Applied Mathematics E-Notes, 4(2004), 54-58 ©
ISSN 1607-2510
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Positive Periodic Solutions of Abstract Difference Equations[*] [†]

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Received 18 August 2003

Abstract

Difference equations in an ordered Banach space are considered. Conditions for the existence of positive periodic solutions are derived.

1 Statement of The Main Result

Periodic solutions of difference equations in a Euclidean space have been considered by many authors, see e.g. [2, 5-7, 9] and the references therein. In particular, the paper [1] should be mentioned, in which nonpositive periodic solutions of abstract difference equations are examined.

Let X be a real Banach space with a normal order cone X_+, which has a nonempty interior (see e.g. [3, 6] for background material). In the present paper we derive conditions for the existence of positive periodic solutions to the following difference equations in X:

$$x_{k+1} = A_k x_k + F_k(x_k),\ k = 0, 1, 2, ..., \tag{1}$$

where $\{A_k\}_{k=0}^{\infty}$ is a sequence of positive linear operators in X such that for an integer $T > 0$,

$$A_k = A_{k+T},\ k \geq 0, \tag{2}$$

and $\{F_k\}_{k=0}^{\infty}$ is a sequence of mappings of X into itself such that

$$F_k(h) = F_{k+T}(h),\ k \geq 0,\ h \in X. \tag{3}$$

Let us set

$$U(k, j) = \prod_{i=j}^{k-1} A_i,\ 0 \leq j < k,$$

[*]Mathematics Subject Classifications: 39A10

[†]Research partially supported by Natural Science Foundation of Shanxi Province (20001001) and by the Yanbei Normal Institute (200304105), the High Science and Technology Foundation of Shanxi Province of China and the Kamea Fund of the Israel.

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and

$$U(j,j) = I_X.$$

Here and below $I = I_X$ is the identity operator in X . It is assumed that the spectral radius $r_s(U(T,0))$ of the operator $U(T,0)$ satisfies the inequality

$$r_s(U(T,0)) < 1. \tag{4}$$

THEOREM 1. In additions to conditions (2)-(4), let there be a linear operator S in X which has a positive inverse operator such that the operators $A_k S^{-1}$ are compact and that the functions SF_k are positive, continuous in X and monotone decreasing. Moreover, suppose there is a $z \in X$ such that

$$SF_k(u) \le z, \; k = 0, 1, ..., T-1 \tag{5}$$

for all $u \in X$. Then equation (1) has at least one positive periodic solution.

2 Proof of Theorem 1

It is easily checked that the unique solution of the equation

$$y_{k+1} = A_k y_k + f_k, \; f_k \in X, \; k = 0, 1, ...,$$

is given by

$$y_k = U(k,0)y_0 + \sum_{j=0}^{k-1} U(k,j+1)f_j, \; k = 1, 2, ... \, .$$

Thus, the periodic boundary value problem

$$\begin{aligned} y_{k+1} &= A_k y_k + f_k, \; f_k \in X, \; k = 0, 1, ..., T-1, \\ y_0 &= y_T \end{aligned}$$

has a solution provided

$$y_0 = y_T = U(T,0)y_0 + \sum_{j=0}^{T-1} U(T,j+1)f_j,$$

or

$$y_0 = (I - U(T,0))^{-1} \sum_{j=0}^{T-1} U(T,j+1)f_j,$$

and in such a case, this solution is given by

$$y_k = U(k,0)\,(I - U(T,0))^{-1} \sum_{j=0}^{T-1} U(T,j+1)f_j + \sum_{j=0}^{k-1} U(k,j+1)f_j, \; 0 \le k \le T.$$

Hence the periodic problem for (1) can be written as

$$\begin{aligned} x_k &= U(k,0)\left(I - U(T,0)\right)^{-1}\sum_{j=0}^{T-1} U(T,j+1)F_j(x_j) \\ &\quad + \sum_{j=0}^{k-1} U(k,j+1)F_j(x_j) \\ &= \sum_{j=0}^{T-1} M_{k,j}F_j(x_j), \end{aligned} \tag{6}$$

where

$$M_{k,j} = U(k,0)\left(I - U(T,0)\right)^{-1} U(T,j+1) + W(k,j),\ 0 \le k \le T, \tag{7}$$

and $W(k,j) = U(k,j+1)$ for $j < k$ and $W(k,j) = 0$ for $j \ge k$. Let $c(T,X)$ be the space of sequences $h = \{h_k \in X\}_{k=1}^{T}$ with the norm

$$\|h\|_c = \max_{k=1,\ldots,T} \|h_k\|_X.$$

Rewrite (6) as

$$x = B\Phi(x),$$

where B is defined by

$$(Bh)_k = \sum_{j=0}^{T-1} M_{k,j}S^{-1}h_j,\ h = \{h_k\}_{k=1}^{T} \in c(T,X)$$

and $\Phi(h) = \{SF_k(h_k)\}_{k=1}^{T}$. Since A_kS^{-1} are compact in X and B is a finite sum of A_kS^{-1}, B is compact in $c(T,X)$. Moreover, due to (4),

$$\left(I - U(T,0)\right)^{-1} = \sum_{k=0}^{\infty} U^k(T,0) \ge 0.$$

So $B \ge 0$.

We now invoke the following result (see Theorem 7.G(c) in [6, pp.309-310]: Let Y and Z be real ordered Banach spaces. Let the order cone Y_+ on Y be normal with nonempty interior. In addition, let an operator $F_0 : Y \to Z$ be continuous and an operator $K : Z \to Y$ be linear, compact and positive. Then the equation

$$u = KF_0(u) \tag{8}$$

has a solution $u \in Y$, provided F is monotone decreasing and there is a $z_0 \in Z$, such that

$$F_0(u) \le z_0$$

for all $u \in Y$.

If we now take $K = B$ and $F_0 = \Phi$ in (8), we arrive at the proof of our result.

3 Example

Let c_0 be the Banach space of bounded sequences of real numbers with the supremum norm. Take $X = c_0$ and consider the system

$$x_{m+1,j} = \sum_{k=1}^{\infty} a_{m,j,k} x_{m,k} + f_{mj}(x_{mj}),\ j,k = 1,2,...;\ m = 0,1,..., \tag{9}$$

where $a_{m,j,k}$ is a positive number sequence of three arguments with the properties

$$a_{m,j,k} = a_{m+T,j,k} \tag{10}$$

and

$$\sup_{m=0,...,T-1, j=1,2,...} \sum_{k=1}^{\infty} a_{m,j,k} < 1. \tag{11}$$

The functions $f_{mj}(v)$ are positive scalar-valued functions which are decreasing as the argument $v \in \mathbf{R}$ increases. In addition

$$f_{mj}(v) = f_{m+T,j}(v),\ v \in \mathbf{R};\ m = 0,1,2,...;\ j = 1,2,..., \tag{12}$$

and

$$\sup\{j f_{mj}(v) : m = 0,...,T-1;\ v \in \mathbf{R};\ j = 1,2,...\} = z < \infty. \tag{13}$$

For instance, we can take

$$f_{mj}(v) = \frac{l_m}{j(1+v^2)},$$

where l_m is a positive constant for each $m = 0,...,T-1$.Then

$$z = \max_{m=0,...,T-1} l_m.$$

Define operator S by

$$(Sh)_j = jh_j,\ j = 1,2,...;\ h = (h_k)_{k=1}^{\infty} \in c_0.$$

Let us apply Theorem 1 to system (9) with A_m defined by

$$(A_m h)_j = \sum_{k=1}^{\infty} a_{m,j,k} h_k$$

and

$$(F_m h)_j = f_{mj}(h_j),\ j = 1,2,...;\ h = (h_k)_{k=1}^{\infty} \in c_0.$$

In view of (11), condition (4) holds (see [4], inequality (16.2)). Operators $A_m S^{-1}$ are compact in c_0. Moreover, in view of (13), condition (5) holds. Now Theorem 1 implies that system (9) under conditions (10)-(13) has at least one positive periodic solution in the space c_0.

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