A Fixed Point Theorem For Multivalued Maps In Symmetric Spaces *

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Abstract

The main purpose of this paper is to give a generalization of the well-known Nadler multivalued contraction fixed point to the setting of symmetric spaces. We apply our main result to obtain a new fixed point theorem for multivalued mappings in probabilistic spaces.

1 Introduction

Let \((X, d)\) be a metric space. Let \((CB(X), H)\) denote the hyperspaces of nonempty closed bounded subsets of \(X\), where \(H\) is the Hausdorff metric induced by \(d\), i.e.,

\[
H(A, B) = \max \left\{ \sup_{b \in B} d(a, B); \sup_{a \in A} d(b, A) \right\}
\]

for all \(A, B \in CB(X)\), where \(d(x, A) = \inf\{d(x, a) | a \in A\}\). In [3], Nadler proved the following important fixed point result, which has been used and extended in many different directions, for contraction multivalued operators: Let \((X, d)\) be a complete metric space and \(T : X \rightarrow CB(X)\) a multivalued mapping such that:

\[
H(Tx, Ty) \leq kd(x, y), \quad k \in [0, 1), \quad \forall x, y \in X
\]

Then, there exists \(u \in X\) such that \(u \in Tu\).

Although the fixed point theory for single valued maps is very rich and well developed, the multivalued case is not. Note that multivalued mappings play a major role in many areas as in studying disjunctive logic programs.

On the other hand, it has been observed (see for example [1], [2]) that the distance function used in certain metric theorems proofs need not satisfy the triangular inequality nor \(d(x, x) = 0\) for all \(x\). Motivated by this fact, Hicks and Rhoades [1] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Recall that a symmetric on a set \(X\) is a nonnegative real valued function \(d\) on \(X \times X\) such that: (1) \(d(x, y) = 0\) if and only if \(x = y\), (2) \(d(x, y) = d(y, x)\).

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Let $d$ be a symmetric on a set $X$ and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $t(d)$ on $X$ is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, r) \subseteq U$ for some $r > 0$. A symmetric $d$ is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of $x$ in the topology $t(d)$. Note that $\lim_{n \to \infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

In order to unify the notation (see Theorem 2.2.1, Corollary 2.2.1 and Remark 2.2.1), we need the following two axioms (W.3) and (W.4) given by Wilson [5] in a symmetric space $(X, d)$:

(W.3) Given $\{x_n\}$, $x$ and $y$ in $X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y) = 0$ imply $x = y$.

(W.4) Given $\{x_n\}$, $\{y_n\}$ and $x$ in $X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y_n) = 0$ imply that $\lim_{n \to \infty} d(y_n, x) = 0$.

It is easy to see that for a semi-metric $d$, if $t(d)$ is Hausdorff, then (W.3) holds.

A sequence in $X$ is called a $d$-Cauchy sequence if it satisfies the usual metric condition. There are several concepts of completeness in this setting (see [1]):

(i) $X$ is S-complete if for every d-Cauchy sequence $(x_n)$, there exists $x$ in $X$ with $\lim_{n \to \infty} d(x, x_n) = 0$.

(ii) $X$ is d-Cauchy complete if for every d-Cauchy sequence $\{x_n\}$, there exists $x$ in $X$ with $x_n \to x$ in the topology $t(d)$.

REMARK 1.1. Let $(X, d)$ be a symmetric space and let $\{x_n\}$ be a d-Cauchy sequence. If $X$ is S-complete, then there exists $x$ in $X$ such that $\lim_{n \to \infty} d(x, x_n) = 0$. Therefore S-completeness implies d-Cauchy completeness.

## 2 Main results

### 2.1 The Hausdorff distance in a symmetric space

DEFINITION 2.1.1. Let $(X, d)$ be a symmetric space and $A$ a nonempty subset of $X$.

(1) We say that $A$ is $d$-closed iff $\overline{A}^d = A$ where

$$\overline{A}^d = \{x \in X : d(x, A) = 0\} \text{ and } d(x, A) = \inf\{d(x, y) : y \in A\}.$$

(2) We say that $A$ is $d$-bounded iff $\delta_d(A) < \infty$ where $\delta_d(A) = \sup\{d(x, y) : x, y \in A\}$.

The following definition is a generalization of the well-known Hausdorff distance to the setting of symmetric case.

DEFINITION 2.1.2. Let $(X, d)$ be a $d$-bounded symmetric space and let $C(X)$ be the set of all nonempty $d$-closed subsets of $(X, d)$. Consider the function $D : 2^X \times 2^X \to \mathbb{R}^+$ defined by

$$D(A, B) = \max \left\{ \sup_{a \in A} d(a, B); \sup_{b \in B} d(A, b) \right\}$$
for all \( A, B \in C(X) \).

**Remark 2.1.1.** It is easy to see that \((C(X), D)\) is a symmetric space.

For our main Theorem we need the following Lemma. It is used in many papers for metric spaces. The proof is straightforward.

**Lemma 2.1.1.** Let \( (X, d) \) be a \( d \)-bounded symmetric space. Let \( A, B \in C(X) \) and \( \alpha > 1 \). For each \( a \in A \), there exists \( b \in B \) such that: \( d(a, b) \leq \alpha D(A, B) \).

### 2.2 Fixed Point Theorem

Now we are ready to prove our main Theorem which yields the Nadler’s fixed point Theorem in a new setting.

**Theorem 2.2.1.** Let \( (X, d) \) be a \( d \)-bounded and \( S \)-complete symmetric space satisfying \((W.4)\) and \( T : X \to C(X) \) be a multivalued mapping such that:

\[
D(Tx, Ty) \leq kd(x, y), \quad k \in [0, 1), \quad \forall x, y \in X \tag{1}
\]

Then there exists \( u \in X \) such that \( u \in Tu \).

**Proof.** Let \( x_1 \in X \) and \( \alpha \in (k, 1) \). Since \( Tx_1 \) is nonempty, there exists \( x_2 \in Tx_1 \) such that \( d(x_1, x_2) > 0 \) (if not, then \( x_1 \) is a fixed point of \( T \)). In view of (1), we have:

\[
d(x_2, Tx_2) \leq D(Tx_1, Tx_2) \leq kd(x_1, x_2) < \alpha d(x_1, x_2)
\]

using \( d(x_2, Tx_2) = \inf \{d(x_2, b) : b \in Tx_2\} \), it follows that there exists \( x_3 \in Tx_2 \) such that

\[
d(x_2, x_3) < \alpha d(x_1, x_2).
\]

Similarly, there exists \( x_4 \in Tx_3 \) such that

\[
d(x_3, x_4) < \alpha d(x_2, x_3).
\]

Continuing in this fashion, there exists a sequence \( \{x_n\} \) in \( X \) satisfying \( x_{n+1} \in Tx_n \) and

\[
d(x_n, x_{n+1}) < \alpha d(x_{n-1}, x_n).
\]

We claim that \( \{x_n\} \) is a \( d \)-Cauchy sequence. Indeed, we have

\[
d(x_n, x_{n+m}) < \alpha d(x_{n-1}, x_{n+m-1}) < \alpha^2 d(x_{n-2}, x_{n+m-2}) < \cdots < \alpha^{n-1} d(x_1, x_{m+1}) < \alpha^{n-1} \delta_d(X).
\]

So \( \{x_n\} \) is a \( d \)-Cauchy sequence. Hence \( \lim_{n \to \infty} d(u, x_n) = 0 \) for some \( u \in X \). Now we are able to show that \( u \in Tu \). Let \( \epsilon > 1 \). From Lemma 2.1.1, for each \( n \in \{1, 2, \ldots\} \) there exists \( y_n \in Tu \) such that:

\[
d(x_{n+1}, y_n) \leq \epsilon D(Tx_n, Tu) \leq \epsilon kd(x_n, u), \quad n = 1, 2, \ldots
\]

which implies that \( \lim_{n \to \infty} d(x_{n+1}, y_n) = 0 \). In view of \((W.4)\), we have \( \lim_{n \to \infty} d(y_n, u) = 0 \) and therefore \( u \in Tu^d = Tu \). The proof is complete.
If $T$ is a single-valued mapping of a symmetric space $(X, d)$ into itself, we obtain the following known result [1, Theorem 1] for $f = Id_X$ which generalizes [2, Proposition 1].

COROLLARY 2.2.1. Let $(X, d)$ be a $d$-bounded and $S$-complete symmetric space satisfying (W.4) and $T$ be a selfmapping of $X$ such that

\[ d(Tx, Ty) \leq kd(x, y), \quad k \in [0, 1], \quad \forall x, y \in X. \]

Then $T$ has a fixed point.

REMARK 2.2.1. It is clear that in corollary 2.2.1, the fixed point is unique. Moreover, it is easy to see that the condition (W.4)[5] implies (W.3)[5] which guarantees the uniqueness of limits of sequences.

### 2.3 Application

Throughout this section, a distribution function $f$ is a nondecreasing, left continuous real-valued function defined on the set of real numbers, with $\inf f = 0$ and $\sup f = 1$.

DEFINITION 2.3.1. Let $X$ be a set and $@$ a function defined on $X \times X$ such that $F_{x,y}$ is a distribution function. Consider the following conditions:

I. $F_{x,y}(0) = 0$ for all $x, y \in X$.

II. $F_{x,y} = H$ if and only if $x = y$, where $H$ denotes the distribution function defined by $H(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 & \text{if } x > 0
\end{cases}$

III. $F_{x,y} = F_{y,x}$.

IV. If $F_{x,y}(\epsilon) = 1$ and $F_{y,z}(\delta) = 1$ then $F_{x,z}(\epsilon + \delta) = 1$.

If $@$ satisfies I and II, then it is called a PPM-structure on $X$ and the pair $(X, @)$ is called a PPM space. An $@$ satisfying III is said to be symmetric. A symmetric PPM-structure $@$ satisfying IV is a probabilistic metric structure and the pair $(X, @)$ is a probabilistic metric space.

Let $(X, @)$ be a symmetric PPM-space. For $\epsilon, \lambda > 0$ and $x \in X$, let $N_x(\epsilon, \lambda) = \{ y \in X : F_{x,y}(\epsilon) > 1 - \lambda \}$. A $T_1$ topology $t(@)$ on $X$ is defined as follows:

\[ t(@) = \{ U \subseteq X \mid \text{for each } x \in U, \text{ there exists } \epsilon > 0, \text{ such that } N_x(\epsilon, \epsilon) \subseteq U \}. \]

Recall that a sequence $\{x_n\}$ is called a fundamental sequence if $\lim_{n \to \infty} F_{x_n,x_n}(t) = 1$ for all $t > 0$. The space $(X, @)$ is called F-complete if for every fundamental sequence $\{x_n\}$ there exists $x \in X$ such that $\lim_{n \to \infty} F_{x_n,x}(t) = 1$, for all $t > 0$. Note that condition (W.4), defined earlier, is equivalent to the following condition:

\[ P(4) \lim_{n \to \infty} F_{x_n,x}(t) = 1 \text{ and } \lim_{n \to \infty} F_{x_n,y_n}(t) = 1 \text{ imply } \lim_{n \to \infty} F_{y_n,x}(t) = 1, \text{ for all } t > 0. \]
In [1], Hicks and Rhoades proved that each symmetric PPM-space admits a compatible symmetric function as follows:

THEOREM 2[1] Let \((X, \mathfrak{S})\) be a symmetric PPM-space. Let \(p : X \times X \rightarrow \mathbb{R}^+\) be a function defined as follows:

\[
p(x, y) = \begin{cases} 
0 & \text{if } y \notin N_x(t, t), \ 0 < t < 1 \\
\sup \{ t : y \notin N_x(t, t), \ 0 < t < 1 \} & \text{otherwise}
\end{cases}
\]

Then

(1) \(p(x, y) < t\) if and only if \(F_{x,y}(t) > 1 - t\).

(2) \(p\) is a compatible symmetric for \(t(\mathfrak{S})\).

(3) \((X, \mathfrak{S})\) is \(F\)-complete if and only if \((X, p)\) is \(S\)-complete.

For our main result in this section, we need the following new Proposition:

PROPOSITION 2.3.1. Let \((X, \mathfrak{S})\) be a symmetric PPM-space and \(p\) a compatible symmetric function for \(t(\mathfrak{S})\). For \(A, B \in C(X)\), set

\[
E_{A,B}(\epsilon) = \min \left\{ \inf_{a \in A} \sup_{b \in B} F_{a,b}(\epsilon), \inf_{b \in B} \sup_{a \in A} F_{a,b}(\epsilon) \right\}, \ \epsilon > 0.
\]

and

\[
P(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} p(a, b); \sup_{b \in B} \inf_{a \in A} p(a, b) \right\}.
\]

If \(T : X \rightarrow C(X)\) is a multivalued mapping, then we have

\[
F_{x,y}(t) > 1 - t \implies E_{Tx,Ty}(kt) > 1 - kt, \ k \in [0,1), \ \forall t > 0, \ \forall x, y \in X.
\]

implies that \(P(Tx, Ty) \leq kp(x, y)\).

PROOF. Let \(t > 0\) be given and set \(\mu = p(x, y) + t\). Then \(p(x, y) = \mu - t < \mu\) gives \(F_{x,y}(\mu) > 1 - \mu\), and therefore \(E_{Tx,Ty}(\mu) > 1 - k\mu\). Then

\[
\left\{ \begin{array}{l}
\inf_{a \in Tx} \sup_{b \in Ty} F_{a,b}(k\mu) > 1 - k\mu \\
\inf_{b \in Ty} \sup_{a \in Tx} F_{a,b}(\mu) > 1 - k\mu
\end{array} \right.
\]

\[
\implies \left\{ \begin{array}{l}
\forall a \in Tx, \exists b \in Ty, \ F_{a,b}(k\mu) > 1 - k\mu \\
\forall b \in Ty, \exists a \in Tx, \ F_{a,b}(\mu) > 1 - k\mu
\end{array} \right.
\]

\[
\implies \left\{ \begin{array}{l}
\forall a \in Tx, \exists b \in Ty, \ p(a, b) < k\mu \\
\forall b \in Ty, \exists a \in Tx, \ p(a, b) < k\mu
\end{array} \right.
\]

then

\[
\sup_{a \in Tx} \inf_{b \in Ty} p(a, b) < k\mu \text{ and } \sup_{a \in Tx, b \in Ty} p(a, b) < k\mu
\]

and therefore \(P(Tx, Ty) < k\mu = k(p(x, y) + t)\). Since \(t > 0\) was arbitrary, it follows \(P(Tx, Ty) \leq kp(x, y)\).
DEFINITION 2.3.2. Let \((X, \mathcal{A})\) be a symmetric PPM space and \(A\) a nonempty subset of \(X\). We say that \(A\) is \(\mathcal{A}\)-closed iff \(\overline{A}^\mathcal{A} = A\) where
\[
\overline{A}^\mathcal{A} = \{x \in X : \sup_{a \in A} F_{x,a}(t) = 1, \text{ for all } t > 0\}.
\]
We denote by \(C_\mathcal{A}(X)\) the set of all nonempty \(\mathcal{A}\)-closed subsets of \(X\).

REMARK 2.3.1. Let \((X, \mathcal{A})\) be a symmetric PPM space and \(C_\mathcal{A}(X)\) be the set of all nonempty \(\mathcal{A}\)-closed subsets of \(X\). It is not hard to see that if \(p\) is a compatible symmetric function for \(t(\mathcal{A})\) then \(C_\mathcal{A}(X) = C(X)\) where \(C(X)\) is the set of all nonempty \(p\)-closed subsets of \((X, p)\).

Now we are able to state and prove an application of our main Theorem 2.2.1 in the following way

THEOREM 2.3.1. Let \((X, \mathcal{A})\) be a \(F\)-complete symmetric PPM space that satisfies \((P.4)\) and \(p\) a compatible symmetric function for \(t(\mathcal{A})\). Let \(T : X \to C_\mathcal{A}(X)\) be a multivalued mapping such that:
\[
F_{x,y}(t) > 1 - t \text{ implies } E_{Tx,Ty}(kt) > 1 - kt, \text{ for all } t > 0, \forall x, y \in X.
\]
Then there exists \(u \in X\) such that \(u \in Tu\).

PROOF. Note that \((X, p)\) is bounded and \(S\)-complete. Also \(p(x, y) < t\) if and only if \(F_{x,y}(t) > 1 - t\). Let \(\epsilon > 0\) be given, and set \(t = p(x, y) + \epsilon\). Then \(p(x, y) < t\) gives \(F_{x,y}(t) > 1 - t\) and therefore \(E_{Tx,Ty}(kt) > 1 - kt\). In view of Proposition 2.3.1, it follows that \(P(Tx, Ty) \leq kt = kp(x, y) + \epsilon\). On letting \(\epsilon \to 0\) (since \(\epsilon > 0\) is arbitrary), we have \(p(Tx, Ty) \leq kp(x, y)\). Now apply Theorem 2.2.1.

For a single-valued selfmapping \(T\), Theorem 2.3.1 is reduced to the following known result:

COROLLARY 2.3.1. Let \((X, \mathcal{A})\) be a \(F\)-complete symmetric PPM space that satisfies \((P.4)\) and \(p\) a compatible symmetric function for \(t(\mathcal{A})\). Let \(T\) be a selfmapping of \(X\) satisfying
\[
F_{x,y}(t) > 1 - t \text{ implies } F_{Tx,Ty}(kt) > 1 - kt, \text{ for all } t > 0, \forall x, y \in X.
\]
Then \(T\) has a fixed point.

References


