Applied Mathematics E-Notes, 4(2004), 142-149 ©　　ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/~amen/

Numerical Solutions Of Ordinary Differential Equations With Quadratic Trigonometric Splines*

Athanassios Nikolis †

Received 16 January 2004

Abstract

In this article we present a numerical method for solving ODE with quadratic trigonometric splines. This is a modification of a method, due to F.R. Loscalzo and T. Talbot, in which the approximations were made using only polynomial splines. If the solution is trigonometric or periodical, then trigonometric splines in general give better results. In the following analysis we are going to prove that the convergence of this numerical method is quadratic. This theoretical result agrees with the numerical experiments.

1 Introduction

In [7] (see also [8]) a procedure for obtaining spline approximations for solutions of initial value problems was presented, using quadratic and cubic polynomial splines. It was also proved in [7] that for splines of degree ≥ 4 the method is divergent. In this article we apply the procedure of [7] to obtain numerical solutions of the initial value problem in ordinary differential equations using *quadratic trigonometric splines*. We use the expression of the splines as a linear combination of the trigonometric Basic splines, which provides an easier computation of the spline approximations. In section 3 the extended Taylor formula is applied in order to prove that the convergence of the method is quadratic. This extended Taylor formula is a generalization for differential operators of the classical Taylor formula.

1.1 Quadratic Trigonometric Splines

The trigonometric splines are a special case of L-splines (see [11]). Consider the differential operator $L_3 y := y''' + y'$ with the following fundamental system: $N_L = \text{span}\{1, \sin(x), \cos(x)\}$.

The associated Green function to the operator L_3 is well known to be

$$G(x;\xi) = \begin{cases} 0 & \text{for } x \leq \xi, \\ 2\sin^2((x-\xi)/2) & \text{for } x > \xi. \end{cases}$$

*Mathematics Subject Classifications: 65D07, 65L05, 65L20.

†Argyrokastrou 17 / 85300 Kos / Greece.

The extended Taylor formula for the operator L_3 and $y \in C^3[a,b]$ is: (see [11], p. 425)

$$y(x) = u_y(x) + \int_a^x 2\sin^2(\frac{x-\xi}{2}) L_3 y(\xi)\, d\xi, \quad x \in [a,b], \tag{1}$$

with $u_y(x) = y(a) + y''(a) + y'(a)\sin(x-a) - y''(a)\cos(x-a)$. The function $u_y \in N_L$ satisfies the following conditions: $u_y(a) = y(a)$, $u'_y(a) = y'(a)$ and $u''_y(a) = y''(a)$. This extended Taylor formula is very important for the error analysis in this article. Error estimates obtained by using it, shows that if the considered ODE has solution that belongs in N_L, then the approximation is exact.

Consider the interval $[a,b]$ and the set of equally spaced knots $\Omega_n = \{x_0, \ldots, x_n\}$, with $x_0 = a$, $x_n = b$, $x_{i+1} - x_i = h$ for $i = 0, \ldots, n-1$ and $h = (b-a)/n$. In the case of quadratic trigonometric splines, there is the restriction that $3h < 2\pi$. This is an easy to satisfy restriction (by increasing n if needed).

The quadratic trigonometric splines are functions $s(x) \in C^1[a,b]$ so that the restriction of them in every subinterval $[x_i, x_{i+1}]$ is a linear combination of functions in N_L. A convenient way to express the quadratic trigonometric splines is to use the **T**rigonometric **B**asic splines, denoted in the following with TB-splines. For the computation of the TB-splines there is a recursion formula, as in the polynomial case (see [6]).

However it is easier to use the exact formula given here:

$$TB_i^2(x) := \theta \cdot \begin{cases} \sin^2(\frac{x-x_i}{2}) & \text{for } x \in [x_i, x_{i+1}), \\ \sin(\frac{x-x_i}{2})\sin(\frac{x_{i+2}-x}{2}) + & \\ +\sin(\frac{x_{i+3}-x}{2})\sin(\frac{x-x_{i+1}}{2}) & \text{for } x \in [x_{i+1}, x_{i+2}), \\ \sin^2(\frac{x_{i+3}-x}{2}) & \text{for } x \in [x_{i+2}, x_{i+3}], \\ 0 & \text{elsewhere}, \end{cases}$$

where $\theta = \frac{1}{\sin(h)\sin(h/2)}$.

If we denote by $TS_2(\Omega_n)$ the space of the quadratic trigonometric splines in $[a,b]$ with respect to the knots Ω_n, then $TS_2(\Omega_n) = \text{span}\{TB_i^2\}_{i=-2}^{n-1}$. Subsequently every spline can be expressed in the form:

$$s(x) = \sum_{i=-2}^{n-1} \alpha_i TB_i^2(x)$$

where $\alpha = (\alpha_i)_{i=-2}^{n-1}$ the vector of the coefficients to be computed.

1.2 A Brief Description of the Method

Consider the differential equation

$$y' = f(x,y),\ a \le x \le b \tag{2}$$

with the initial value $y(a) = y_a$. Assume also $f \in C^2$ in T where $T = \{(x,y)/x \in [a,b]\}$ and f satisfies the Lipschitz condition:

$$\exists L > 0\ \ \forall x \in [a,b]\ \ \forall y_1, y_2 \in \mathbf{R} \quad |f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|. \tag{3}$$

This is satisfied if the partial derivatives of f are bounded in T.

We construct the numerical solution $s(x)$ of (2) requiring the following: $s(a) = y_a$, $s'(a) = f(a, s(a)) = f(a, y_a)$ and that $s(x)$ satisfies the differential equation (2) at all the knots $a + kh$, for $k = 1, 2, \ldots, n$. See also [7] and [8].

2 Computation of the Spline Approximations

The condition $s(a) = y_a$ and $s'(a) = f(a, s(a)) = f(a, y_a)$ enable us to compute the first two coefficients of the quadratic trigonometric spline approximation $s(x)$. Thus

$$\alpha_{-2} + \alpha_{-1} = 2\cos(\frac{h}{2})y_a \quad \text{and} \quad -\alpha_{-2} + \alpha_{-1} = 2\sin(\frac{h}{2})f(a, y_a).$$

The above system gives

$$\alpha_{-1} = \cos(\frac{h}{2})y_a + \sin(\frac{h}{2})f(a, y_a) \quad \text{and} \quad \alpha_{-2} = \cos(\frac{h}{2})y_a - \sin(\frac{h}{2})f(a, y_a).$$

To determine the next coefficient α_0 we require that $s'(x_1) = f(x_1, s(x_1))$ where $x_1 = a + h$. Thus we obtain the equation

$$-\frac{1}{2\sin(h/2)}\alpha_{-1} + \frac{1}{2\sin(h/2)}\alpha_0 = f\left(a + h, \frac{1}{2\cos(h/2)}\alpha_{-1} + \frac{1}{2\cos(h/2)}\alpha_0\right)$$

which is a nonlinear equation with respect to α_0 that can be solved with an iteration method, for example with Newton method. According to the previous construction and having determined the coefficient α_k, we can compute α_{k+1} from

$$-\frac{1}{2\sin(h/2)}\alpha_k + \frac{1}{2\sin(h/2)}\alpha_{k+1} = f\left(a + (k+2)h, \frac{\alpha_k + \alpha_{k+1}}{2\cos(h/2)}\right), \tag{4}$$

setting $x := \alpha_{k+1}$ we derive the following:

$$\varphi(x) := 2\sin(h/2)f\left(a + (k+2)h, \frac{\alpha_k + x}{2\cos(h/2)}\right) + \alpha_k = x \tag{5}$$

or $\varphi(x) = x$. We are going to prove that the function $\varphi(x)$ has a unique fixed point, i.e. the equation (4) has a unique solution. Since $\varphi(x)$ satisfies condition (3), it follows

$$\begin{aligned}
& |\varphi(x_1) - \varphi(x_2)| \\
= \;& 2\sin(h/2)\left|f\left(a + (k+2)h, \frac{\alpha_k + x_1}{2\cos(h/2)}\right) - f\left(a + (k+2)h, \frac{\alpha_k + x_2}{2\cos(h/2)}\right)\right| \\
\leq \;& \frac{\sin(h/2)}{\cos(h/2)} L\,|x_1 - x_2|\,.
\end{aligned}$$

.

From which it follows $|\varphi(x_1) - \varphi(x_2)| \leq Lh\,|x_1 - x_2|$, since $3h < 2\pi$ has as consequences that $\sin(h/2) < h/2$ and $\cos(h/2) > 1/2$. So if $h < \frac{1}{L}$ then

$$|\varphi(x_1) - \varphi(x_2)| < |x_1 - x_2|$$

i.e. that $\varphi(x)$ is a strong contraction mapping for an adequate h. This completes the proof of the following.

PROPOSITION 1. If $h < 1/L$, then there is a unique quadratic trigonometric spline approximation of the solution of the initial value problem.

3 Error Estimates for the Spline Approximations

Let $s_i := s(x_i)$ and $y_i := y(x_i)$ for $i = 0, 1, \ldots, n$, where $y(x)$ is the solution of the initial value problem (2) and $s(x)$ is the quadratic trigonometric spline approximation. Let also $y'_i := y'(x_i)$, $y''_i := y''(x_i)$, $s'_i := s'(x_i)$, $s''_i := s''_i(x_i + 0)$ where $s_i(x) = s(x)|_{[x_i, x_{i+1}]}$. From the definition of the splines it follows that $s_i(x) \in N_L$. This means that $L_3 s_i(x) = 0$. So

$$\begin{aligned} s_0 &= y_0 \\ s'_0 &= y'_0 \\ s'_1 &= f(x_1, s_1). \end{aligned}$$

The extended Taylor formula gives for $s_i(x)$:

$$s_i(x) = s_i + s''_i + s'_i \cdot \sin(x - x_i) - s''_i \cdot \cos(x - x_i)$$

which yields

$$s_1 = s_0(x_1) = s_0 + s''_0 + s'_0 \cdot \sin(h) - s''_0 \cdot \cos(h).$$

For the solution of (2), we have

$$y_1 = y(x_1) = y_0 + y''_0 + y'_0 \cdot \sin(h) - y''_0 \cdot \cos(h) + K$$

where $K = \int_{x_0}^{x_1} 2\sin^2(\frac{x_1 - \xi}{2}) L_3 y(\xi)\, d\xi$. Subtracting the two last equations we obtain

$$e_1 := s_1 - y_1 = (s_0 - y_0) + (s''_0 - y''_0) + (s'_0 - y'_0) \cdot \sin(h) - (s''_0 - y''_0) \cdot \cos(h) - K$$

or

$$e_1 = (s''_0 - y''_0) - (s''_0 - y''_0) \cdot \cos(h) - K$$

which is equivalent to

$$e_1 = 2(s''_0 - y''_0) \cdot \sin^2(\frac{h}{2}) - K, \tag{6}$$

where e_i is the discrete error at the point x_i. Now for e_1 we must compute $s''_0 - y''_0$. Differentiating the extended Taylor formula (1) we obtain

$$y'(x) = u'_y(x) + \int_a^x \sin(x - \xi) L_3 y(\xi)\, d\xi,\ x \in [a, b], \tag{7}$$

with $u'_y(x) = y'(a)\cos(x - a) + y''(a)\sin(x - a)$. Thus

$$\begin{aligned} s'_1 &= s'_0(x_1) = s'_0 \cos(h) + s''_0 \sin(h), \\ y'_1 &= y'_0 \cos(h) + y''_0 \sin(h) + \int_{x_0}^{x_1} \sin(x_1 - \xi) L_3 y(\xi)\, d\xi. \end{aligned}$$

Subtracting these equations we obtain

$$s_0'' - y_0'' = \frac{s_1' - y_1'}{\sin(h)} + \int_{x_0}^{x_1} \frac{\sin(x_1 - \xi)}{\sin(h)} L_3 y(\xi)\, d\xi.$$

Substituting in (6) we have

$$e_1 = e_1' \frac{\sin(h/2)}{\cos(h/2)} + \int_{x_0}^{x_1} A(\xi) L_3 y(\xi)\, d\xi \tag{8}$$

where

$$\begin{aligned} A(\xi) &= \frac{\sin(h/2)\sin(x_1-\xi)}{\cos(h/2)} - 2\sin^2\left(\frac{x_1-\xi}{2}\right) \\ &= \frac{\cos(x_1-\xi-h/2)}{\cos(h/2)} - 1 \geq 0 \end{aligned}$$

and applying the mean value theorem for integrals, the equation (8) becomes

$$\begin{aligned} e_1 &= e_1' \tan(h/2) + L_3 y(\gamma)\left[-\frac{\sin(x_1-\xi-h/2)}{\cos(h/2)} - \xi\right]_{x_0}^{x_1} \Leftrightarrow \\ e_1 &= e_1' \tan(h/2) + L_3 y(\gamma)\left[2\tan(h/2) - h\right]. \end{aligned} \tag{9}$$

where $\gamma \in [x_0, x_1]$ and $e_i' := s_i' - y_i'$ for $i = 1, 2, \ldots, n$. Thus

$$|e_1| \leq |e_1'| O(h) + |L_3 y(\gamma)| O(h^3)$$

using the Lipschitz condition (3):

$$|e_1'| = |f(x_1, s_1) - f(x_1, y_1)| \leq L|s_1 - y_1| = L|e_1|$$

we obtain

$$|e_1|(1 - L \cdot O(h)) \leq \|L_3 y\|_{\infty,[a,b]} O(h^3)$$

which means $|e_1| = O(h^3)$. Using the same analysis it can be derived that

$$e_k = e_{k-1} + e_{k-1}' \sin(h) + e_k' \tan(h/2) + L_3 y(\xi) O(h^3)$$

for some $\xi \in [x_k, x_{k+1}]$, so the conclusion is:

$$\begin{aligned} |e_k|(1 - L \cdot O(h)) &\leq |e_{k-1}|(1 + L \cdot O(h)) + \|L_3 y\|_{\infty,[a,b]} O(h^3) \\ \text{or} \quad |e_k| &\leq |e_{k-1}| + |e_{k-1}| O(h) + O(h^3) \quad \text{for} \quad k = 1, \ldots, n. \end{aligned}$$

Which means that at each next knot, the error of the previous knot is added. So the error at the last knot is $|e_n| = n\, O(h^3) = O(h^2)$.

LEMMA 1. The discrete error at the knots of the approximation with trigonometric quadratic splines is $|e_k| = O(h^2)$ for $k = 0, \ldots, n$.

We are going to prove also that, the convergence order of the global error of the spline approximations is quadratic.

In $[x_i, x_{i+1}]$ applying the Taylor formula (1) it holds

$$\begin{aligned} s_i(x) &= s_i + s_i'' + s_i' \cdot \sin(x - x_i) - s_i'' \cdot \cos(x - x_i), \\ y(x) &= y_i + y_i'' + y_i' \cdot \sin(x - x_i) - y_i'' \cdot \cos(x - x_i) + I, \end{aligned}$$

where $I = \int_{x_i}^{x} 2\sin^2(\frac{x-\xi}{2}) L_3 y(\xi)\, d\xi$. Subtracting the above equations and applying the mean value theorem for integrals we obtain:

$$s_i(x) - y(x) = (s_i - y_i) + (s_i'' - y_i'')\,(1 - \cos(x - x_i)) + (s_i' - y_i') \cdot \sin(x - x_i) - I$$

with $I = L_3 y(\gamma)((x - x_i) - \sin(x - x_i))$ for some $\gamma \in [x_i, x]$. Now the estimation for $e_i'' = s_i'' - y_i''$: From (7) we obtain

$$\begin{aligned} e_{i+1}' &= e_i' \cos(h) + e_i'' \sin(h) - \int_{x_i}^{x_{i+1}} \sin(x_{i+1} - \xi) L_3 y(\xi)\, d\xi \\ \text{or}\quad e_i'' \sin(h) &= e_{i+1}' - e_i' \cos(h) + L_3 y(\zeta)(1 - \cos(h)) \quad \text{for a} \quad \zeta \in [x_i, x_{i+1}] \end{aligned}$$

this means: $|e_i''| = O(h)$. Thus:

$$\begin{aligned} |e_i| = |s_i - y_i| &= O(h^2), \\ |s_i' - y_i'| = L|e_i| &= O(h^2), \\ |s_i'' - y_i''| &= O(h), \\ \text{and} \quad |I| &= O(h^3). \end{aligned}$$

THEOREM 1. The convergence of the spline approximations of the solution of the initial value problem (2) is quadratic, i.e. that $\|s(x) - y(x)\|_{[a,b]} = O(h^2)$.

REMARK: The analysis that was used here to obtain error estimates for the trigonometric splines approximations is in principal the same as the analysis used in [9] to obtain error estimates for the interpolation with trigonometric splines. Particularly in [9] error estimates were obtained for the interpolation with linear, quadratic and cubic trigonometric splines. For the cubic case a super convergence theorem was also proved in [9], which can also be found with some other parts of [9] in [4]. More about splines and polynomial spline interpolation can be found for example in [1, 2, 3, 5, 10, 11, 12].

4 Numerical results

We are going to test numerically in $[0, 1]$ the following three examples.

Example 1: $y' = 1 + y^2$, with $y(0) = 0$ and exact solution $y(x) = \tan(x)$.

Example 2: $y' = xy^{-2/3}$, with $y(0) = 1$ and exact solution $y(x) = \left(\frac{5}{6}x^2 + 1\right)^{3/5}$.

Example 3: $y' = \sqrt{1 - y^2}$, with $y(0) = 0$ and exact solution $y(x) = \sin(x)$.

Example	n	max. absolute error	num. convergence order
1	40	0.001133968452	2.002469
	60	0.000503481658	2.001217
	80	0.000283109324	2.000726
	100	0.000181160629	
2	40	0.000004867986	2.000005
	60	0.000002163545	2.001666
	80	0.000001216411	1.998115
	100	0.000000778830	
3	40	0	
	60	0	
	80	0	
	100	0	

where the numerical convergence order in the tables above is computed using:

$$\ln\left(\frac{|e_k|}{|e_m|}\right) / \ln\left(\frac{m}{k}\right).$$

This numerical convergence order confirms the theoretical error estimates, proved in previous paragraph in Theorem 1. The numerical solution of example 3 is exact, as expected from the error analysis; since the solution $y(x) = \sin(x)$ belongs in N_L.

References

[1] C. de Boor, A Practical Guide to Splines, Springer-Verlag 1978.

[2] C. de Boor, On Bounding Spline Interpolation, Journal of Approximation Theory 14(1975), 191–203.

[3] G. Hämmerlin and K-H. Hoffmann, Numerische Mathematik, Springer-Verlag, 1989.

[4] G. Hämmerlin and A. Nikolis, Superkonvergenz bei trigonometrischen Splines und Spline-on-Splines. Beiträge zur Angewandten Analysis und Informatik: Helmut Brakhage zu Ehren. hrsg. von Eberhard Schock. Verlag Shaker, 1994, pp. 112–122.

[5] C. A. Hall, On error bounds for spline interpolation, Journal of Approximation Theory 1(1968), 209–218.

[6] T. Lyche and R. Winter, A stable recurrence relation for trigonometric splines, Journal of Approximation Theory 25(1979), 266–279.

[7] F. R. Loscalzo and T. Talbot, Spline function approximations for solutions of ordinary differential equations, SIAM J. Numer. Anal. 4(3)(1967), 433–445.

[8] Ch. Micula, Numerical integration of differential equation $y^{(n)} = f(x, y)$ by spline function, Rev. Roum. Math. Pures et Appl. (Bucarest), 17(1972), 1385–1390.

[9] A. Nikolis, Trigonometrische Splines und Ihre Anwendung zur numerischen Behandlung von Integralgleichungen. Dissertation, Fak. für Mathematik der Ludwig-Maximilians- Universität München, 103 S., 1993.

[10] G. Nürnberger, Approximation by Spline Functions, Springer-Verlag, 1989.

[11] L. Schumaker, Spline Functions: Basic Theory, John Wiley and Sons, Inc., New York, 1981.

[12] R. A. Usmani, On quadratic spline interpolation, BIT 27(1987), 615-622.