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A Fixed Point Theorem For Multivalued Maps In Symmetric Spaces *

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Abstract

The main purpose of this paper is to give a generalization of the well-known Nadler multivalued contraction fixed point to the setting of symmetric spaces. We apply our main result to obtain a new fixed point theorem for multivalued mappings in probabilistic spaces.

1 Introduction

Let (X, d) be a metric space. Let $(CB(X), H)$ denote the hyperspaces of nonempty closed bounded subsets of X, where H is the Hausdorff metric induced by d, i.e.,

$$H(A, B) = \max\left\{\sup_{b\in B} d(b, A);\ \sup_{a\in A} d(a, B)\right\}$$

for all $A, B \in CB(X)$, where $d(x, A) = \inf\{d(x, a)|\ a \in A\}$. In [3], Nadler proved the following important fixed point result, which has been used and extended in many different directions, for contraction multivalued operators: Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multivalued mapping such that:

$$H(Tx, Ty) \le kd(x, y),\ k \in [0, 1),\ \forall x, y \in X$$

Then, there exists $u \in X$ such that $u \in Tu$.

Although the fixed point theory for single valued maps is very rich and well developed, the multivalued case is not. Note that multivalued mappings play a major role in many areas as in studying disjunctive logic programs.

On the other hand, it has been observed (see for example [1], [2]) that the distance function used in certain metric theorems proofs need not satisfy the triangular inequality nor $d(x, x) = 0$ for all x. Motivated by this fact, Hicks and Rhoades [1] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that: (1) $d(x, y) = 0$ if and only if $x = y$, (2) $d(x, y) = d(y, x)$.

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Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x,r) = \{y \in X : d(x,y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x,r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x,r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n,x) = 0$ if and only if $x_n \longrightarrow x$ in the topology $t(d)$.

In order to unify the notation (see Theorem 2.2.1, Corollary 2.2.1 and Remark 2.2.1), we need the following two axioms $(W.3)$ and $(W.4)$ given by Wilson [5] in a symmetric space (X,d):

$(W.3)$ Given $\{x_n\}, x$ and y in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y) = 0$ imply $x = y$.

$(W.4)$ Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n,x) = 0$.

It is easy to see that for a semi-metric d, if $t(d)$ is Hausdorff, then $(W.3)$ holds.

A sequence in X is called a d-Cauchy sequence if it satisfies the usual metric condition. There are several concepts of completeness in this setting (see [1]):

(i) X is S-complete if for every d-Cauchy sequence (x_n), there exists x in X with $\lim_{n\to\infty} d(x,x_n) = 0$.

(ii) X is d-Cauchy complete if for every d-Cauchy sequence $\{x_n\}$, there exists x in X with $x_n \to x$ in the topology $t(d)$.

REMARK 1.1. Let (X,d) be a symmetric space and let $\{x_n\}$ be a d-Cauchy sequence. If X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} d(x,x_n) = 0$. Therefore S-completeness implies d-Cauchy completeness.

2 Main results

2.1 The Hausdorff distance in a symmetric space

DEFINITION 2.1.1. Let (X,d) be a symmetric space and A a nonempty subset of X.

(1) We say that A is d-closed iff $\overline{A}^d = A$ where

$$\overline{A}^d = \{x \in X : d(x,A) = 0\} \text{ and } d(x,A) = \inf\{d(x,y) : y \in A\}.$$

(2) We say that A is d-bounded iff $\delta_d(A) < \infty$ where $\delta_d(A) = \sup\{d(x,y) : x,y \in A\}$.

The following definition is a generalization of the well-known Hausdorff distance to the setting of symmetric case.

DEFINITION 2.1.2. Let (X,d) be a d-bounded symmetric space and let $C(X)$ be the set of all nonempty d-closed subsets of (X,d). Consider the function $D : 2^X \times 2^X \longrightarrow I\!R^+$ defined by

$$D(A,B) = \max\left\{\sup_{a\in A} d(a,B);\ \sup_{b\in B} d(A,b)\right\}$$

for all $A, B \in C(X)$.

REMARK 2.1.1. It is easy to see that $(C(X), D)$ is a symmetric space.

For our main Theorem we need the following Lemma. It is used in many papers for metric spaces. The proof is straightforward.

LEMMA 2.1.1. Let (X, d) be a d-bounded symmetric space. Let $A, B \in C(X)$ and $\alpha > 1$. For each $a \in A$, there exists $b \in B$ such that: $d(a, b) \leq \alpha D(A, B)$.

2.2 Fixed point Theorem

Now we are ready to prove our main Theorem which yields the Nadler's fixed point Theorem in a new setting.

THEOREM 2.2.1. Let (X, d) be a d-bounded and S-complete symmetric space satisfying $(W.4)$ and $T : X \longrightarrow C(X)$ be a multivalued mapping such that:

$$D(Tx, Ty) \leq kd(x, y), \quad k \in [0, 1), \quad \forall x, y \in X \tag{1}$$

Then there exists $u \in X$ such that $u \in Tu$.

PROOF. Let $x_1 \in X$ and $\alpha \in (k, 1)$. Since Tx_1 is nonempty, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) > 0$ (if not, then x_1 is a fixed point of T). In view of (1), we have:

$$d(x_2, Tx_2) \leq D(Tx_1, Tx_2) \leq kd(x_1, x_2) < \alpha d(x_1, x_2)$$

using $d(x_2, Tx_2) = \inf\{d(x_2, b) : b \in Tx_2\}$, it follows that there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < \alpha d(x_1, x_2).$$

Similarly, there exists $x_4 \in Tx_3$ such that

$$d(x_3, x_4) < \alpha d(x_2, x_3).$$

Continuing in this fashion, there exists a sequence $\{x_n\}$ in X satisfying $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) < \alpha d(x_{n-1}, x_n).$$

We claim that $\{x_n\}$ is a d-Cauchy sequence. Indeed, we have

$$\begin{aligned} d(x_n, x_{n+m}) \quad & < \alpha d(x_{n-1}, x_{n+m-1}) \\ & < \alpha^2(d(x_{n-2}, x_{n+m-2}))... \\ & < ... < \alpha^{n-1}(d(x_1, x_{m+1})) \\ & < \alpha^{n-1}\delta_d(X). \end{aligned}$$

So $\{x_n\}$ is a d-Cauchy sequence. Hence $\lim_{n\to\infty} d(u, x_n) = 0$ for some $u \in X$. Now we are able to show that $u \in Tu$. Let $\epsilon > 1$. From Lemma 2.1.1, for each $n \in \{1, 2, ...\}$ there exists $y_n \in Tu$ such that:

$$d(x_{n+1}, y_n) \leq \epsilon D(Tx_n, Tu) \leq \epsilon kd(x_n, u), \ n = 1, 2, ... \ .$$

which implies that $\lim_{n\to\infty} d(x_{n+1}, y_n) = 0$. In view of $(W.4)$, we have $\lim_{n\to\infty} d(y_n, u) = 0$ and therefore $u \in \overline{Tu}^d = Tu$. The proof is complete.

If T is a single-valued mapping of a symmetric space (X,d) into itself, we obtain the following known result [1, Theorem1] for $f = Id_X$ which generalizes [2, Proposition 1].

COROLLARY 2.2.1. Let (X,d) be a d-bounded and S-complete symmetric space satisfying $(W.4)$ and T be a selfmapping of X such that

$$d(Tx,Ty) \leq kd(x,y), \quad k \in [0,1[, \quad \forall x,y \in X.$$

Then T has a fixed point.

REMARK 2.2.1. It is clear that in corollary 2.2.1, the fixed point is unique. Moreover, it is easy to see that the condition (W.4)[5] implies (W.3)[5] which guarantees the uniqueness of limits of sequences.

2.3 Application

Throughout this section, a distribution function f is a nondecreasing, left continuous real-valued function defined on the set of real numbers, with $\inf f = 0$ and $\sup f = 1$.

DEFINITION 2.3.1. Let X be a set and $\Im$ a function defined on $X \times X$ such that $\Im(x,y) = F_{x,y}$ is a distribution function. Consider the following conditions:

I. $F_{x,y}(0) = 0$ for all $x,y \in X$.

II. $F_{x,y} = H$ if and only if $x = y$, where H denotes the distribution function defined by $H(x) = \begin{cases} 0 & if\ x \leq 0 \\ 1 & if\ x > 0 \end{cases}$

III. $F_{x,y} = F_{y,x}$.

IV. If $F_{x,y}(\epsilon) = 1$ and $F_{y,z}(\delta) = 1$ then $F_{x,z}(\epsilon + \delta) = 1$.

If $\Im$ satisfies I and II, then it is called a PPM-structure on X and the pair $(X,\Im)$ is called a PPM space. An $\Im$ satisfying III is said to be symmetric. A symmetric PPM-structure $\Im$ satisfying IV is a probabilistic metric structure and the pair $(X,\Im)$ is a probabilistic metric space.

Let $(X,\Im)$ be a symmetric PPM-space. For $\epsilon, \lambda > 0$ and x in X, let $N_x(\epsilon,\lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}$. A T_1 topology $t(\Im)$ on X is defined as follows:

$$t(\Im) = \{U \subseteq X |\ \text{for each } x \in U, \text{ there exists } \epsilon > 0, \text{ such that } N_x(\epsilon,\epsilon) \subseteq U\}.$$

Recall that a sequence $\{x_n\}$ is called a fundamental sequence if $\lim_{n\to\infty} F_{x_n,x_m}(t) = 1$ for all $t > 0$. The space $(X,\Im)$ is called F-complete if for every fundamental sequence $\{x_n\}$ there exists x in X such that $\lim_{n\to\infty} F_{x_n,x}(t) = 1$, for all $t > 0$. Note that condition $(W.4)$, defined earlier, is equivalent to the following condition:

$P(4)$ $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ and $\lim_{n\to\infty} F_{x_n,y_n}(t) = 1$ imply $\lim_{n\to\infty} F_{y_n,x}(t) = 1$, for all $t > 0$.

In [1], Hicks and Rhoades proved that each symmetric PPM-space admits a compatible symmetric function as follows:

THEOREM 2[1] Let $(X, \Im)$ be a symmetric PPM-space. Let $p : X \times X \longrightarrow I\!R^+$ be a function defined as follows:

$$p(x,y) = \begin{cases} 0 & \text{if } y \in N_x(t,t) \text{ for all } t > 0. \\ \sup\{t:\ y \notin N_x(t,t),\ 0 < t < 1\} & \text{otherwise.} \end{cases}$$

Then

(1) $p(x,y) < t$ if and only if $F_{x,y}(t) > 1 - t$.

(2) p is a compatible symmetric for $t(\Im)$.

(3) $(X, \Im)$ is F-complete if and only if (X, p) is S-complete.

For our main result in this section, we need the following new Proposition:

PROPOSITION 2.3.1. Let $(X, \Im)$ be a symmetric PPM-space and p a compatible symmetric function for $t(\Im)$. For $A, B \in C(X)$, set

$$E_{A,B}(\epsilon) = \min\left\{\inf_{a\in A}\sup_{b\in B}F_{a,b}(\epsilon),\ \inf_{b\in B}\sup_{a\in A}F_{a,b}(\epsilon)\right\},\ \epsilon > 0.$$

and

$$P(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}p(a,b); \sup_{b\in B}\inf_{a\in A}p(a,b)\right\}.$$

If $T : X \to C(X)$ is a multivalued mapping, then we have

$$F_{x,y}(t) > 1 - t \text{ implies } E_{Tx,Ty}(kt) > 1 - kt,\ k \in [0,1),\ \forall t > 0,\ \forall x, y \in X.$$

implies that $P(Tx, Ty) \leq kp(x, y)$.

PROOF. Let $t > 0$ be given and set $\mu = p(x,y) + t$. Then $p(x,y) = \mu - t < \mu$ gives $F_{x,y}(\mu) > 1 - \mu$, and therefore $E_{Tx,Ty}(k\mu) > 1 - k\mu$. Then

$$\begin{cases} \inf_{a\in Tx}\sup_{b\in Ty}F_{a,b}(k\mu) > 1 - k\mu \\ \inf_{b\in Ty}\sup_{a\in Tx}F_{a,b}(\mu) > 1 - k\mu \end{cases}$$

$$\Longrightarrow \begin{cases} \forall a \in Tx, \exists b \in Ty,\ F_{a,b}(k\mu) > 1 - k\mu \\ \forall b \in Ty, \exists a \in Tx,\ F_{a,b}(k\mu) > 1 - k\mu \end{cases}$$

$$\Longrightarrow \begin{cases} \forall a \in Tx, \exists b \in Ty,\ p(a,b) < k\mu \\ \forall b \in Ty, \exists a \in Tx,\ p(a,b) < k\mu\mu \end{cases}$$

then

$$\sup_{a\in Tx}\inf_{b\in Ty}p(a,b) < k\mu \text{ and } \sup_{b\in Ty}\inf_{a\in Tx}p(a,b) < k\mu$$

and therefore $P(Tx, Ty) < k\mu = k(p(x,y) + t)$. Since $t > 0$ was arbitrary, it follows $P(Tx, Ty) \leq kp(x, y)$.

DEFINITION 2.3.2. Let $(X, \Im)$ be a symmetric PPM space and A a nonempty subset of X. We say that A is $\Im$-closed iff $\overline{A}^{\Im} = A$ where

$$\overline{A}^{\Im} = \{x \in X : \sup_{a \in A} F_{x,a}(t) = 1, \text{ for all } t > 0\}.$$

We denote by $C_{\Im}(X)$ the set of all nonempty $\Im$-closed subsets of X.

REMARK 2.3.1. Let $(X, \Im)$ be a symmetric PPM space and $C_{\Im}(X)$ be the set of all nonempty $\Im$-closed subsets of X. It is not hard to see that if p is a compatible symmetric function for $t(\Im)$ then $C_{\Im}(X) = C(X)$ where $C(X)$ is the set of all nonempty p-closed subsets of (X, p).

Now we are able to state and prove an application of our main Theorem 2.2.1 in the following way

THEOREM 2.3.1. Let $(X, \Im)$ be a F-complete symmetric PPM space that satisfies $(P.4)$ and p a compatible symmetric function for $t(\Im)$. Let $T : X \longrightarrow C_{\Im}(X)$ be a multivalued mapping such that:

$$F_{x,y}(t) > 1 - t \text{ implies } E_{Tx,Ty}(kt) > 1 - kt, \; k \in [0,1), \; \forall t > 0, \; \forall x, y \in X.$$

Then there exists $u \in X$ such that $u \in Tu$.

PROOF. Note that (X, p) is bounded and S-complete. Also $p(x, y) < t$ if and only if $F_{x,y}(t) > 1 - t$. Let $\epsilon > 0$ be given, and set $t = p(x, y) + \epsilon$. Then $p(x, y) < t$ gives $F_{x,y}(t) > 1 - t$ and therefore $E_{Tx,Ty}(kt) > 1 - kt$. In view of Proposition 2.3.1, it follows that $P(Tx, Ty) \leq kt = k(p(x, y) + \epsilon)$. On letting ϵ to 0 (since $\epsilon > 0$ is arbitrary), we have $p(Tx, Ty) \leq kp(x, y)$. Now apply Theorem 2.2.1.

For a single-valued selfmapping T, Theorem 2.3.1 is reduced to the following known result:

COROLLARY 2.3.1. Let $(X, \Im)$ be a F-complete symmetric PPM space that satisfies $(P.4)$ and p a compatible symmetric function for $t(\Im)$. Let T be a selfmapping of X satisfying

$$F_{x,y}(t) > 1 - t \text{ implies } F_{Tx,Ty}(kt) > 1 - kt, \; k \in [0,1), \; \forall t > 0, \; \forall x, y \in X.$$

Then T has a fixed point.

References

[1] T. L. Hicks and B. E. Rhoades, Fixed point theory in symmetric spaces with applications to probabilistic spaces, Nonlinear Analysis 36(1999), 331–344.

[2] J. Jachymski, J. Matkowski and T. Swiatkowski, Nonlinear contractions on semimetric spaces, J. Appl. Anal. 1(1995), 125–134.

[3] S. B. Nadler, Multivalued contraction mappings, Pacific. J. Math. 30(1969), 475–488.

[4] B. Schweizer and A. Sklar, Probabilistic metric spaces, North-Holland, Amsterdam, 1983.

[5] W. A. Wilson, On semi-metric spaces, Amer. J. Math. 53(1931), 361-373.