Attractivity In A Nonlinear Delay Difference Equation *

Wan-Sheng He†, Wan-Tong Li‡§

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Abstract

In this paper, we study the global stability and periodic character of the positive solution of the difference equation

$$x_{n+1} = \frac{(a - bx_n - k)}{(A + x_n)},$$

where $a \geq 0$, $b > 0$, and $k \in \{1, 2, \cdots \}$, and initial conditions $x_{-k}, \cdots, x_0$ are arbitrary real numbers. We show that the positive equilibrium of the equation is a global attractor with a basin that depends on certain conditions posed on the coefficients.

1 Introduction

The global asymptotic stability of the rational recursive relation

$$x_{n+1} = \frac{(\alpha - \beta x_n)}{(\gamma + x_{n-k})}, \quad n = 0, 1, \ldots, \quad (1)$$

and

$$x_{n+1} = \frac{(\alpha - \beta x_n)}{(\gamma - x_{n-k})}, \quad n = 0, 1, \ldots, \quad (2)$$

is investigated when $\alpha, \beta, \gamma$ are nonnegative real numbers and $k \in \{1, 2, \ldots \}$, and sufficient conditions for the global attractivity of the positive equilibriums of (1) and (2) are obtained, see [1, 3, 7]. Also, Yan et al. [8] studied the rational recursive equation

$$x_{n+1} = \frac{(\alpha + \beta x_n)}{(\gamma - x_{n-1})}, \quad n = 0, 1, \ldots, \quad (3)$$

where $\alpha \geq 0$, $\beta, \gamma > 0$ are real numbers, and obtained the global attractivity of positive equilibrium of (3).

Other related results can be found in [2, 4, 5, 6].

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†Department of Mathematics, Tianshui Normal University, Tianshui, Gansu 741001, P. R. China
‡Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, P. R. China
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Our aim in this paper is study the global attractivity and periodic character of positive solution of the rational recursive relation

\[ x_{n+1} = \frac{a - bx_{n-k}}{A + x_n}, \quad n = 0, 1, \ldots, \] (4)

where \(a \geq 0, A, b > 0\) are real numbers and the initial values \(x_{-k}, \ldots, x_0\) are arbitrary real numbers. We show that the nonnegative equilibrium point of the equation is a global attractor with a basin that depends on certain conditions of the coefficients.

We first recall some results which will be useful in the sequel.

Let \(I\) be some real interval and let \(F\) be a continuous function defined on \(I^{k+1}\). Then, for initial conditions \(x_{-k}, \ldots, x_0 \in I\), it is easy to see that the difference equation

\[ x_{n+1} = F(x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \] (5)

has a unique solution \(\{x_n\}\).

A point \(\mathbf{x}\) is called an equilibrium of (5) if \(\mathbf{x} = F(\mathbf{x}, \ldots, \mathbf{x})\). That is, \(x_n = \mathbf{x}\) for \(n \geq 0\) is a solution of (5), or equivalently, is fixed point of \(F\).

An interval \(J \subset I\) is called an invariant interval of (5) if \(x_{-k}, \ldots, x_0 \in J \Rightarrow x_n \in J, \quad n > 0\).

That is, every solution of Eq.(5) with initial conditions in \(J\) remains in \(J\).

**DEFINITION 1.1.** The difference equation (5) is said to be permanent, if there exist numbers \(P\) and \(Q\) with \(0 < P \leq Q < \infty\) such that for any initial conditions \(x_{-k}, \ldots, x_0\) there exists a positive integer \(N\) which depends on the initial conditions such that \(P \leq x_n \leq Q\) for \(n \geq N\).

The linearized equation associated with (5) about the equilibrium \(\mathbf{x}\) is

\[ y_{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\mathbf{x}, \ldots, \mathbf{x})y_{n-i}, \quad n = 0, 1, \ldots. \] (6)

Its characteristic equation is

\[ \lambda^{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\mathbf{x}, \ldots, \mathbf{x})\lambda^{n-i}. \] (7)

**THEOREM A [5].** Assume that \(F\) is a \(C^1\) function and let \(\mathbf{x}\) be an equilibrium of (5). Then the following statements are true:

(a) If all the roots of the equation (7) lie in the open unit disk \(|\lambda| < 1\), then the equilibrium \(\mathbf{x}\) of (5) is asymptotically stable.

(b) If at least one root of (5) has absolute value greater than one, then the equilibrium \(\mathbf{x}\) of (5) is unstable.

**THEOREM B [2, 5].** Assume that \(p, q \in R\) and \(k \in \{1, 2, \ldots\}\). Then

\[ |p| + |q| < 1 \] (8)
is a sufficient condition for asymptotic stability of the difference equation
\[ x_{n+1} - px_n + qx_{n-k} = 0, \quad n = 0, 1, \ldots \] (9)
Suppose in addition that one of the following two cases holds: (a) \( k \) is odd and \( q < 0 \), or, (b) \( k \) is even and \( pq < 0 \). Then (8) is also a necessary condition for asymptotic stability of (9).

2 The Case \( a > 0 \)

In this section, we discuss the periodic character and global attractivity of positive solutions of (4).

Consider the difference equation (4) with \( a > 0 \) and \( A, b > 0 \).

The unique positive equilibrium point of (4) is
\[ x = \frac{-A + \sqrt{(A + b)^2 + 4a}}{2} \]
The linearized equation associated with (4) about the equilibrium \( x \) is
\[ y_{n+1} + \frac{x}{A+b} y_n + \frac{b}{A+x} y_{n-k} = 0, \quad n = 0, 1, \ldots \]
Its characteristic equation is
\[ \lambda^{k+1} + \frac{-A + \sqrt{(A + b)^2 + 4a}}{A - b + \sqrt{(A + b)^2 + 4a}} \lambda^k + \frac{2b}{A - b + \sqrt{(A + b)^2 + 4a}} = 0. \]
By using Theorem B, we have the following result.

LEMMA 2.1. The following statements are true.
(i) Assume that \( k \) is even. Then the positive equilibrium \( x \) of (4) is locally asymptotically stable if and only if \( A > b \).
(ii) Assume that \( k \) is odd. Then the positive equilibrium \( x \) of (4) is locally asymptotically stable if \( A > b \).

In the following, we always assume that
\[ a > 0 \quad \text{and} \quad A > b > 0. \] (11)

Set \( f(u, v) = (a-bv)/(A+u) \), then it is easy to see that \( f(u, v) \) satisfies the following properties.

LEMMA 2.2. Assume that (11) holds. Then the following statements are true.
(i) \( 0 < x < \frac{a}{b} < \frac{A}{x} \).
(ii) \( f(x, x) \) is a strictly decreasing function in \([0, \infty)\).
(iii) If \((u, v) \in [0, \infty] \times (-\infty, a/b)\), then \( f(u, v) \) is a strictly decreasing function in each of its arguments.
THEOREM 2.1. Assume that (11) holds. Then Eq.(4) has no positive solution with prime period two for all $a \in [0, \infty)$.

PROOF. Assume for the sake of contradiction that there exist distinctive positive real numbers $\phi$ and $\psi$, such that

..., $\phi, \psi, \phi, \psi, ...$

forms a period-two solution of Eq.(4). There are two cases to consider.

Case (a) $k$ is odd.
In this case $x_{n+1} = x_{n-k}$, $\phi$ and $\psi$ satisfy the system

$\phi(A + \psi) = a - b\phi$ and $\psi(A + \phi) = a - b\psi$.

Subtracting these equations, we get $(A + b)(\phi + \psi) = 0$. Since $\phi \neq \psi$, then we have $A + b = 0$, this is a contradiction.

Case (b) $k$ is even.
In this case $x_n = x_{n-k}$, $\phi$ and $\psi$ satisfy the system

$\phi(A + \psi) = a - b\psi$ and $\psi(A + \phi) = a - b\phi$.

Subtracting these equations, we obtain $(A - b)(\phi - \psi) = 0$, so $\phi = \psi$, which contradicts the hypothesis $\phi \neq \psi$. The proof is complete.

THEOREM 2.2. Assume that (11) holds, and let initial conditions $x_{-k}, \ldots, x_0 \in [0, a/b]$. Then Eq.(4) is permanent, that is, there exist constants $P$ and $Q$ with $0 < P < Q < \infty$ such that $P \leq x_n \leq Q$, for $n \geq 0$.

PROOF. Set $Q = f(0, 0), P = f(Q, Q)$. Then we have

$0 < P < Q = f(0, 0) = a/A < a/b$.

By part (iii) of Lemma 2.1, we have

$0 = f(a/b, a/b) \leq x_1 = f(x_0, x_{-k}) \leq f(0, 0) = Q,$

$0 = f(Q, a/b) \leq x_2 = f(x_1, x_{-k+1}) \leq f(0, 0) = Q,$

and

$0 < P = f(Q, Q) \leq x_2 = f(x_1, x_{-k+1}) \leq f(0, 0) = Q$.

Hence, the result follows by induction. The proof is complete.

By Theorem 2.2, we know that the interval $[0, a/b]$ is an invariant interval of Eq.(4).

THEOREM 2.3. Assume that (11) holds. Then the positive equilibrium $\mathbf{e}$ of Eq.(4) is a global attractor with the basin $S = [0, a/b]^{k+1}$.

PROOF. Let $\{x_n\}$ be a solution of Eq.(4) with initial condition $(x_{-k}, \ldots, x_0) \in S$. Then, by part (iii) of Lemma 2.1, for any $u, v \in [0, a/b]$, we have

$0 < f(u, v) = \frac{a - bv}{A + a} < a/b$. 
Hence, \( f \in C([0,a/b], [0,a/b]) \) and is strictly decreasing in each of its arguments.

Let \( \lambda = \lim \inf_{n \to \infty} x_n, \Lambda = \lim \sup_{n \to \infty} x_n \), and let \( \varepsilon > 0 \) such that \( \varepsilon < \min\{a/b - \Lambda, \lambda\} \). Then there exist \( n_0 \in \mathbb{N} \) such that \( \lambda - \varepsilon \leq x_n \leq \Lambda + \varepsilon \). Thus

\[
\frac{a - b(\Lambda + \varepsilon)}{A + (\Lambda + \varepsilon)} < x_{n+1} < \frac{a - b(\lambda - \varepsilon)}{A + (\lambda - \varepsilon)}, \quad n \geq n_0 + 1.
\]

Then we get the following inequality

\[
\frac{a - b(\Lambda + \varepsilon)}{A + (\Lambda + \varepsilon)} \leq \lambda \leq \Lambda \leq \frac{a - b(\lambda - \varepsilon)}{A + (\lambda - \varepsilon)}.
\]

This inequality yields

\[
\frac{a - b\Lambda}{A + \Lambda} \leq \lambda \leq \frac{a - b\lambda}{A + \lambda},
\]

which implies that \( a - b\Lambda - A\lambda \leq \lambda A \leq a - b\lambda - A\Lambda \). In view of \( A > b, \Lambda \leq \lambda \). Hence \( \lambda = \Lambda = \pi \), that is \( \lim_{n \to \infty} x_n = \pi \). This completes the proof.

3 The Case \( a = 0 \)

In the section, we study the asymptotic stability for the difference equation

\[
x_{n+1} = \frac{-bx_{n-k}}{A + x_n}, \quad n = 0, 1, ..., \tag{12}
\]

where

\[
b, A \in (0, \infty), \quad k \in \{1, 2, ...\}, \tag{13}
\]

and the initial condition \( x_{-k}, ..., x_0 \) are arbitrary real numbers.

By putting \( x_n = by_n \), Eq.(4) yields

\[
y_{n+1} = \frac{-y_{n-k}}{C + y_n}, \quad n = 0, 1, ..., \tag{14}
\]

where \( C = A/b > 0 \). Eq.(14) has two equilibria \( \overline{y}_1 = 0 \) and \( \overline{y}_2 = -(C + 1) \). The linearized equations of the Eq.(14) about the equilibria \( \overline{y}_1 \) and \( \overline{y}_2 \) are

\[
Z_{n+1} + \frac{\overline{y}_1}{C + \overline{y}_1}Z_n + \frac{1}{C + \overline{y}_1}Z_{n-k} = 0, \quad i = 1, 2, \quad n = 0, 1, ... .
\]

For \( \overline{y}_2 = -(C + 1) \), by Theorem A we can see that it is unstable. For \( \overline{y}_1 = 0 \), we have

\[
Z_{n+1} + \frac{1}{C}Z_{n-k} = 0, \quad n = 0, 1, ... . \tag{15}
\]

The characteristic equation of Eq.(15) is \( \lambda^{k+1} + 1/C = 0 \). Hence, by Theorem A , we have
(i) if \( A > b \), then \( \mathcal{y}_1 \) is locally asymptotically stable.
(ii) if \( A < b \), then \( \mathcal{y}_1 \) is unstable.
(iii) if \( A = b \), then linearized stability analysis fails.

In the sequel, we discuss the global attractivity of the zero equilibrium of Eq.(14).

So, we assume that \( A > b \), namely, \( C > 1 \).

**LEMMA 3.1.** Assume that the initial conditions \( y_{-k}, \ldots, y_0 \in [-C + 1, C - 1] \). Then \( y_n \in [-C + 1, C - 1] \) for \( n \geq -1 \).

**PROOF.** Suppose \( y_{-k}, \ldots, y_0 \in [-C + 1, C - 1] \). Then we have

\[
-C + 1 = \frac{-C + 1}{C - C + 1} \leq \frac{-C + 1}{C + y_0} \leq y_1 = \frac{-y_{k}}{C + y_0} \leq \frac{C - 1}{C + y_0} \leq \frac{C - 1}{C - C + 1} = C - 1,
\]

and

\[
-C + 1 = \frac{-C + 1}{C - C + 1} \leq y_2 = \frac{-y_{k+1}}{C + y_1} \leq \frac{C - 1}{C + y_1} \leq \frac{C - 1}{C - C + 1} = C - 1.
\]

Our result now follows by induction.

By Lemma 3.1, we know that the interval \( [-C + 1, C - 1] \) is an invariant interval of Eq.(14). Also, Lemma 3.1 implies that the following is true.

**THEOREM 3.1.** The equilibrium \( \mathcal{y}_1 = 0 \) of Eq.(14) is a global attractor with a basin \( S = [-C + 1, C - 1]^{k+1} \).

**References**


