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One Aspect On Bellman's Equation To Optimal Control Theory*

Samir Lahrech†, Ahmed Addou ‡

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Abstract

In this present work, we develop the idea of the dynamic programming approach. The main observation is that the Bellman function $\omega(x,t)$ which is the function that provides, for any given state x at any given time t, the smallest possible cost among all possible trajectories starting at this event is in general not differentiable, and consequently we cannot use the Hamilton-Jacobi-Bellman (HJB)equation. By the classical Hamilton-Jacobi-Bellman theory if the value function ω is continuously differentiable, then it is the unique solution of the (HJB) equation. It is well known that the value function ω is in general discontinuous, even if all the data of the problem is continuously differentiable. Using the (HJB) equation in some nonclassical sense (e.g. using generalized gradients, in the framework of viscosity solutions, proximal solutions, etc.) has become a very active research area. Here, we give techniques based on the differential set $\partial\omega(x,l)$ of the function ω at the element x along the direction l for the analysis of such problems.

1 Description of the problem

Let X a Banach space, B a closed unit ball in X, W a nonempty subset of X. Let $l \in X$, $x \in W$. Assume that l, x are such that $x + \delta l \in W$ for δ small enough. Let $\rho : W \longrightarrow R$. We define $\partial\rho(x,l)$ as follows :

$$\partial\rho(x,l) = \bigcap_{\varepsilon>0} \overline{\bigcup_{0<\delta<\varepsilon} \delta^{-1}\{\rho(x+\delta l) - \rho(x)\}}.$$

$\partial\rho(x,l)$ is called the differential set of the function ρ at the element x along the direction l. If $\partial\rho(x,l)$ is a one-element set, we will then say that ρ is differentiable along the direction l, $(l \neq 0)$ at x, (see [5]). In this case, we write simply $\partial\rho(x,l) = \rho'(x,l)$. Let

*Mathematics Subject Classifications: 49K24, 49K27.

†Department of Mathematics, Faculty of Sciences, University Mohamed I, Box 524, 60000 oujda, Morocco

‡Department of Mathematics, Faculty of Sciences, University Mohamed I, Box 524, 60000 oujda, Morocco

$P : W \to 2^X \setminus \{\emptyset\}$ a multivalued function. We are interested in systems described by the differential inclusion

$$\dot{x}(t) \in P(x(t)) \tag{1}$$

with the initial condition

$$x(0) = x_0, \tag{2}$$

where $x : [0,T) \to W$ is assumed to be locally absolutely continuous on $[0,T)$ and $\exists \dot{x}(t) \in P(x(t))$ almost everywhere on each compact interval of $[0,T)$, $x_0 \in W$. Pick any $x_1 \in W$. Denote by $\mathcal{D}(\omega)$ the set of all $x_0 \in W$ such that x_0 can be controlled to x_1 along some trajectory $x(.)$ in some time $t(x_0, x(.)) < T$. Put

$$\omega(x_0) = \sup(-t(x_0, x(.))), \text{ for } x_0 \in \mathcal{D}(\omega).$$

So the problem we wish to study is : Given a system (1), a trajectory cost function $t(x_0, x(.))$, and a state x_0, find an admissible trajectory $\bar{x}$ for x_0 such that x_0 can be controlled to x_1 along the trajectory $\bar{x}$ and $\bar{x}$ minimizes the cost function $t(x_0, x(.))$.

DEFINITION 1. Assume that $x \in \mathcal{D}(\omega)$, $l \in P(x)$. l is said to be an admissible direction for x, if $l \neq 0$ and $\forall \varepsilon > 0\ \exists \delta \in (0, \varepsilon)\ \ \forall x' \in x + (0,\delta)l\ \ \exists t' \in (0,\varepsilon)\ \exists x(.) \in W_1^1([0,t'], X)$ such that *a)* $\dot{x}(t) \in P(x(t))$ almost everywhere on $[0,t']$; *b)* $x(0) = x$; *c)* $x(t') = x'$; and *d)* $\|x' - x - lt'\| \leq t'\varepsilon$, where $W_1^1([0,t'], X)$ is the Sobolev space.

We denote by $V(x)$ the set of all directions $l \in P(x)$ such that the assumptions in Definition 1 hold.

2 On Bellman's equation.

Assume that all the hypotheses of the above Section hold. Then we have the following result.

THEOREM 1. Let $x \in \mathcal{D}(\omega)$ such that $V(x) \neq \emptyset$. Then

$$\sup_{l \in V(x)} \sup \partial\omega(x,l) \leq 1.$$

PROOF. Suppose to the contrary that $\sup\limits_{l \in V(x)} \sup \partial\omega(x,l) > 1$. Then

$$\exists \sigma > 0\ \ \exists l \in V(x)\ \ \forall \xi > 0\ \ \exists \eta \in (0,\xi)$$

such that

$$x + \eta l \in \mathcal{D}(\omega)$$

and

$$\omega(x + \eta l) - \omega(x) > (1 + 2\sigma)\eta.$$

Put $\xi_n = n^{-1}$ where $n \in N$. We may assume that $\exists \eta_n \in (0, \xi_n)$ such that $x + \eta_n l \in \mathcal{D}(\omega)$ and

$$\omega(x + \eta_n l) - \omega(x) > (1 + 2\sigma)\eta_n.$$

Let $\varepsilon_k = k^{-1}$, $k \in N$. Since $l \in V(x)$, it follows that $\exists \delta_k \in (0, \varepsilon_k)$ such that the conditions of Definition 1 holds. For every $k \in N$, let us take $n_k \geq k$ such that $\xi_{n_k} < \delta_k$ and $\eta_{n_k} < \delta_k$. Put $x'_k = x + \eta_{n_k} l$. Then, there exists $t_k \in (0, \varepsilon_k)$ and $x_k(.) \in W_1^1([0, t_k], X)$ such that

$$\dot{x_k}(t) \in P(x_k(t)) \text{ almost everywhere on } [0, t_k]$$

and

$$x(0) = x, \ x(t_k) = x'_k, \ \|x'_k - x - lt_k\| \leq t_k \varepsilon_k.$$

Let us remark that $\exists h'_k \in B$ such that $x - x'_k + lt_k = t_k \varepsilon_k h'_k$. We now establish that $\eta_{n_k}^{-1} t_k \to 1$, as $k \to \infty$. We claim that the sequence $\eta_{n_k}^{-1} t_k$ is bounded. Indeed, suppose the contrary holds, then we can suppose without loss of generality that $\eta_{n_k}^{-1} t_k \to \infty$. Next, we have

$$\begin{aligned} t_k^{-1} l(\eta_{n_k} - t_k) &= t_k^{-1}[-x + x'_k + x - x'_k - t_k \varepsilon_k h'_k)] \\ &= -t_k^{-1}(t_k \varepsilon_k h'_k) \\ &= -\varepsilon_k h'_k \to 0 \end{aligned}$$

as $k \to \infty$. But, on the other hand, $t_k^{-1} l(\eta_{n_k} - t_k) = (\eta_{n_k} t_k^{-1} - 1)l \to -l$, contradicting the assumption $l \neq 0$. Thus $\eta_{n_k}^{-1} t_k$ is bounded.

Analogously for $\eta_{n_k} t_k^{-1}$, we claim that $\eta_{n_k} t_k^{-1}$ is also bounded. Otherwise, computing the value of $\eta_{n_k}^{-1} l(\eta_{n_k} - t_k)$, we obtain

$$\eta_{n_k}^{-1} l(\eta_{n_k} - t_k) = -\eta_{n_k}^{-1} t_k \varepsilon_k h'_k \to 0.$$

So, as $\eta_{n_k}^{-1} l(\eta_{n_k} - t_k) = (1 - \eta_{n_k}^{-1} t_k) l \to l \neq 0$, we conclude that $\eta_{n_k} t_k^{-1}$ is bounded.

Observing that

$$(t_k^{-1} \eta_{n_k} - 1)l = t_k^{-1} l(\eta_{n_k} - t_k) = -\varepsilon_k h'_k \to 0,$$

we deduce that $\eta_{n_k} t_k^{-1} \to 1$. Finally, $(1 + 2\sigma)\eta_{n_k} > (1 + \sigma)t_k$ for k large enough.

Note that x can be controlled to x'_k in time t_k. Thus, the system (1) will be controlled from the initial state x to the state x'_k in time t_k along the trajectory $x_k(.)$, and the system (1) will be also controlled from the state x'_k to the final state x_1 in time $\tilde{t}_k = -\omega(x'_k) < -\omega(x'_k) + \sigma t_k$. It will then follow that the system (1) will be controlled from the initial state x to the final state x_1 in time

$$\begin{aligned} t_k + \tilde{t}_k &< t_k - \omega(x'_k) + \sigma t_k = -\omega(x'_k) + (1 + \sigma)t_k \\ &< -\omega(x'_k) + (1 + 2\sigma)\eta_{n_k} \leq -\omega(x), \end{aligned}$$

contradicting the definition of ω.

Suppose now that $x_0 \neq x_1$. Let $\bar{x}$ the optimal trajectory that achieves the transfer of the system (1) from the initial state x_0 to the final state x_1 on $[0, -\omega(x_0)]$. Let us remark that if $t \neq t'$ and $t, t' \leq -\omega(x_0)$, then $\bar{x}(t) \neq \bar{x}(t')$, i.e.e the time length $t \leq -\omega(x_0)$ to achieve the transfer of the system (1) along the trajectory $\bar{x}$ from the initial state $x_0 = \bar{x}(0)$ to the final state $x = \bar{x}(t)$ is an invertible function $t = t(x)$ on the optimal trajectory $\bar{x}$.

Put $M(\bar{x}(.)) = \{x \in \bigcup_{0\le t\le -\omega(x_0)} \bar{x}(t) : \exists l_1 \in P(x)\ \forall\varepsilon > 0\ \exists t_1 \in [t(x), t(x)+\varepsilon)$ $\|r(t_1 - t(x)) \equiv \bar{x}(t_1) - x - (t_1 - t(x))l_1\| < \varepsilon|t_1 - t(x)|\}$.

If $x \in M(\bar{x}(.))$, then we put $\bar{l}(x) = l_1$. Let us remark that l_1 is not unique.

We now state the main Theorem.

THEOREM 2. Let $x \in M(\bar{x}(.))$ and suppose that $r \equiv 0$. Then $1 \in \partial\omega(x, \bar{l}(x))$.

PROOF. Let t_n a sequence such that $t_n \to 0$ and $0 < t_n < -\omega(x)$. Then $\bar{x}(t(x) + t_n) = x + \bar{l}(x)t_n \in \mathcal{D}(\omega)$, where $\varepsilon_n > 0$, and $\varepsilon_n \to 0$. On the other hand , by Bellman's principle it follows that $-\omega(\bar{x}(t(x) + t_n)) = -\omega(x) - t_n$. Consequently, $\forall\varepsilon > 0\ \exists n_0 \in N\ \forall n > n_0$:

$$\begin{aligned}
\bigcup_{0<\delta<\varepsilon} \delta^{-1}[\omega(x+\delta\bar{l}(x)) - \omega(x)] \quad &\ni \quad t_n^{-1}\big(\omega[x + t_n\bar{l}(x)] - \omega(x)\big) \\
&= \quad t_n^{-1}(\omega(\bar{x}(t(x)+t_n)) - \omega(x)) \\
&= \quad t_n^{-1}t_n \\
&= \quad 1.
\end{aligned}$$

The proof is complete.

COROLLARY 1. Let $x \in M(\bar{x}(.))$. Suppose that $\bar{l}(x) \in V(x)$, $1 \in \partial\omega(x, \bar{l}(x))$. Then

$$\sup_{l\in V(x)} \sup \partial\omega(x, l) = \sup \partial\omega(x, \bar{l}(x)) = 1.$$

In the particular case when $r \equiv 0$ and $\bar{l}(x) \in V(x)$, the same conclusion holds also.

The proof follows from Theorems 1 and 2.

COROLLARY 2. Assume that $x \in M(\bar{x}(.))$, $r \equiv 0$, ω is differentiable at x along every direction $l \in \tilde{U}(x) \subset V(x)$. Assume also that $\bar{l}(x) \in \tilde{U}(x)$. Then

$$\sup_{l\in\tilde{U}(x)} \omega'(x, l) = \omega'(x, \bar{l}(x)) = 1.$$

The proof follows from Corollary 1 and the definition of differential set.

Now we give an example of optimal control problem when the Hamilton-Jacobi-Bellman equation (HJB) cannot be used, but with the new dynamic programming approach given here, the above problem can be resolved.

EXAMPLE 1. Consider the following system

$$(\mathcal{S}) \begin{cases} \dot{x_1} = -x_2 = f_1(x,u), \\ \dot{x_2} = -u = f_2(x,u), \\ \mid u \mid \le 1. \end{cases}$$

In this case, $V(x) \supseteq \{l \in R^2 \setminus \{0\} / l_1 = -x_2, \mid l_2 \mid \le 1\}$,

$$\omega(x) = \begin{cases} -[-x_2 + 2(x_1 + 2^{-1}x_2^2)^{\frac{1}{2}}], & x_2 \le \varphi(x_1), \\ -[x_2 + 2(-x_1 + 2^{-1}x_2^2)^{\frac{1}{2}}], & x_2 \ge \varphi(x_1), \end{cases}$$

where

$$\varphi(x_1) = \begin{cases} -(-2x_1)^{\frac{1}{2}}, & x_1 \le 0, \\ (2x_1)^{\frac{1}{2}}, & x_1 \ge 0, \end{cases}$$

$$\bar{l}(x) = \begin{cases} (-x_2, -1), \ (x_2 > \varphi(x_1), \ x_1 \le 0) \text{ or } (x_2 \ge \varphi(x_1), \ x_1 > 0), \\ (-x_2, 1), \ (x_2 \le \varphi(x_1), \ x_1 < 0) \text{ or } (x_2 < \varphi(x_1), \ x_1 \ge 0). \end{cases}$$

On the optimal trajectory $x_2 = \varphi(x_1)$, the Bellman's equation

$$\sup_{|u| \le 1} \sum_{i=1}^{2} \frac{\partial \omega}{\partial x_i} f_i(x, u) = 1$$

has no sense, as the partial derivative $\frac{\partial \omega}{\partial x_i}$ of ω with respect to x_i do not exist. However, on this optimal trajectory (for $x \neq 0$) $\partial\omega(x, \bar{l}(x)) = \omega'(x, \bar{l}(x)) = \{1\}$. Thus, using Corollary 1, we conclude that

$$\sup_{l \in V(x)} \sup \partial\omega(x, l) = \sup \partial\omega(x, \bar{l}(x)) = \sup \omega'(x, \bar{l}(x)) = 1.$$

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