2))^{\frac{1}{k}}$ is monotone increasing with respect to k in $\mathbf{N}$, we only need to prove that: if $b_1 \geqslant b_2 > 0$, $a_1/b_1 \geqslant a_2/b_2 > 0$ and k is a natural number, then

$$\left(\sum_{i=0}^{k} a_1^{k-i} a_2^{i} \Big/ \sum_{i=0}^{k} b_1^{k-i} b_2^{i}\right)^{\frac{1}{k}} \leqslant \left(\sum_{i=0}^{k+1} a_1^{k+1-i} a_2^{i} \Big/ \sum_{i=0}^{k+1} b_1^{k+1-i} b_2^{i}\right)^{\frac{1}{k+1}}. \tag{31}$$

Taking $x_1 = \frac{a_1}{a_2}, x_2 = \frac{b_1}{b_2}$, we have $x_1 \geqslant x_2 \geqslant 1$, and inequality (31) is equivalent to

$$\left(\sum_{i=0}^{k} x_1^{k-i}\right)^{\frac{1}{k}} \Big/ \left(\sum_{i=0}^{k+1} x_1^{k+1-i}\right)^{\frac{1}{k+1}} \leqslant \left(\sum_{i=0}^{k} x_2^{k-i}\right)^{\frac{1}{k}} \Big/ \left(\sum_{i=0}^{k+1} x_2^{k+1-i}\right)^{\frac{1}{k+1}}. \tag{32}$$

From Lemma 6, we find (32) or (31). Thus, Theorem 7 is proved.

The monotonicity of $(D_k(a_1,a_2)/D_k(b_1,b_2))^{\frac{1}{k}}$ in the above Theorem was obtained by Wang et al. in 1988 (see [9]). By proof similar to that of Theorem 6, we may obtain

THEOREM 8. If $0 < a \leqslant b \leqslant \frac{1}{2}$, then $(D_k(a,b)/D_k(1-a,1-b))^{\frac{1}{k}}$ and $(h_k(a,b)/h_k(1-a,1-b))^{\frac{1}{k+1}}$ are monotone increasing with respect to r.

REMARK 3. Let $k \to +\infty$, from Proposition 1(d), we have

$$\lim_{k\to+\infty} h_r(a,b;k) = \lim_{k\to+\infty} H_r(a,b;k) = [L(a^r,b^r)]^{\frac{1}{r}}. \tag{33}$$

We may also obtain some similar results for $[L(a^r,b^r)]^{\frac{1}{r}}$:

(a) $[L(a^r,b^r)]^{\frac{1}{r}}$ is monotone increasing with respect to a and b for fixed real numbers r, or to r for fixed positive numbers a and b; and are logarithmical concave on $(0,+\infty)$ with respect to r; and logarithmical convex on $(-\infty,0)$ with respect to r;

(b) If a and b are fixed positive numbers, then $L(a^r,b^r)$ is a logarithmic convex function with respect to $r > 0$;

(c) If $b_1 \geqslant b_2 > 0$ and $a_1/b_1 \geqslant a_2/b_2 > 0$, then $[L(a_1^r,a_2^r)/L(b_1^r,b_2^r)]^{\frac{1}{r}}$ is monotone increasing with respect to r in $\mathbf{R}$;

(d) If $0 < a \leqslant b \leqslant \frac{1}{2}$, then $[L(a^r,b^r)/L((1-a)^r,(1-b)^r)]^{\frac{1}{r}}$ is monotone increasing with respect to $r \in \mathbf{R}$.

References

[1] J. C. Kuang, Applied Inequalities, Hunan Eduation Press, 2nd. Ed., 1993 (in Chinese).

[2] G. Jia and J. D. Cao, A new upper bound of the logarithmic mean, J. Ineq. Pure Appl. Math., 4(4)(2003), Article 80.

[3] Z. G. Xiao and Z. H. Zhang, The inequalities $G \leq L \leq I \leq A$ in n variables, J. Ineq. Pure Appl. Math., 4(2)(2003), Article 39.

[4] D. W. Detemple and J. M. Robertson, On generalized symmetric means of two varibles, Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.634–672(1979), 236–238.

[5] F. Qi, Logarithmic convexities of the extended mean values, RGMIA Resarch Report Collection 5(2)(1999), Article5.

[6] E. B. Lenach and M. Sholander, Extended mean values, Amer. Math. Monthly, 85(1978), 84–90.

[7] Zs. Páles, Inequalities for differences of powers, J. Math. Anal. Appl., 131(1988), 271–281.

[8] A. W. Marsall, I. Olkin and F. Proschan, Monotonicty of ratios of means and other applications of majorization, in Inequalities, edited by O. Shisha. New York London 1967, 177–190.

[9] W. L. Wang, G. X. Li and J. Chen, Inequalities involving ratios of means, J. Chendu University of Science and Technology, 42(6) (1988), 83–88 (in Chinese)