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PERIODIC WAVE SOLUTIONS FOR THE GENERALIZED SHALLOW WATER WAVE EQUATION BY THE IMPROVED JACOBI ELLIPTIC FUNCTION METHOD *

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Abstract

In this paper, improved Jacobi elliptic function method is used to construct new exact doubly periodic wave solutions of the generalized shallow water wave equation (GSWW). The method can also be applied to other nonlinear partial differential equations (PDEs) or systems in mathematical physics.

1 Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many workers who are interested in nonlinear physical phenomena. Many powerful methods have been presented, such as the homogeneous balance method [1], the hyperbolic tangent expansion method [2], the trial function method [3], the tanh method [4], the nonlinear transformation method [5], inverse scattering transformation [6], Bäcklund transformation [7], Hirota's bilinear method [8], the generalized Riccati equation method [9], the Weierstrass elliptic function method [10], the theta function method [11], sine-cosine method [12] and the Jacobi elliptic function expansion method [13, 14] and so on.

In this paper, we will use the improved Jacobi elliptic function method to construct new doubly Jacobi periodic wave solutions of the generalized shallow water wave equation

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - \gamma u_{xx} = 0, \tag{1}$$

where subscripts indicate partial derivatives, u is a real scalar function of the two independent variables x and t, while α, β and γ are all model parameters and they are arbitrary, nonzero constants. This equation can be derived from the classical shallow water theory in the so-called Boussinesq approximation [15]. Two special cases of (1) have been discussed in the literature, $\alpha = \beta$ and $\alpha = 2\beta$ [15]. Hietarinta [16] discussed the GSWW equation and he showed that it can be expressed in Hirota' bilinear form [17]

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if and only if either $\alpha = \beta$ or $\alpha = 2\beta$. Gao and Tian [18] obtained soliton like solutions of (1) for $\alpha = \beta = -3$ and $\gamma = 0$ by the generalized tanh method. Yan and Zhang [19] obtained new families of soliton like solutions of (1) for $\alpha = -4$, $\beta = -2$ and $\gamma = 0$, which is named $(2+1)$-dimensional breaking soliton equation, by using computerized symbolic computation. Recently, Elwakil et al. [20] presented solitary wave solutions of (1) by using homogeneous balance method and auto-Bäcklund transformation and they also applied modified extented tanh-function method for obtaining new exact travelling wave solutions.

2 Improved Jacobi Elliptic Function Method

For the partial differential equation

$$H\left(u, u_x, u_t, u_{xx}, \dots\right) = 0, \tag{2}$$

we seek the formal travelling wave solutions of the form

$$u\left(x,t\right) = u(\xi),\ \xi = x - \lambda t. \tag{3}$$

Such a solution has important physical significance, where λ is constant to be determined later. The improved Jacobi elliptic function method is defined by Chen and Zhang [21] in the following form. The main idea of this method is to take full advantage of the equation that Jacobi elliptic functions satisfy and use its solutions F which are snξ, cnξ, dnξ, csξ and tnξ. Let

$$(F')^2 = \left(1 + \varepsilon F^2\right)\left(aF^2 + b\right), \tag{4}$$

where $' = d/d\xi$, and a, b, ε are constants. The solutions of (1) can be expressed in the form

$$u\left(x,t\right) = u(\xi) = \sum_{i=0}^{n} a_i F^i, \tag{5}$$

where $a_0, \dots, a_n$ are parameters to be determined.

Case I: $\varepsilon = -1$.

A. If $a = -m^2$ and $b = 1$, then (4) becomes

$$(F')^2 = \left(1 - F^2\right)\left(1 - m^2F^2\right), \tag{6}$$

which has solution snξ.

B. If $a = m^2$ and $b = 1 - m^2$, then (4) becomes

$$(F')^2 = \left(1 - F^2\right)\left(m^2F^2 + 1 - m^2\right), \tag{7}$$

which has solution cnξ.

C. If $a = 1$ and $b = m^2 - 1$, then (4) becomes

$$(F')^2 = \left(1 + F^2\right)\left(F^2 + m^2 - 1\right), \tag{8}$$

which has solution dnξ.

D. If $a=0$ and $b=1$, then (4) becomes

$$(F')^2=\left(1+\varepsilon F^2\right), \tag{9}$$

which has solution $\sin\xi$ and $\cos\xi$.

Case II: $\varepsilon=1$.

A. If $a=1$ and $b=1-m^2$, then (4) becomes

$$(F')^2=\left(1+F^2\right)\left(F^2+1-m^2\right), \tag{10}$$

which has solution $cs\xi$.

B. If $a=1-m^2$ and $b=1$, then (4) becomes

$$(F^{'})^2=\left(1+F^2\right)\left(1+F^2-m^2F^2\right), \tag{11}$$

which has solution tnξ.

The functions snξ, cnξ, dnξ, csξ and tnξ are Jacobi elliptic functions, which are double periodic and possess the following properties:

$$\begin{aligned} \mathrm{cn}^2\xi &= 1-\sin^2\xi,\ \mathrm{sn}^2\xi+\mathrm{cn}^2\xi=1, \\ \mathrm{dn}^2\xi &= 1-m^2\mathrm{sn}^2\xi,\ \frac{d}{d\xi}\mathrm{sn}\xi=\mathrm{cn}\xi\mathrm{dn}\xi, \\ \frac{d}{d\xi}\mathrm{sn}\xi &= \mathrm{cn}\xi\mathrm{dn}\xi,\ \frac{d}{d\xi}\mathrm{cn}\xi=-\mathrm{sn}\xi\mathrm{dn}\xi,\ \frac{d}{d\xi}\mathrm{dn}\xi=-m^2\mathrm{sn}\xi\mathrm{cn}\xi, \end{aligned} \tag{12}$$

where m is the modulus $0<m<1$. When $m\to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions, i.e.,

$$\mathrm{sn}\xi\to\tanh\xi\ ,\ \mathrm{cn}\xi\to\mathrm{sech}\xi\ ,\ \mathrm{dn}\xi\to\mathrm{sech}\xi. \tag{13}$$

When $m\to 0$, the Jacobi functions degenerate to the triangular functions, i.e.,

$$\mathrm{sn}\xi\to\sin\xi\ ,\ \mathrm{cn}\xi\to\cos\xi\ ,\ \mathrm{dn}\xi\to 1. \tag{14}$$

The highest power order of $u(\xi)$ is equal to n,

$$O(u(\xi))=n, \tag{15}$$

and the highest power order of $du/d\xi$ can be taken as

$$O\left(\frac{du}{d\xi}\right)=n+1. \tag{16}$$

We have

$$O\left(\frac{d^pu}{d\xi^p}\right)=n+p,\ p=1,2,3,\dots, \tag{17}$$

and

$$O\left(u^q \frac{d^p u}{d\xi^p}\right) = (q+1)\,n + p,\ \ q = 0, 1, 2, \ldots \tag{18}$$

Substituting (3) into (1), we have

$$\lambda\left(u'''' + \alpha u' u'' + \beta u' u'' - u''\right) + \gamma u'' = 0. \tag{19}$$

Integrating (19) once and setting integration constant to zero, we have

$$\lambda\left(u''' + \frac{1}{2}\alpha (u')^2 + \frac{1}{2}\beta\left(u'\right)^2 - u'\right) + \gamma u' = 0. \tag{20}$$

Balancing u''' with $(u')^2$ yields $n = 2$. Therefore, we use

$$u = a_0 + a_1 F + a_2 F^2. \tag{21}$$

Substituting (20) into (19) and making use of (4), with the help of Mathematica, we get a system of algebraic equations for a_0, a_1 and $\lambda:$

$$\begin{aligned}
\frac{1}{2}\lambda b a_1^2\left(\alpha+\beta\right) + \lambda a_1\left(a + \varepsilon b - 1\right) &= 0,\\
4\lambda\left(a+\varepsilon b\right) a_2 + 4 b a_1 a_2 - \lambda a_2 &= 0,\\
6\lambda\varepsilon a a_1 + \frac{1}{2}\lambda\alpha a a_1^2 + \frac{1}{2}\lambda\alpha\varepsilon b + 8 b a_2^2 + \frac{1}{2}\lambda\beta a a_1^2 + \frac{1}{2}\lambda\beta b\varepsilon a_1^2 &= 0,\\
3\lambda\varepsilon a a_2 + 3\lambda\varepsilon a a_2^2 + 2 a a_1 a_2 + 2 b\varepsilon a_1 a_2 &= 0,\\
\frac{1}{2}\lambda\alpha a\varepsilon + \frac{1}{2}\lambda\beta a\varepsilon a_1^2 + 8 a a_2^2 + 8 b\varepsilon a_2^2 &= 0,\\
8 a\varepsilon^2 a_1 a_2 &= 0,\\
8 a\varepsilon a_2^2 &= 0.
\end{aligned}$$

From which, we obtain:

Case I. $a_2 = 0,\ a_1 = \frac{\lambda}{b}\left[\frac{1}{4} - (a+\varepsilon b)\right],\ \lambda = -\frac{8(a+\varepsilon b-1)}{1-4(a+\varepsilon b)},$

Case II. $a_2 = 0,\ a_1 = \mp\sqrt{\frac{\alpha}{\beta}}$, where $\lambda = -\frac{E}{\frac{\alpha b}{2}\left(\alpha+\frac{1}{\beta}\right)+\sqrt{\frac{\alpha}{\beta}}\left[(a+\varepsilon b)-1\right]},$

Case III. $a_2 = 0,\ a_1 = -\frac{3\lambda\varepsilon a}{2(a+\varepsilon b)}$, if $E = 0$, then $\lambda = \frac{4(a+\varepsilon b)(a+\varepsilon b-1)}{3\varepsilon a b(\alpha+\beta)},$

Case IV. $a_2 = 0,\ a_1 = \frac{-6\varepsilon a \mp\sqrt{36\varepsilon^2 a^2 - \alpha\varepsilon\beta a b - \beta\varepsilon^2 b^2}}{\beta(a+\varepsilon b)}.$

If $\varepsilon = -1, a = -m^2, b = 1$, then we obtain the exact periodic solution of (1)

$$u_1 = \frac{2m^2}{(\alpha+\beta)}\mathrm{sn}\left(x - \left(\frac{4\left(1+m^2\right)}{3\left(\alpha+\beta\right)}\right)t\right). \tag{22}$$

If $m = 1$, then (22) degenerates and become

$$u_1' = \frac{2}{(\alpha+\beta)}\tanh\left(x - \left(\frac{8}{3\left(\alpha+\beta\right)}\right)t\right). \tag{23}$$

If $\varepsilon = -1, a = m^2, b = 1 - m^2$, then we obtain the following wave solution of (11)

$$u_2 = \frac{2}{(\alpha+\beta)} \mathrm{cn}\left(x - \left(\frac{4\left(2m^2-1\right)}{3m^2\left(\alpha+\beta\right)}\right)t\right). \tag{24}$$

If $m = 1$, then (24) becomes the solitary wave solution of (1)

$$u_2' = \frac{2}{(\alpha+\beta)} \mathrm{sech}\left(x - \left(\frac{4}{3\left(\alpha+\beta\right)}\right)t\right). \tag{25}$$

If $\varepsilon = -1, a = 1, b = m^2 - 1$, then we obtain the other exact periodic solution of (1)

$$u_3 = \frac{2}{(\alpha+\beta)} \mathrm{dn}\left(x - \left(\frac{4\left(2-m^2\right)}{3m^2\left(\alpha+\beta\right)}\right)t\right). \tag{26}$$

If $\varepsilon = 1, a = 1, b = 1 - m^2$, then we obtain another exact solution of (1)

$$u_4 = -\frac{2}{(\alpha+\beta)} \mathrm{cs}\left(x - \left(\frac{4\left(2-m^2\right)}{3\left(\alpha+\beta\right)}\right)t\right). \tag{27}$$

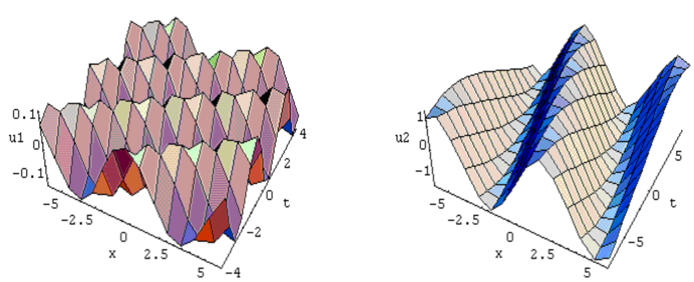
If $m = 1$, then (27) become the other solitary wave solution of (1)

$$u_4' = -\frac{2}{(\alpha+\beta)} \mathrm{csch}\left(x - \left(\frac{4}{3\left(\alpha+\beta\right)}\right)t\right). \tag{27}$$

If $\varepsilon = 1, a = 1 - m^2, b = 1$, then we obtain another exact periodic solution of (1)

$$u_5 = -\frac{2\left(1-m^2\right)}{(\alpha+\beta)} \mathrm{tn}\left(x - \left(\frac{4\left(2-m^2\right)}{3\left(\alpha+\beta\right)}\right)t\right). \tag{28}$$

Remark. If we consider Case I, II and IV, then other new periodic wave solutions can be obtained for equation (1). For simplicity, we omit them here.



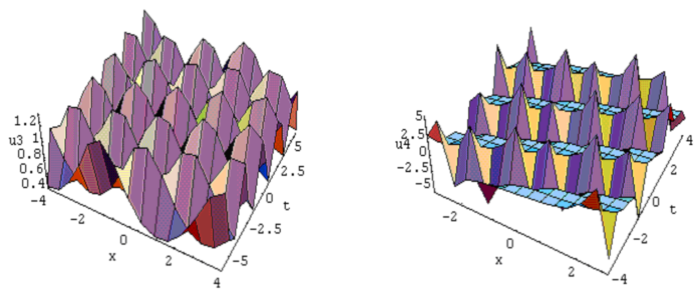


Figure 1. The surfaces show the double periodic wave solutions of (1), where $\alpha = 1$ and $\beta = 0.5$.

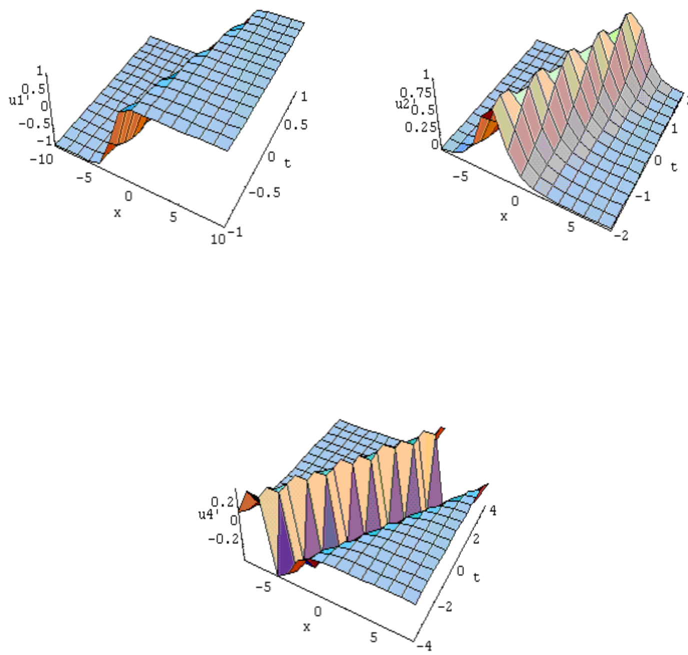


Figure 2. The surfaces show the solitary wave solutions of (1), $u_1^{'}$, $u_2^{'}$ and $u_4^{'}$ respectively, where $\alpha = \beta = 1$.

3 Conclusions

In this paper, the Improved Jacobi elliptic function expansion method is applied to the generalized shallow water wave equation. The aim to obtain Jacobi doubly periodic-wave and solitary wave solutions of the GSWW equation by using this method has been achieved. Some of the behaviors of the solutions can be inferred from Figures 1 and 2. The physical relevance of soliton solutions and periodic solutions is clear to us. We can also see that some solutions obtained in this paper develop a singularity at a finite point, i.e. for any fixed $t = t_0$, there exist x_0 at which these solutions blow up. There is much current interest in the formation of so-called "hot spots" or "blow up" of solutions [22,23]. It appears that these singular solutions will model these physical phenomena. It is clear that all the obtained solutions are valid for all values of α and β except for $\alpha \neq \beta$.

The present method provides a reliable technique that requires less work if compared with the Jacobi elliptic function method. The method is relatively easy when applied to nonlinear differential equations and differential systems. The method avoids the difficulties and massive computational work that usually arise from inverse scattering method, the homogeneous balance method, the hyperbolic tangent expansion method and the Jacobi elliptic function method.

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