Positive Periodic Solutions of Abstract Difference Equations* †

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Abstract

Difference equations in an ordered Banach space are considered. Conditions for the existence of positive periodic solutions are derived.

1 Statement of The Main Result

Periodic solutions of difference equations in a Euclidean space have been considered by many authors, see e.g. [2, 5-7, 9] and the references therein. In particular, the paper [1] should be mentioned, in which nonpositive periodic solutions of abstract difference equations are examined.

Let $X$ be a real Banach space with a normal order cone $X_+$, which has a nonempty interior (see e.g. [3, 6] for background material). In the present paper we derive conditions for the existence of positive periodic solutions to the following difference equations in $X$:

$$x_{k+1} = A_k x_k + F_k(x_k), \quad k = 0, 1, 2, ..., \quad (1)$$

where $\{A_k\}_{k=0}^\infty$ is a sequence of positive linear operators in $X$ such that for an integer $T > 0$,

$$A_k = A_{k+T}, \quad k \geq 0, \quad (2)$$

and $\{F_k\}_{k=0}^\infty$ is a sequence of mappings of $X$ into itself such that

$$F_k(h) = F_{k+T}(h), \quad k \geq 0, \quad h \in X. \quad (3)$$

Let us set

$$U(k, j) = \prod_{i=j}^{k-1} A_i, \quad 0 \leq j < k,$$
and

\[ U(j, j) = I_X. \]

Here and below \( I = I_X \) is the identity operator in \( X \). It is assumed that the spectral radius \( r_s(U(T, 0)) \) of the operator \( U(T, 0) \) satisfies the inequality

\[ r_s(U(T, 0)) < 1. \]  \hspace{1cm} (4)

**THEOREM 1.** In additions to conditions (2)-(4), let there be a linear operator \( S \) in \( X \) which has a positive inverse operator such that the operators \( A_kS^{-1} \) are compact and that the functions \( SF_k \) are positive, continuous in \( X \) and monotone decreasing. Moreover, suppose there is a \( z \in X \) such that

\[ SF_k(u) \leq z, \quad k = 0, 1, ..., T - 1 \]  \hspace{1cm} (5)

for all \( u \in X \). Then equation (1) has at least one positive periodic solution.

**2 Proof of Theorem 1**

It is easily checked that the unique solution of the equation

\[ y_{k+1} = A_ky_k + f_k, \quad f_k \in X, \quad k = 0, 1, ..., \]

is given by

\[ y_k = U(k, 0)y_0 + \sum_{j=0}^{k-1} U(k, j + 1)f_j, \quad k = 1, 2, ... . \]

Thus, the periodic boundary value problem

\[
\begin{align*}
y_{k+1} &= A_ky_k + f_k, \quad f_k \in X, \quad k = 0, 1, ..., T - 1, \\
y_0 &= y_T
\end{align*}
\]

has a solution provided

\[ y_0 = y_T = U(T, 0)y_0 + \sum_{j=0}^{T-1} U(T, j + 1)f_j, \]

or

\[ y_0 = (I - U(T, 0))^{-1} \sum_{j=0}^{T-1} U(T, j + 1)f_j, \]

and in such a case, this solution is given by

\[
\begin{align*}
y_k &= U(k, 0) (I - U(T, 0))^{-1} \sum_{j=0}^{T-1} U(T, j + 1)f_j + \sum_{j=0}^{k-1} U(k, j + 1)f_j, \quad 0 \leq k \leq T.
\end{align*}
\]
Hence the periodic problem for (1) can be written as

\[ x_k = U(k,0) (I - U(T,0))^{-1} \sum_{j=0}^{T-1} U(T, j + 1) F_j(x_j) \]

\[ + \sum_{j=0}^{k-1} U(k, j + 1) F_j(x_j) \]

\[ = \sum_{j=0}^{T-1} M_{k,j} F_j(x_j), \]

(6)

where

\[ M_{k,j} = U(k,0) (I - U(T,0))^{-1} U(T, j + 1) + W(k,j), \quad 0 \leq k \leq T, \]

(7)

and \( W(k,j) = U(k, j + 1) \) for \( j < k \) and \( W(k,j) = 0 \) for \( j \geq k \). Let \( c(T,X) \) be the space of sequences \( h = \{ h_k \in X \}_{k=1}^{T} \) with the norm

\[ \| h \|_c = \max_{k=1,\ldots,T} \| h_k \|_X. \]

Rewrite (6) as

\[ x = B\Phi(x), \]

where \( B \) is defined by

\[ (Bh)_k = \sum_{j=0}^{T-1} M_{k,j} S^{-1} h_j, \quad h = \{ h_k \}_{k=1}^{T} \in c(T,X) \]

and \( \Phi(h) = \{ SF_k(h_k) \}_{k=1}^{T} \). Since \( A_k S^{-1} \) are compact in \( X \) and \( B \) is a finite sum of \( A_k S^{-1} \), \( B \) is compact in \( c(T,X) \). Moreover, due to (4),

\[ (I - U(T,0))^{-1} = \sum_{k=0}^{\infty} U^k(T,0) \geq 0. \]

So \( B \geq 0 \).

We now invoke the following result (see Theorem 7.G(c) in [6, pp.309-310]: Let \( Y \) and \( Z \) be real ordered Banach spaces. Let the order cone \( Y_+ \) on \( Y \) be normal with nonempty interior. In addition, let an operator \( F_0 : Y \to Z \) be continuous and an operator \( K : Z \to Y \) be linear, compact and positive. Then the equation

\[ u = KF_0(u) \]

(8)

has a solution \( u \in Y \), provided \( F \) is monotone decreasing and there is a \( z_0 \in Z \), such that

\[ F_0(u) \leq z_0 \]

for all \( u \in Y \).

If we now take \( K = B \) and \( F_0 = \Phi \) in (8), we arrive at the proof of our result.
3 Example

Let $c_0$ be the Banach space of bounded sequences of real numbers with the supremum norm. Take $X = c_0$ and consider the system

$$x_{m+1,j} = \sum_{k=1}^{\infty} a_{m,j,k} x_{m,k} + f_{mj}(x_{mj}), \; j, k = 1, 2, \ldots; \; m = 0, 1, \ldots,$$

where $a_{m,j,k}$ is a positive number sequence of three arguments with the properties

$$a_{m,j,k} = a_{m+T,j,k}$$

and

$$\sup_{m=0, \ldots, T-1, j=1, 2, \ldots} \sum_{k=1}^{\infty} a_{m,j,k} < 1.$$  \hspace{1cm} (10)

The functions $f_{mj}(v)$ are positive scalar-valued functions which are decreasing as the argument $v \in \mathbb{R}$ increases. In addition

$$f_{mj}(v) = f_{mj+T}(v), \; v \in \mathbb{R}; \; m = 0, 1, 2, \ldots; \; j = 1, 2, \ldots,$$  \hspace{1cm} (12)

and

$$\sup\{jf_{mj}(v) : m = 0, \ldots, T-1; \; v \in \mathbb{R}; \; j = 1, 2, \ldots\} = z < \infty.$$  \hspace{1cm} (13)

For instance, we can take

$$f_{mj}(v) = \frac{l_m}{j(1 + v^2)},$$

where $l_m$ is a positive constant for each $m = 0, \ldots, T - 1$. Then

$$z = \max_{m=0, \ldots, T-1} l_m.$$  \hspace{1cm} (14)

Define operator $S$ by

$$(Sh)_j = jh_j, \; j = 1, 2, \ldots; \; h = (h_k)_{k=1}^{\infty} \in c_0.$$  \hspace{1cm} (15)

Let us apply Theorem 1 to system (9) with $A_m$ defined by

$$(A_m h)_j = \sum_{k=1}^{\infty} a_{m,j,k} h_k$$

and

$$(F_m h)_j = f_{mj}(h_j), \; j = 1, 2, \ldots; \; h = (h_k)_{k=1}^{\infty} \in c_0.$$  \hspace{1cm} (16)

In view of (11), condition (4) holds (see [3], inequality 16.2)). Operators $A_m S^{-1}$ are compact in $c_0$. Moreover, in view of (13), condition (5) holds. Now Theorem 1 implies that system (9) under conditions (10)-(14) has at least one positive periodic solution in the space $c_0$.\hspace{1cm}
References


