Numerical Solutions Of Ordinary Differential Equations With Quadratic Trigonometric Splines*

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Received 16 January 2004

Abstract

In this article we present a numerical method for solving ODE with quadratic trigonometric splines. This is a modification of a method, due to F.R. Loscalzo and T. Talbot, in which the approximations were made using only polynomial splines. If the solution is trigonometric or periodical, then trigonometric splines in general give better results. In the following analysis we are going to prove that the convergence of this numerical method is quadratic. This theoretical result agrees with the numerical experiments.

1 Introduction

In [7] (see also [8]) a procedure for obtaining spline approximations for solutions of initial value problems was presented, using quadratic and cubic polynomial splines. It was also proved in [7] that for splines of degree \( \geq 4 \) the method is divergent. In this article we apply the procedure of [7] to obtain numerical solutions of the initial value problem in ordinary differential equations using quadratic trigonometric splines. We use the expression of the splines as a linear combination of the trigonometric Basic splines, which provides an easier computation of the spline approximations. In section 3 the extended Taylor formula is applied in order to prove that the convergence of the method is quadratic. This extended Taylor formula is a generalization for differential operators of the classical Taylor formula.

1.1 Quadratic Trigonometric Splines

The trigonometric splines are a special case of L-splines (see [11]). Consider the differential operator \( L_3y := y'' + y' \) with the following fundamental system: \( N_L = \text{span}\{1, \sin(x), \cos(x)\} \).

The associated Green function to the operator \( L_3 \) is well known to be

\[
G(x; \xi) = \begin{cases} 
0 & \text{for } x \leq \xi, \\
2\sin^2((x - \xi)/2) & \text{for } x > \xi.
\end{cases}
\]

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*Mathematics Subject Classifications: 65D07, 65L05, 65L20.
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The extended Taylor formula for the operator $L_3$ and $y \in C^3[a, b]$ is: (see [11], p. 425)

$$y(x) = u_y(x) + \int_a^x 2 \sin^2\left(\frac{x - \xi}{2}\right)L_3y(\xi) \, d\xi, \quad x \in [a, b], \quad (1)$$

with $u_y(x) = y(a) + y''(a) + y'(a)\sin(x - a) - y''(a)\cos(x - a)$. The function $u_y \in N_L$ satisfies the following conditions: $u_y(a) = y(a)$, $u_y'(a) = y'(a)$ and $u_y''(a) = y''(a)$. This extended Taylor formula is very important for the error analysis in this article. Error estimates obtained by using it, shows that if the considered ODE has solution that belongs in $N_L$, then the approximation is exact.

Consider the interval $[a, b]$ and the set of equally spaced knots $\Omega_n = \{x_0, \ldots, x_n\}$, with $x_0 = a$, $x_n = b$, $x_{i+1} - x_i = h$ for $i = 0, \ldots, n - 1$ and $h = (b - a)/n$. In the case of quadratic trigonometric splines, there is the restriction that $3h < 2\pi$. This is an easy to satisfy restriction (by increasing $n$ if needed).

The quadratic trigonometric splines are functions $s(x) \in C^2[a, b]$ so that the restriction of them in every subinterval $[x_i, x_{i+1}]$ is a linear combination of functions in $N_L$. A convenient way to express the quadratic trigonometric splines is to use the Trigonometric Basic splines, denoted in the following with TB-splines. For the computation of the TB-splines there is a recursion formula, as in the polynomial case (see [6]).

However it is easier to use the exact formula given here:

$$TB^2_i(x) := \theta \cdot \begin{cases} 
\sin^2\left(\frac{x - x_i}{2}\right) & \text{for } x \in [x_i, x_{i+1}], \\
\sin\left(\frac{x - x_i}{2}\right)\sin\left(\frac{x_{i+2}-x}{2}\right) + \\
\sin\left(\frac{x_{i+3}-x}{2}\right) \sin\left(\frac{x - x_{i+1}}{2}\right) & \text{for } x \in [x_{i+1}, x_{i+2}], \\
\sin^2\left(\frac{x_{i+3}-x}{2}\right) & \text{for } x \in [x_{i+2}, x_{i+3}], \\
0 & \text{elsewhere},
\end{cases}$$

where $\theta = \frac{1}{\sin(\pi/3)\sin(\pi/2)}$.

If we denote by $TS_2(\Omega_n)$ the space of the quadratic trigonometric splines in $[a, b]$ with respect to the knots $\Omega_n$, then $TS_2(\Omega_n) = \text{span}\{TB^2_i\}_{i=-2}^{n-1}$. Subsequently every spline can be expressed in the form:

$$s(x) = \sum_{i=-2}^{n-1} \alpha_i TB^2_i(x)$$

where $\alpha = (\alpha_i)_{i=-2}^{n-1}$ the vector of the coefficients to be computed.

### 1.2 A Brief Description of the Method

Consider the differential equation

$$y' = f(x, y), \quad a \leq x \leq b \quad (2)$$

with the initial value $y(a) = y_0$. Assume also $f \in C^2$ in $T$ where $T = \{(x, y)/x \in [a, b]\}$ and $f$ satisfies the Lipschitz condition:

$$\exists L > 0 \quad \forall x \in [a, b] \quad \forall y_1, y_2 \in \mathbb{R} \quad |f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|. \quad (3)$$
This is satisfied if the partial derivatives of \( f \) are bounded in \( T \).

We construct the numerical solution \( s(x) \) of (2) requiring the following: \( s(a) = y_a \), \( s'(a) = f(a, s(a)) = f(a, y_a) \) and that \( s(x) \) satisfies the differential equation (2) at all the knots \( a + kh \), for \( k = 1, 2, \ldots, n \). See also [7] and [8].

## Computation of the Spline Approximations

The condition \( s(a) = y_a \) and \( s'(a) = f(a, s(a)) = f(a, y_a) \) enable us to compute the first two coefficients of the quadratic trigonometric spline approximation \( s(x) \). Thus

\[
\alpha_{-2} + \alpha_{-1} = 2 \cos\left(\frac{h}{2}\right)y_a \quad \text{and} \quad -\alpha_{-2} + \alpha_{-1} = 2 \sin\left(\frac{h}{2}\right)f(a, y_a).
\]

The above system gives

\[
\alpha_{-1} = \cos\left(\frac{h}{2}\right)y_a + \sin\left(\frac{h}{2}\right)f(a, y_a) \quad \text{and} \quad \alpha_{-2} = \cos\left(\frac{h}{2}\right)y_a - \sin\left(\frac{h}{2}\right)f(a, y_a).
\]

To determine the next coefficient \( \alpha_0 \) we require that \( s'(x_1) = f(x_1, s(x_1)) \) where \( x_1 = a + h \). Thus we obtain the equation

\[
-\frac{1}{2 \sin(h/2)}\alpha_{-1} + \frac{1}{2 \sin(h/2)}\alpha_0 = f \left( a + h, \frac{1}{2 \cos(h/2)}\alpha_{-1} + \frac{1}{2 \cos(h/2)}\alpha_0 \right)
\]

which is a nonlinear equation with respect to \( \alpha_0 \) that can be solved with an iteration method, for example with Newton method. According to the previous construction and having determined the coefficient \( \alpha_k \), we can compute \( \alpha_{k+1} \) from

\[
-\frac{1}{2 \sin(h/2)}\alpha_k + \frac{1}{2 \sin(h/2)}\alpha_{k+1} = f \left( a + (k + 1)h, \frac{\alpha_k + \alpha_{k+1}}{2 \cos(h/2)} \right), \tag{4}
\]

setting \( x := \alpha_{k+1} \) we derive the following:

\[
\varphi(x) := 2 \sin(h/2)f \left( a + (k + 2)h, \frac{\alpha_k + x}{2 \cos(h/2)} \right) + \alpha_k = x \tag{5}
\]

or \( \varphi(x) = x \). We are going to prove that the function \( \varphi(x) \) has a unique fixed point, i.e. the equation (4) has a unique solution. Since \( \varphi(x) \) satisfies condition (3), it follows

\[
|\varphi(x_1) - \varphi(x_2)| = 2 \sin(h/2) \left| f \left( a + (k + 2)h, \frac{\alpha_k + x_1}{2 \cos(h/2)} \right) - f \left( a + (k + 2)h, \frac{\alpha_k + x_2}{2 \cos(h/2)} \right) \right|
\]

\[
\leq \frac{\sin(h/2)}{\cos(h/2)}L |x_1 - x_2|.
\]

From which it follows \( |\varphi(x_1) - \varphi(x_2)| \leq Lh |x_1 - x_2| \), since \( 3h < 2\pi \) has as consequences that \( \sin(h/2) < h/2 \) and \( \cos(h/2) > 1/2 \). So if \( h < \frac{1}{T} \) then

\[
|\varphi(x_1) - \varphi(x_2)| < |x_1 - x_2|
\]
i.e. that $\varphi(x)$ is a strong contraction mapping for an adequate $h$. This completes the proof of the following.

**PROPOSITION 1.** If $h < 1/L$, then there is a unique quadratic trigonometric spline approximation of the solution of the initial value problem.

### 3 Error Estimates for the Spline Approximations

Let $s_i := s(x_i)$ and $y_i := y(x_i)$ for $i = 0,1,\ldots,n$, where $y(x)$ is the solution of the initial value problem (2) and $s(x)$ is the quadratic trigonometric spline approximation.

Let also $y'_i := y'(x_i)$, $y''_i := y''(x_i)$, $s'_i := s'(x_i)$, $s''_i := s''(x_i) + 0$ where $s_i(x) = s(x)|_{x_i,x_{i+1}}$. From the definition of the splines it follows that $s_i(x) \in N_L$. This means that $L_3s_i(x) = 0$. So

\[
\begin{align*}
  s_0 &= y_0 \\
  s'_0 &= y'_0 \\
  s'_1 &= f(x_1,s_1).
\end{align*}
\]

The extended Taylor formula gives for $s_i(x)$:

\[
s_i(x) = s_i + s''_i \cdot \sin(x-x_i) - s'_i \cdot \cos(x-x_i)
\]

which yields

\[
s_1 = s_0(x_1) = s_0 + s'_0 \cdot \sin(h) - s''_0 \cdot \cos(h).
\]

For the solution of (2), we have

\[
y_1 = y(x_1) = y_0 + y'_0 \cdot \sin(h) - y''_0 \cdot \cos(h) + K
\]

where $K = \int_{x_0}^{x_1} 2 \sin^2\left(\frac{x-\xi}{2}\right)L_3y(\xi) d\xi$. Subtracting the two last equations we obtain

\[
e_1 := s_1 - y_1 = (s_0 - y_0) + (s''_0 - y''_0) + (s'_0 - y'_0) \cdot \sin(h) - (s''_0 - y''_0) \cdot \cos(h) - K
\]

or

\[
e_1 = (s''_0 - y''_0) - (s'_0 - y'_0) \cdot \cos(h) - K
\]

which is equivalent to

\[
e_1 = 2(s''_0 - y''_0) \cdot \sin^2\left(\frac{h}{2}\right) - K, \tag{6}
\]

where $e_1$ is the discrete error at the point $x_i$. Now for $e_1$ we must compute $s''_0 - y''_0$.

Differentiating the extended Taylor formula (1) we obtain

\[
y'(x) = u'_i(x) + \int_a^x \sin(x-\xi)L_3y(\xi) d\xi, \ x \in [a,b], \tag{7}
\]

with $u'_y(x) = y'(a) \cos(x-a) + y''(a) \sin(x-a)$. Thus

\[
\begin{align*}
  s'_1 &= s'_0(x_1) = s'_0 \cos(h) + s''_0 \sin(h), \\
  y'_1 &= y'_0 \cos(h) + y''_0 \sin(h) + \int_{x_0}^{x_1} \sin(x_1-\xi)L_3y(\xi) d\xi.
\end{align*}
\]
Subtracting these equations we obtain

\[ s''_0 - y''_0 = \frac{s'_1 - y'_1}{\sin(h)} + \int_{x_0}^{x_1} \frac{\sin(x_1 - \xi)}{\sin(h)} L_3 y(\xi) d\xi. \]

Substituting in (6) we have

\[ e_1 = e'_1 \frac{\sin(h/2)}{\cos(h/2)} + \int_{x_0}^{x_1} A(\xi) L_3 y(\xi) d\xi \]

where

\[ A(\xi) = \frac{\sin(h/2) \sin(x_1 - \xi)}{\cos(h/2)} - 2 \sin^2 \left( \frac{x_1 - \xi}{2} \right) \]

\[ = \frac{\cos(x_1 - \xi - h/2)}{\cos(h/2)} - 1 \geq 0 \]

and applying the mean value theorem for integrals, the equation (8) becomes

\[ e_1 = e'_1 \tan(h/2) + L_3 y(\gamma) \left[ - \frac{\sin(x_1 - \xi - h/2)}{\cos(h/2)} - \xi \right]_{x_0}^{x_1} \]

\[ e_1 = e'_1 \tan(h/2) + L_3 y(\gamma) [2 \tan(h/2) - h]. \]

where \( \gamma \in [x_0, x_1] \) and \( e'_i := s'_i - y'_i \) for \( i = 1, 2, \ldots, n \). Thus

\[ |e_1| \leq |e'_1| O(h) + |L_3 y(\gamma)| O(h^3) \]

using the Lipschitz condition (3):

\[ |e'_1| = |f(x_1, s_1) - f(x_1, y_1)| \leq L |s_1 - y_1| = L |e_1| \]

we obtain

\[ |e_1|(1 - L \cdot O(h)) \leq |L_3 y|_{\infty, [a, b]} O(h^3) \]

which means \( |e_1| = O(h^3) \). Using the same analysis it can be derived that

\[ e_k = e_{k-1} + e'_{k-1} \sin(h) + e'_k \tan(h/2) + L_3 y(\xi) O(h^3) \]

for some \( \xi \in [x_k, x_{k+1}] \), so the conclusion is:

\[ |e_k|(1 - L \cdot O(h)) \leq |e_{k-1}||1 + L \cdot O(h)| + |L_3 y|_{\infty, [a, b]} O(h^3) \]

or \[ |e_k| \leq |e_{k-1}| + |e_{k-1}| O(h) + O(h^3) \] for \( k = 1, \ldots, n \).

Which means that at each next knot, the error of the previous knot is added. So the error at the last knot is \( |e_n| = n O(h^3) = O(h^2) \).

**Lemma 1.** The discrete error at the knots of the approximation with trigonometric quadratic splines is \( |e_k| = O(h^2) \) for \( k = 0, \ldots, n \).

We are going to prove also that, the convergence order of the global error of the spline approximations is quadratic.
In $[x_i, x_{i+1}]$ applying the Taylor formula (1) it holds

\[ s_i(x) = s_i + s_i'' + s_i' \cdot \sin(x - x_i) - s_i'' \cdot \cos(x - x_i), \]
\[ y(x) = y_i + y_i'' + y_i' \cdot \sin(x - x_i) - y_i'' \cdot \cos(x - x_i) + \Delta, \]

where $\Delta = \int_{x_i}^{x_{i+1}} 2\sin^2\left(\frac{x - \xi}{2}\right) L_3(y(\xi)) d\xi$. Subtracting the above equations and applying the mean value theorem for integrals we obtain:

\[ s_i(x) - y(x) = (s_i - y_i) + (s_i'' - y_i'') (1 - \cos(x - x_i)) + (s_i' - y_i') \cdot \sin(x - x_i) - \Delta, \]

with $\Delta = L_3(y(\gamma)((x - x_i) - \sin(x - x_i)))$ for some $\gamma \in [x_i, x]$. Now the estimation for $e''_i = s_i'' - y_i''$: From (7) we obtain

\[ e''_{i+1} = e_i' \cos(h) + e_i'' \sin(h) - \int_{x_i}^{x_{i+1}} \sin(x - x_{i+1} - \xi) L_3(y(\xi)) d\xi, \]

or $e''_i \sin(h) = e'_{i+1} - e_i' \cos(h) + L_3(y(\xi)(1 - \cos(h))$ for any $\xi \in [x_i, x_{i+1}]$.

This means: $|e''_i| = O(h)$. Thus:

\[ |e_i| = |s_i - y_i| = O(h^2), \]
\[ |s_i' - y_i'| = L|e_i| = O(h^2), \]
\[ |s_i'' - y_i''| = O(h), \]
and $|\Delta| = O(h^3)$.

**Theorem 1.** The convergence of the spline approximations of the solution of the initial value problem (2) is quadratic, i.e. that $\|s(x) - y(x)\|_{[a,b]} = O(h^2)$.

**Remark:** The analysis that was used here to obtain error estimates for the trigonometric splines approximations is in principal the same as the analysis used in [9] to obtain error estimates for the interpolation with trigonometric splines. Particularly in [9] error estimates were obtained for the interpolation with linear, quadratic and cubic trigonometric splines. For the cubic case a super convergence theorem was also proved in [9], which can also be found with some other parts of [9] in [4]. More about splines and polynomial spline interpolation can be found for example in [1, 2, 3, 5, 10, 11, 12].

### 4 Numerical results

We are going to test numerically in $[0, 1]$ the following three examples.

**Example 1:** $y' = 1 + y^2$, with $y(0) = 0$ and exact solution $y(x) = \tan(x)$.

**Example 2:** $y' = xy - 2/3$, with $y(0) = 1$ and exact solution $y(x) = (5/3x^2 + 1)^{3/5}$.

**Example 3:** $y' = \sqrt{1 - y^2}$, with $y(0) = 0$ and exact solution $y(x) = \sin(x)$. 
Quadratic Trigonometric Splines

<table>
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<th>Example</th>
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<th>max. absolute error</th>
<th>num. convergence order</th>
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<tr>
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</tr>
</tbody>
</table>

where the numerical convergence order in the tables above is computed using:

\[ \ln \left( \frac{|e_k|}{|e_m|} \right) / \ln \left( \frac{m}{k} \right). \]

This numerical convergence order confirms the theoretical error estimates, proved in previous paragraph in Theorem 1. The numerical solution of example 3 is exact, as expected from the error analysis; since the solution \( y(x) = \sin(x) \) belongs in \( N_L \).

References


