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New Exact Solutions For A Reaction-Diffusion Equation And A Quasi-Camassa Holm Equation*

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Abstract

In this paper, the projective Riccati equation method, presented by Yan in [1], is used for obtaining exact solutions, solitary solutions as well as periodic solutions of the reaction-diffusion equation and the quasi-Camassa Holm equation.

1 Introduction

Since nonlinear partial differential equations are widely used to describe complex phenomena in various fields of science, it is important to their seek exact solutions. Exact solutions may describe not only the propagation of nonlinear waves but also spatially localized structures of permanent shape that may be of interest in experimentation [1, 2]. Many powerful methods have been developed to explore exact solutions. In particular, Yan in [1] has put forward the generally projective Riccati equation method which can briefly be described as follows.

For a given nonlinear partial differential equation (NLPDE), say, in two variables x and t

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{1}$$

by means of the transformation $u(x,t) = u(\xi)$ and $\xi = x + \lambda t$, (1) can be reduced to

$$Q(u, u', u'', u''', ...) = 0. \tag{2}$$

Assumed that (1) or (2) has a solution of the form

$$u = \sum_{i=1}^{n} \sigma^{i-1}(\xi)[A_i\sigma(\xi) + B_i\tau(\xi)] + A_0, \tag{3}$$

where σ and τ satisfy

$$\sigma'(\xi) = e\sigma(\xi)\tau(\xi), \quad \tau'(\xi) = e\tau^2(\xi) - \mu\sigma(\xi) + r, \tag{4}$$

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where $' = \frac{d}{d\xi}$, $e = \pm 1$, $r \neq 0$ and μ are constants. Then it is easy to see that (4) admits the first integral

$$\tau^2(\xi) = -e\left[r + \frac{\mu^2 - 1}{r}\sigma^2(\xi) - 2\mu\sigma(\xi)\right]. \tag{5}$$

In particular, when $r = \mu = 0$, we can assume that (1) has the solution

$$u = \sum_{i=1}^{n} \tau^i(\xi), \tag{6}$$

where τ satisfies

$$\tau'(\xi) = \tau^2(\xi),$$

and the parameter n can be found by balancing the highest-order linear term and nonlinear term of (1).

Thus, (4) may yield the following solutions:

Case 1. When $e = -1$ and $r \neq 0$, we have

$$\sigma_1 = \frac{\sqrt{r}\mathrm{sech}(\sqrt{r}\xi)}{\mu\mathrm{sech}(\sqrt{r}\xi) + 1}, \tau_1 = \frac{\sqrt{r}\tanh(\sqrt{r}\xi)}{\mu\tanh(\sqrt{r}\xi) + 1}, \tag{7}$$

$$\sigma_2 = \frac{\sqrt{r}\mathrm{csch}(\sqrt{r}\xi)}{\mu\mathrm{csch}(\sqrt{r}\xi) + 1}, \tau_2 = \frac{\sqrt{r}\coth(\sqrt{r}\xi)}{\mu\coth(\sqrt{r}\xi) + 1}. \tag{8}$$

Case 2. When $e = 1$ and $r \neq 0$, we have

$$\sigma_3 = \frac{\sqrt{r}\sec(\sqrt{r}\xi)}{\mu\sec(\sqrt{r}\xi) + 1}, \tau_3 = \frac{\sqrt{r}\tan(\sqrt{r}\xi)}{\mu\tan(\sqrt{r}\xi) + 1}, \tag{9}$$

$$\sigma_4 = \frac{\sqrt{r}\csc(\sqrt{r}\xi)}{\mu\csc(\sqrt{r}\xi) + 1}, \tau_4 = \frac{\sqrt{r}\cot(\sqrt{r}\xi)}{\mu\cot(\sqrt{r}\xi) + 1}. \tag{10}$$

Case 3. When $r = \mu = 0$, we see that

$$\sigma_5(\xi) = C\xi, \tau_5(\xi) = \frac{1}{e\xi}. \tag{11}$$

This method can be used to find exact solutions for many NLPDEs. In this paper, we apply it to a reaction-diffusion equation and the quasi-Camassa Holm equation to find new exact solutions.

2 Exact Solutions for Reaction-Diffusion Equation

For the following reaction-diffusion equation [4,5,6]

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0, \tag{12}$$

if we apply the transformation $u(x,t) = u(\xi)$ and $\xi = x + \lambda t$, it is reduced to

$$u'' + mu + nu^3 = 0, \tag{13}$$

where

$$m = \frac{\beta}{\alpha + \lambda^2}, \ n = \frac{\gamma}{\alpha + \lambda^2}.$$

According to the method described above, we may assume that (12) has solutions of the form

$$u = a\sigma(\xi) + b\tau(\xi) + c, \tag{14}$$

where a, b, c are constants to be determined later and $\sigma(\xi)$ as well as $\tau(\xi)$ satisfy (4) and (5) with $r \neq 0$. With the aid of the symbolic Maple software, we can substitute (14) along with (4) and (5) into (13) and collect all terms with the same power in $\sigma^i(\xi)\tau^j(\xi)$ for $i = 0, 1, 2, 4$ and $j = 0, 1$. Setting these coefficients to zero yields a set of over-determined algebraic system with respect to a, b, c, r, μ, namely,

$$2ae^3 - 3nab^2e\mu^2 + na^3r - 2ae^3\mu^2 + 3nab^2e = 0,$$

$$3na^2cr + 3nb^2ec - 3nb^2ec\mu^2 + 6nab^2e\mu r - ae\mu r + 4ae^3\mu r = 0,$$

$$nb^3e + 3na^2br - 2be^3\mu^2 + 2be^3 - nb^3e\mu^2 = 0,$$

$$aer^2 + 6nb^2ec\mu r - 3nab^2er^2 + 3nac^2r - 2ae^3r^2 + mar = 0,$$

$$2be^3\mu r + 6nabcr + 2nb^3e\mu r - be\mu r = 0,$$

$$mbr + 3nbc^2r - nb^3er^2 = 0,$$

and

$$nc^3r + mcr - 3nb^2ecr^2 = 0.$$

With the aid of Maple, we can get the following solutions:

Case 1:

$$e = -1, r = \frac{m}{2}, \mu = 0, b = \sqrt{-\frac{2}{n}};$$

Case 2:

$$e = -1, r = -m, \mu = 0, a = \sqrt{-\frac{2}{mn}};$$

Case 3:

$$e = -1, r = 2m, \mu = \pm 1, b = \sqrt{-\frac{1}{2n}};$$

Case 4:

$$e = -1, r = 2m, \mu = \mu, a = \sqrt{\frac{1-\mu^2}{4mn}}, b = \sqrt{-\frac{1}{2n}};$$

Case 5:

$$e = 1, r = -\frac{m}{2}, \mu = 0, b = \sqrt{-\frac{2}{n}};$$

Case 6:

$$e = 1, r = m, \mu = 0, a = \sqrt{-\frac{2}{mn}};$$

Case 7:

$$e = 1, r = -2m, \mu = \pm 1, b = \sqrt{-\frac{1}{2n}};$$

Case 8:

$$e = 1, r = -2m, \mu = \mu, a = \sqrt{\frac{1-\mu^2}{4mn}}, b = \sqrt{-\frac{1}{2n}}.$$

Therefore, from (7)-(10) and the above solutions, we can obtain many families of exact travelling wave solutions for (12):

1. Soliton solutions:

$$u_1 = \sqrt{-\frac{\beta}{\gamma}}\tanh\left[\sqrt{\frac{\beta}{2(\alpha+\lambda^2)}}(x+\lambda t)\right],$$

$$u_2 = \sqrt{\frac{2(\alpha+\lambda^2)}{\gamma}}\operatorname{sech}\left[\sqrt{-\frac{\beta}{\alpha+\lambda^2}}(x+\lambda t)\right],$$

$$u_3 = \sqrt{-\frac{\beta}{\gamma}}\coth\left[\sqrt{\frac{\beta}{2(\alpha+\lambda^2)}}(x+\lambda t)\right],$$

and

$$u_4 = \sqrt{\frac{2(\alpha+\lambda^2)}{\gamma}}\operatorname{csch}\left[\sqrt{-\frac{\beta}{\alpha+\lambda^2}}(x+\lambda t)\right].$$

2. Periodic solutions:

$$u_5 = \sqrt{\frac{\beta}{\gamma}}\tan\left[\sqrt{-\frac{\beta}{2(\alpha+\lambda^2)}}(x+\lambda t)\right],$$

$$u_6 = \sqrt{-\frac{2(\alpha+\lambda^2)}{\gamma}}\sec\left[\sqrt{\frac{\beta}{\alpha+\lambda^2}}(x+\lambda t)\right],$$

$$u_7 = \sqrt{\frac{\beta}{\gamma}}\cot\left[\sqrt{-\frac{\beta}{2(\alpha+\lambda^2)}}(x+\lambda t)\right],$$

and

$$u_8 = \sqrt{-\frac{2(\alpha+\lambda^2)}{\gamma}}\csc\left[\sqrt{\frac{\beta}{\alpha+\lambda^2}}(x+\lambda t)\right].$$

3. New soliton solutions:

$$u_9 = \frac{\sqrt{-\frac{\beta}{\gamma}}\tanh\left[\sqrt{\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1 \pm \tanh\left[\sqrt{\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]},$$

$$u_{10} = \sqrt{\frac{(1-\mu^2)(\alpha+\lambda^2)}{2\gamma}}\frac{\operatorname{sech}\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1+\mu\operatorname{sech}\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]} + \sqrt{-\frac{\beta}{\gamma}}\frac{\tanh\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1+\mu\tanh\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]},$$

$$u_{11} = \frac{\sqrt{-\frac{\beta}{\gamma}}\coth\left[\sqrt{\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1\pm\coth\left[\sqrt{\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]},$$

and

$$u_{12} = \sqrt{\frac{(1-\mu^2)(\alpha+\lambda^2)}{2\gamma}}\frac{\operatorname{csch}\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1+\mu\operatorname{csch}\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]} + \sqrt{-\frac{\beta}{\gamma}}\frac{\coth\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1+\mu\coth\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}.$$

4. New periodic solutions:

$$u_{13} = \frac{\sqrt{\frac{\beta}{\gamma}}\tan\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1\pm\tan\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]},$$

$$u_{14} = \sqrt{\frac{(1-\mu^2)(\alpha+\lambda^2)}{2\gamma}}\frac{\sec\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1+\mu\sec\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]} + \sqrt{\frac{\beta}{\gamma}}\frac{\tan\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1+\mu\tan\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]},$$

$$u_{15} = \frac{\sqrt{\frac{\beta}{\gamma}}\cot\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1\pm\cot\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]},$$

and

$$u_{16} = \sqrt{\frac{(1-\mu^2)(\alpha+\lambda^2)}{2\gamma}}\frac{\csc\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1+\mu\csc\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]} + \sqrt{\frac{\beta}{\gamma}}\frac{\cot\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}{1+\mu\cot\left[\sqrt{-\frac{2\beta}{\alpha+\lambda^2}}(x+\lambda t)\right]}.$$

3 Exact Solutions of the Quasi-Camassa Holm Equation

The method introduced above can also be applied to the quasi-Camassa Holm equation

$$u_t - u_{xt} + uu_x + 2ku_x = 3u_x u_{xx} + uu_{xxx}. \tag{15}$$

By means of the transformation $u(x,t) = u(\xi)$ and $\xi = x + \lambda t$, (15) is changed into

$$\lambda u' - \lambda u'' + uu' - 2ku' - 3u'u'' - uu''' = 0. \tag{16}$$

We assume (15) has solutions of the form

$$u = A_1\sigma(\xi) + B_1\tau(\xi) + A_2\sigma^2(\xi) + B_2\tau^2(\xi) + C_2\sigma(\xi)\tau(\xi) + C_1. \tag{17}$$

Then by means of similar methods described above, we get:
Case 1:

$$\begin{aligned} e &= -1, r = r, \mu = \mu, B_1 = C_2 = 0, A_1 = 2\mu B_2, \\ A_2 &= \frac{1-\mu^2}{r} B_2, B_2 = B_2, C_1 = C_1; \end{aligned}$$

Case 2:

$$\begin{aligned} e &= 1, r = r, \mu = \mu, B_1 = C_2 = 0, A_1 = -2\mu B_2, \\ A_2 &= \frac{(\mu^2-1)B_2}{r}, B_2 = B_2, C_1 = C_1. \end{aligned}$$

Then we can obtain the following new soliton and periodic solutions for (15):

$$\begin{aligned} u_1 &= \frac{2\mu B_2\sqrt{r}\mathrm{sech}[\sqrt{r}(x+\lambda t)]}{1+\mu\mathrm{sech}[\sqrt{r}(x+\lambda t)]} + \frac{(1-\mu^2)B_2\mathrm{sech}^2[\sqrt{r}(x+\lambda t)]}{(1+\mu\mathrm{sech}[\sqrt{r}(x+\lambda t)])^2} \\ &\quad + \frac{B_2 r\tanh^2[\sqrt{r}(x+\lambda t)]}{(1+\mu\tanh[\sqrt{r}(x+\lambda t)])^2} + C_1, \end{aligned}$$

$$\begin{aligned} u_2 &= \frac{2\mu B_2\sqrt{r}\mathrm{csch}[\sqrt{r}(x+\lambda t)]}{1+\mu\mathrm{csch}[\sqrt{r}(x+\lambda t)]} + \frac{(1-\mu^2)B_2\mathrm{csch}^2[\sqrt{r}(x+\lambda t)]}{(1+\mu\mathrm{csch}[\sqrt{r}(x+\lambda t)])^2} \\ &\quad + \frac{B_2 r\coth^2[\sqrt{r}(x+\lambda t)]}{(1+\mu\coth[\sqrt{r}(x+\lambda t)])^2} + C_1, \end{aligned}$$

and

$$\begin{aligned} u_3 &= -\frac{2\mu B_2\sqrt{r}\sec[\sqrt{r}(x+\lambda t)]}{1+\mu\sec[\sqrt{r}(x+\lambda t)]} + \frac{(\mu^2-1)B_2\sec^2[\sqrt{r}(x+\lambda t)]}{(1+\mu\sec[\sqrt{r}(x+\lambda t)])^2} \\ &\quad + \frac{B_2 r\tan^2[\sqrt{r}(x+\lambda t)]}{(1+\mu\tan[\sqrt{r}(x+\lambda t)])^2} + C_1, \end{aligned}$$

$$\begin{aligned} u_4 &= -\frac{2\mu B_2\sqrt{r}\csc[\sqrt{r}(x+\lambda t)]}{1+\mu\csc[\sqrt{r}(x+\lambda t)]} + \frac{(\mu^2-1)B_2\csc^2[\sqrt{r}(x+\lambda t)]}{(1+\mu\csc[\sqrt{r}(x+\lambda t)])^2} \\ &\quad + \frac{B_2 r\cot^2[\sqrt{r}(x+\lambda t)]}{(1+\mu\cot[\sqrt{r}(x+\lambda t)])^2} + C_1, \end{aligned}$$

where the constants $r > 0, \mu, B_2$ are arbitrary and independent of the wave speed λ and k in the (15).

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