One Aspect On Bellman’s Equation To Optimal Control Theory*

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Abstract

In this present work, we develop the idea of the dynamic programming approach. The main observation is that the Bellman function \( \omega(x, t) \) which is the function that provides, for any given state \( x \) at any given time \( t \), the smallest possible cost among all possible trajectories starting at this event is in general not differentiable, and consequently we cannot use the Hamilton-Jacobi-Bellman (HJB) equation. By the classical Hamilton-Jacobi-Bellman theory if the value function \( \omega \) is continuously differentiable, then it is the unique solution of the (HJB) equation. It is well known that the value function \( \omega \) is in general discontinuous, even if all the data of the problem is continuously differentiable. Using the (HJB) equation in some nonclassical sense (e.g. using generalized gradients, in the framework of viscosity solutions, proximal solutions, etc.) has become a very active research area. Here, we give techniques based on the differential set \( \partial \omega(x, l) \) of the function \( \omega \) at the element \( x \) along the direction \( l \) for the analysis of such problems.

1 Description of the problem

Let \( X \) a Banach space, \( B \) a closed unit ball in \( X \), \( W \) a nonempty subset of \( X \). Let \( l \in X \), \( x \in W \). Assume that \( l, x \) are such that \( x + \delta l \in W \) for \( \delta \) small enough. Let \( \rho : W \rightarrow R \). We define \( \partial \rho(x, l) \) as follows :

\[
\partial \rho(x, l) = \bigcap_{\varepsilon > 0} \bigcup_{\delta < \varepsilon} \delta^{-1}(\rho(x + \delta l) - \rho(x)).
\]

\( \partial \rho(x, l) \) is called the differential set of the function \( \rho \) at the element \( x \) along the direction \( l \). If \( \partial \rho(x, l) \) is a one-element set, we will then say that \( \rho \) is differentiable along the direction \( l \), \( (l \neq 0) \) at \( x \), (see [5]). In this case, we write simply \( \partial \rho(x, l) = \rho'(x, l) \). Let

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Put $\xi$ and such that with the initial condition

$$\dot{x}(t) \in P(x(t))$$

(1)

with the initial condition

$$x(0) = x_0,$$

(2)

where $x : [0, T) \to W$ is assumed to be locally absolutely continuous on $[0, T)$ and $\exists \dot{x}(t) \in P(x(t))$ almost everywhere on each compact interval of $[0, T)$, $x_0 \in W$. Pick any $x_1 \in W$. Denote by $D(\omega)$ the set of all $x_0 \in W$ such that $x_0$ can be controlled to $x_1$ along some trajectory $x(.)$ in some time $t(x_0, x(.)) < T$. Put

$$\omega(x_0) = \sup(-t(x_0, x(.))), \text{ for } x_0 \in D(\omega).$$

So the problem we wish to study is: Given a system (1), a trajectory cost function $t(x_0, x(.))$, and a state $x_0$, find an admissible trajectory $\bar{x}$ for $x_0$ such that $x_0$ can be controlled to $x_1$ along the trajectory $\bar{x}$ and $\bar{x}$ minimizes the cost function $t(x_0, x(.)).$

DEFINITION 1. Assume that $x \in D(\omega)$, $l \in P(x)$. $l$ is said to be an admissible direction for $x$, if $l \neq 0$ and $\forall \epsilon > 0 \exists \delta \in (0, \epsilon) \forall x' \in x + (0, \delta) l \exists t' \in (0, \epsilon) \exists x(.) \in W^1_1([0, t'], X)$ such that $a) \dot{x}(t) \in P(x(t))$ almost everywhere on $[0, t'];$ $b) x(0) = x;$ $c) x(t') = x'$; and $d) \|x' - x - lt'\| \leq t' \epsilon$, where $W^1_1([0, t'], X)$ is the Sobolev space.

We denote by $V(x)$ the set of all directions $l \in P(x)$ such that the assumptions in Definition 1 hold.

2 On Bellman’s equation.

Assume that all the hypotheses of the above Section hold. Then we have the following result.

THEOREM 1. Let $x \in D(\omega)$ such that $V(x) \neq \emptyset$. Then

$$\sup_{l \in V(x)} \sup_{l \in V(x)} \partial \omega(x, l) \leq 1.$$ 

PROOF. Suppose to the contrary that $\sup_{l \in V(x)} \sup_{l \in V(x)} \partial \omega(x, l) > 1$. Then

$$\exists \sigma > 0 \forall l \in V(x) \forall \xi > 0 \exists \eta \in (0, \xi)$$

such that

$$x + \eta l \in D(\omega)$$

and

$$\omega(x + \eta l) - \omega(x) > (1 + 2\sigma) \eta.$$ 

Put $\xi_n = n^{-1}$ where $n \in N$. We may assume that $\exists \eta_n \in (0, \xi_n)$ such that $x + \eta_n l \in D(\omega)$ and

$$\omega(x + \eta_n l) - \omega(x) > (1 + 2\sigma) \eta_n.$$
Let $\varepsilon_k = k^{-1}$, $k \in N$. Since $l \in V(x)$, it follows that $\exists \delta_k \in (0, \varepsilon_k)$ such that the conditions of Definition 1 holds. For every $k \in N$, let us take $n_k \geq k$ such that $\xi_{n_k} < \delta_k$ and $\eta_{n_k} < \delta_k$. Put $x_k' = x + \eta_{n_k} l$. Then, there exists $t_k \in (0, \varepsilon_k)$ and $x_k(\cdot) \in W_1^t([0, t_k], X)$ such that

$$x_k(t) \in P(x_k(t)) \text{ almost everywhere on } [0, t_k]$$

and

$$x(0) = x, \ x(t_k) = x_k', \ |x_k' - x - lt_k| \leq t_k \varepsilon_k.$$  

Let us remark that if $\eta_{n_k}^{-1}l \to 1$, as $k \to \infty$. We claim that the sequence $\eta_{n_k}^{-1}l \to 1$, as $k \to \infty$. Indeed, suppose that the contrary holds, then we can suppose without loss of generality that $\eta_{n_k}^{-1}t_k \to \infty$. Next, we have

$$t_k^{-1}(\eta_{n_k} - t_k) = t_k^{-1}[x + x_k' + x - x_k' - t_k \varepsilon_k h_k'] = -t_k^{-1}(t_k \varepsilon_k h_k') = -\varepsilon_k h_k' \to 0$$

as $k \to \infty$. But, on the other hand, $t_k^{-1}(\eta_{n_k} - t_k) = (\eta_{n_k}^{-1}l - 1)l \to 0$, contradicting the assumption $l \neq 0$. Thus $\eta_{n_k}^{-1}t_k$ is bounded.

Analogously for $\eta_{n_k}^{-1}l$, we claim that $\eta_{n_k}^{-1}l$ is also bounded. Otherwise, computing the value of $\eta_{n_k}^{-1}l(\eta_{n_k} - t_k)$, we obtain

$$\eta_{n_k}^{-1}l(\eta_{n_k} - t_k) = -\eta_{n_k}^{-1}l t_k \varepsilon_k h_k' \to 0.$$  

So, as $\eta_{n_k}^{-1}l(\eta_{n_k} - t_k) = (1 - \eta_{n_k}^{-1}l t_k)l \to l \neq 0$, we conclude that $\eta_{n_k}^{-1}l$ is bounded.

Observing that

$$(t_k^{-1} - 1)l = t_k^{-1}(\eta_{n_k} - t_k) = -\varepsilon_k h_k' \to 0,$$

we deduce that $\eta_{n_k}^{-1}l \to 1$. Finally, $(1 + 2\sigma)\eta_{n_k} > (1 + \sigma)t_k$ for $k$ large enough.

Note that $x$ can be controlled to $x_k'$ in time $t_k$. Thus, the system (1) will be controlled from the initial state $x$ to the state $x_k'$ in time $t_k$ along the trajectory $x_k(\cdot)$, and the system (1) will be also controlled from the state $x_k'$ to the final state $x_1$ in time $\tilde{t}_k = -\omega(x_k') < -\omega(x_k') + \sigma t_k$. It will then follow that the system (1) will be controlled from the initial state $x$ to the final state $x_1$ in time

$$t_k + \tilde{t}_k < t_k - \omega(x_k') + \sigma t_k = -\omega(x_k') + (1 + \sigma)t_k < -\omega(x_k') + (1 + 2\sigma)\eta_{n_k} \leq -\omega(x),$$  

contradicting the definition of $\omega$.

Suppose now that $x_0 \neq x_1$. Let $\bar{x}$ the optimal trajectory that achieves the transfer of the system (1) from the initial state $x_0$ to the final state $x_1$ on $[0, -\omega(x_0)]$. Let us remark that if $t \neq t'$ and $t, t' \leq -\omega(x_0)$, then $\bar{x}(t) \neq \bar{x}(t')$, i.e.e the time length $t \leq -\omega(x_0)$ to achieve the transfer of the system (1) along the trajectory $\bar{x}$ from the initial state $x_0 = \bar{x}(0)$ to the final state $x = \bar{x}(t)$ is an invertible function $t = t(x)$ on the optimal trajectory $\bar{x}$.
In this case, Bellman’s principle it follows that approach given here, the above problem can be resolved. Bellman equation (HJB) cannot be used, but with the new dynamic programming

\[ \forall t_n \in \mathbb{D}(\omega), \text{ where } \varepsilon_n \to 0. \]

Then the proof follows from Corollary 1 and the definition of differential set. Consequently, \( \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n > n_0: \]

\[
\bigcup_{0 < \delta < \varepsilon} \delta^{-1}[\omega(x + \delta \tilde{l}(x)) - \omega(x)] \ni t_n^{-1}(\omega[x + t_n \tilde{l}(x)] - \omega(x)) = t_n^{-1}(\tilde{l}(x) + t_n) - \omega(x)) = t_n^{-1}t_n = 1.
\]

The proof is complete.

**COROLLARY 1.** Let \( x \in M(\tilde{x}(.)) \). Suppose that \( \tilde{l}(x) \in V(x), 1 \in \partial \omega(x, \tilde{l}(x)). \) Then

\[
\sup_{l \in V(x)} \sup_{t \in V(x)} \partial \omega(x, l) = \sup_{l \in V(x)} \partial \omega(x, \tilde{l}(x)) = 1.
\]

In the particular case when \( r \equiv 0 \) and \( \tilde{l}(x) \in V(x) \), the same conclusion holds also.

The proof follows from Theorems 1 and 2.

**COROLLARY 2.** Assume that \( x \in M(\tilde{x}(.)), r \equiv 0, \omega \) is differentiable at \( x \) along every direction \( l \in \tilde{U}(x) \subset V(x) \). Assume also that \( \tilde{l}(x) \in \tilde{U}(x) \). Then

\[
\sup_{l \in \tilde{U}(x)} \omega'(x, l) = \omega'(x, \tilde{l}(x)) = 1.
\]

The proof follows from Corollary 1 and the definition of differential set.

Now we give an example of optimal control problem when the Hamilton-Jacobi-Bellman equation (HJB) cannot be used, but with the new dynamic programming approach given here, the above problem can be resolved.

**EXAMPLE 1.** Consider the following system

\[
(S) \begin{cases} 
  x_1 &= -x_2 = f_1(x, u), \\
  x_2 &= -u = f_2(x, u), \\
  u &\leq 1.
\end{cases}
\]

In this case, \( V(x) \supseteq \{ l \in \mathbb{R}^2 \setminus \{0\} \mid l_1 = -x_2, |l_2| \leq 1 \}, \)

\[
\omega(x) = \begin{cases} 
  -[x_2 + 2(x_1 + 2^{-1}x_2^{\frac{1}{2}})], & x_2 \leq \varphi(x_1), \\
  -[x_2 + 2(-x_1 + 2^{-1}x_2^{\frac{1}{2}})], & x_2 \geq \varphi(x_1),
\end{cases}
\]
where
\[
\varphi(x_1) = \begin{cases} 
-(-2x_1)^{\frac{1}{2}}, & x_1 \leq 0, \\
(2x_1)^{\frac{1}{2}}, & x_1 \geq 0,
\end{cases}
\]
\[
\bar{l}(x) = \begin{cases} 
(-x_2, -1), & (x_2 > \varphi(x_1), x_1 \leq 0) \text{ or } (x_2 \geq \varphi(x_1), x_1 > 0), \\
(-x_2, 1), & (x_2 \leq \varphi(x_1), x_1 < 0) \text{ or } (x_2 < \varphi(x_1), x_1 \geq 0).
\end{cases}
\]

On the optimal trajectory \( x_2 = \varphi(x_1) \), the Bellman’s equation
\[
\sup_{|u| \leq 1} \sum_{i=1}^{2} \frac{\partial \omega}{\partial x_i} f_i(x, u) = 1
\]
has no sense, as the partial derivative \( \frac{\partial \omega}{\partial x_i} \) of \( \omega \) with respect to \( x_i \) do not exist. However, on this optimal trajectory (for \( x \neq 0 \)) \( \partial \omega(x, l(x)) = \omega'(x, l(x)) = \{1\} \). Thus, using Corollary 1, we conclude that
\[
\sup_{l \in V(x)} \sup_{i \in V(x)} \partial \omega(x, l) = \partial \omega(x, l(x)) = \omega'(x, l(x)) = 1.
\]

References