Applied Mathematics E-Notes, 3(2003), 16-37 © ISSN 1607-2510
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A Survey On Homoclinic And Heteroclinic Orbits *†

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Received 2 January 2002

Abstract

The study of homoclinic and heteroclinic orbits has a long history. This paper surveys some of the recent advances in this topic. We present some of the main results obtained in recent years, and in addition we indicate some possible research directions and some problems for further studies.

1 Introduction

Homoclinic and heteroclinic orbits arise in the study of bifurcation and chaos phenomena (see e.g. [1–7], [10], [48, 49] and [55]) as well as their applications in mechanics, biomathematics and chemistry (see e.g. [29], [108–111]). Many works related to these topics have been done in recent years. In this paper, we intend to survey some of the results, methods and problems that have recently been reported.

Consider the dynamical system

$$\dot{X} = f(X),\ t \in R, \tag{1}$$

where $X \in R^n$, $f \in C^r$ and $r \geq 1$. Recall that a point X_0 is a singular or equilibrium point of (1) if $f(X_0) = 0$. A singular point X_0 is a hyperbolic point if all of the eigenvalues of $Df(X_0)$ have nonzero real parts and is an elementary saddle point if $\det Df(X)\mid_{X=X_0} < 0$. Let p, q be hyperbolic points of the above system (1). A solution path $s_p : X = X(t)$ of (1) is called a homoclinic orbit connecting with p if $X(t) \to p$ as $t \to \pm\infty$ (See Figure 1),

Figure 1

*Mathematics Subject Classifications: 34C05, 34C23, 34C37.

†This research is supported by the NNSF of China (No.10171099)

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and a solution path $s_{pq} : X = X(t)$ of (1) is called a heteroclinic orbit connecting with p and q if $X(t) \to p$ as $t \to -\infty$ and $X(t) \to q$ as $t \to +\infty$ (See Figure 2).

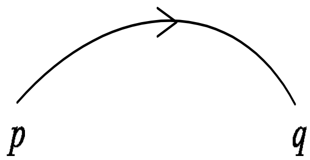

Figure 2

Some heteroclinic orbits may form a cycle. We will denote by $S^n_{O_1,O_2,\cdots,O_n}$ the heteroclinic cycle passing through the singular points $O_1, O_2, \cdots, O_n$ (See Figure 3).

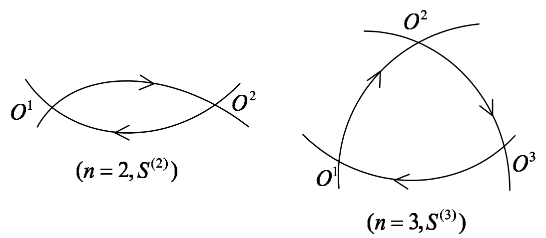

Figure 3

There are several natural aspects related to these orbits which are worthy of discussions:

I. The existence of homoclinic and heteroclinic cycles.

II. The stability of a homoclinic or a heteroclinic cycle.

III. The mutual position between the stable and the unstable manifold of a saddle point.

IV. The bifurcation of a homoclinic or heteroclinic orbit.

2 The Existence of Homoclinic and Heteroclinic Cycles

The research of this problem provides the foundation for the research of the others. There are several well known methods dealing with the existence problems.

The first method is to study a planar Hamiltonian system, because it is basically an analytic geometry problem and it is comparatively easy to determine the existence of the homoclinic cycles or heteroclinic cycles of Hamiltonian systems.

The second method is the so-called structured approach. For example, consider the plane system

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y). \end{aligned} \tag{2}$$

Select a function $F(x, y)$ such that $F(x, y) = 0$ for all (x, y) which belongs to a closed curve L passing through a saddle point (i.e. a singular point (x_0, y_0) of (2) satisfying

$\det \partial(P(x_0,y_0),Q(x_0,y_0))/\partial(x,y)<0$) and

$$\frac{dF}{dt}|_{(2)} = F_x P + F_y Q = FB$$

where B is a regular (or analytic) function in the neighborhood of L. Then we can conclude that there exists a saddle separatrix cycle L. This method can work for non-Hamiltonian system. Furthermore, by constructing a rotational vector field [7], we can also get a saddle separatrix cycle of a system. For example, let $L: F(x,y)=0$ be the separatrix cycle of system (2), then we can conclude that L is the separatrix cycle of the system

$$\begin{aligned} \dot{x} &= P(x,y) + FP_1(x,y), \\ \dot{y} &= Q(x,y) + FQ_1(x,y). \end{aligned} \tag{3}$$

Using this method, we can get a complicated vector field in which there exists a separatrix cycle through a simple vector field in which there exists a separatrix cycle. There are many examples given in [7], [8], [9], [13], [15], [16], [20], [21], [22], [37], [38], [43], [50], [58].

The third method is to study some special separatrix cycle such as the separatrix cycle formed by the conic curve of a quadratic system. This method can be found in [7], [59] and [60]. It is also important to study the existence of separatrix cycles of systems such as the Lorenz system. Many results about the existence of the separatrix cycles, whose solution path cannot be given explicitly, are obtained by perturbation methods (see for examples [37] and [38]).

3 The Stability of a Homoclinic or a Heteroclinic Cycle

Let us denote the system (2) by I. This is convenient since we will introduce a related system later which will be indicated by $I(\varepsilon)$. dependence on ε. Assume that there is a homoclinic cycle L passing through the saddle point $O(0,0)$ in system (2).

In the investigation of the stability of a homoclinic cycle, Dulac in 1923 studied the analytic system by using the so-called semi-regular function and obtained the following result.

THEOREM 1. In system (2), assume that P and Q are analytic. Then L is stable (unstable) if $\sigma_0 = (P_x + Q_y)|_{(x,y)=O(0,0)} < 0$ (respectively > 0).

The proof is rather complicated. In 1958, by using the success function, Andronov et al. assumed the system is C^1 and proved the theorem mentioned above in [2]. Under the assumption that P and Q are C^1, Melnikov [11] gave a proof of this theorem in 1966. Chow proved this result by another method in [15]. The sign of $\int_0^\tau \mathrm{div}(I)|_{(x,y)\in\gamma} dt$ (where γ is a path near L, and τ is the time in which γ finishes a return or Poincare map) were used to determine the stability of L in [11] and [15], and this term was obtained when Andronov et al. [2] computed the success function. Therefore, the work of [2] is the base of the others.

When $\sigma_0 = 0$, the first critical case appears. Andronov et al. [3] first considered this problem. From the proof of Theorem 1 in [11], we may infer the reason why the stability of L can be determined by the information of the saddle point itself when $\sigma_0 \neq 0$. Andronov et al. [3] once tried to expand the saddle point by the Taylor expansion, and use the first non-zero term of the expansion to determine the stability, but finally, they gave a counter-example instead. Today, it is well known that for the first critical case it is impossible to determine the stability of the homoclinic cycle only by using the information of the saddle point itself.

The following result is given in [3].

LEMMA 1. $\sigma_1 = \int_{-\infty}^{+\infty}(P_x + Q_y)|_{(x,y)\in L}dt$ is convergent ($+\infty$ or $-\infty$) if $\sigma_0 = 0$ (respectively $\sigma_0 \neq 0$).

This Lemma enables us to state the criterion for determining the separatrix cycle for the critical case in simpler form. In analogy with the fact that the stability of a limit cycle Γ is determined by the integral $\oint_\Gamma \mathrm{div}(I)dt$, it is natural to guess that the integral $\int_{-\infty}^{+\infty} \mathrm{div}(I)|_{(x,y)\in L}dt$ can be used to determine the stability of the homoclinic cycle L. In 1985, using the C^1-Hartman theorem [77], Feng and Qian in [4] proved the following result.

THEOREM 2. If $\sigma_0 = 0$ and $\sigma_1 = \int_{-\infty}^{+\infty}(P_x + Q_y)|_{(x,y)\in L}dt < 0$ (> 0), then L is stable (respectively unstable).

We remark that Luo et al. simplified the proof of this theorem in [5] (see also [11], [79]). The work of [4] has the following significance: 1. The stability of a homoclinic cycle in the critical case must be determined by all the information of the homoclinic cycle instead of by the information of the saddle point itself. 2. A new method is presented (dealing with success function in the regular domain and singular domain respectively) which is later used in [17], [22] and [38].

When $\sigma_0 = \sigma_1 = 0$, the second critical case appears. Han and Zhu [73] pointed out that the first order saddle quantity can determine the stability of L for this case. In 2001, by using the property that the successor function is intrinsic and it has no relation with the curvilinear coordinate system, Hu and Feng [72] obtained the limiting property of the successor function by constructing a series of special curvilinear coordinate systems, and consequently, they obtain the criterion below.

THEOREM 3. Assume that in system (2), $\sigma_0 = \sigma_1 = 0$. Let

$$\sigma_2 = -\int_{-\infty}^{+\infty} e^{\int_0^t (P_x+Q_y)|_L d\tau} \cdot V|_L dt \tag{4}$$

where

$$\begin{aligned} V \;=\; & \frac{1}{(P^2+Q^2)^2}[(3P_xQ_x + P_xP_y + Q_xQ_y + 3P_yQ_y)(Q^2 - P^2) \\ & +2PQ(P_y^2 - Q_x^2 + 2P_x^2 - 2Q_y^2)] \\ & +\frac{1}{P^2+Q^2}[-Q(2P_{xx} + Q_{xy} + P_{yy}) + P(P_{xy} + Q_{xx} + 2Q_{yy})]. \end{aligned}$$

If $\sigma_2 < 0$, then L is stable from inside (so that every solution curves near and inside L tend to L as $t \to +\infty$). If $\sigma_2 > 0$, then L is unstable from inside.

The integral on the right hand side of (4) equals infinity when the first order saddle quantity does not equal 0, so the result in [72] also pointed out that the first order saddle quantity can determine the stability of the homoclinic cycle for the second critical case. Luo and Zhu [22] gave some examples to point out that it is useful to study the higher critical cases.

As for the stability of heteroclinic cycles, Cerkas [14] in 1968 obtained the following theorem.

THEOREM 4. Assume that in system (2), $S^{(n)}$ is the heteroclinic cycle passing through the elementary saddle points $O_1, O_2, \cdots, O_n$, and

$$\lambda_i^+ > 0 > \lambda_i^-$$

are the eigenvalues of the system at the saddle point O_i. Let

$$\lambda_i = -\frac{\lambda_i^-}{\lambda_i^+},\ \lambda = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Then, when $\lambda > 1$ (< 1), $S^{(n)}$ is stable (respectively unstable).

When $n = 1$, the heteroclinic cycle degenerates into a homoclinic cycle $S^{(1)}$, and the criterion here is that when $\lambda = -\lambda^-/\lambda^+ > 1$ (< 1), $S^{(1)}$ is stable (respectively unstable). In view of $\sigma_0 = \lambda^+ + \lambda^-$, it is clear that

$$\sigma_0 < 0\ (> 0) \Longleftrightarrow \lambda > 1\ (< 1).$$

This fact shows that Theorem 2 is a special case of Theorem 4.

The results of [14] not only gave the criterion for determining the stability of a heteroclinic cycle, but also change the thoughts of some mathematicians. σ_0 is important for the study of a homoclinic cycle, so, some people want to determine the stability of $S^{(n)}$ by $\sigma_0^{(1)}, \sigma_0^{(2)} \cdots, \sigma_0^{(n)}$. From Theorem 4, we learn that instead of $\sigma_0^{(i)}$, λ_i^+ and λ_i^- are the essential quantities, and the reason that σ_0 can be used to determine the stability of $S^{(1)}$ is that σ_0 happens to be equivalent to λ when $n = 1$.

When $\lambda = 1$, the critical case for a heteroclinic cycle appears. Some mathematicians once suspected that the integral $(\int_{s_{12}} + \int_{s_{23}} + \cdots + \int_{s_{n1}})(\mathrm{div}(I))dt$ (where s_{ij} is the heteroclinic path from the saddle point O_i to the saddle point O_j) can be used to determine the stability of $S^{(n)}$. But in fact, this conjecture can be right only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$, so it is a special case. Before the right answer has been given, many people may be misled by this conjecture.

By means of the methods used in [12], [14] and [15], Feng [17] obtained the following result.

THEOREM 5. In system (2), let M_{i1}, M_{i2} be the points on the stable manifold s_i^+ and unstable manifold s_i^-, and the arc length $\widehat{O_i M_{i1}} = \widehat{O_i M_{i2}} = c_i \rho$ (where $\rho > 0$ is small). Then, S^n is stable (unstable) if $\lambda > 1$ (respectively < 1) or $\lambda = 1$ and $\Lambda_{n1} < 0$ (> 0), where

$$\Lambda_{n1} = \lim_{\rho \to 0} (J_{n1} + \lambda_n J_{n-1,n} + \cdots + \lambda_2 \cdots \lambda_n J_{12}),$$

$$J_{ij} = \int_{\widehat{M_{i2} M_{j1}}} \mathrm{div}(I) dt,\ j = i+1, \quad \text{and when} \quad i = n, j = 1,$$

and c_i are determined by the equations

$$\begin{array}{rcl} c_1^{2-\lambda_1}(\lambda_1^+)\cos\varphi_1 &=& c_2(-\lambda_2^-)\cos\varphi_2 \\ c_2^{2-\lambda_2}(\lambda_2^+)\cos\varphi_2 &=& c_3(-\lambda_3^-)\cos\varphi_3 \\ .... &=& ... \\ c_n^{n-\lambda_n}(\lambda_n^+)\cos\varphi_n &=& c_1(-\lambda_1^-)\cos\varphi_1 \end{array}$$

here $\varphi_i = |90^\circ - \theta_i|$, and θ_i is the angle subtended by $s_{i-1,i}$ and $s_{i,i+1}$.

When $\lambda \neq 0$, corresponding to $n = 1$ and $n > 1$, Theorem 5 is equivalent to Theorems 2 and 4, when $\lambda = 1$ and $n = 1$, Theorem 5 yields Theorem 3, and when $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$, Theorem 5 extends Theorem 2.1 in [25], i.e. the conjecture above. So the results about the stability of a homoclinic cycle and heteroclinic cycle in [1]-[3], [12], [14] and [23]-[25] are special cases of the results in [17]. Theorem 5 not only contains the quantities $\lambda_i^+, \lambda_i^-, \lambda_i, \int_{s_{ij}} \mathrm{div}(I)dt$, but also indicates that the brevity of the criterion depends on the special proportion between the positions of the arcs which are not tangent with any solution paths (i.e. arcs without contact). (The proof in [14] shows that if the proportion between the positions of the arcs without contact is not appropriate, the complicated expression would appear in the criterion so that the criterion cannot be applied.) When it comes to the convergence of the integral in the theorem, new situations about the heteroclinic cycle arise. Lemma 1 indicates that $\int_L \mathrm{div}(I)dt$ is convergent if $\sigma = 0$ or $\lambda = 1$. Feng [17] proved that this property is right for $n = 2$. In other words, the integrals $\int_{s_{12}} \mathrm{div}(I)dt$ and $\int_{s_{21}} \mathrm{div}(I)dt$ are convergent, but the convergence here is weaker than that when $n = 1$, and it is based on the proportion between the positions of the arcs without contact in Theorem 5 and is the convergence under Cauchy's principal value. When $n \geq 3$, we can only guarantee the convergence of some special linear combination of them instead of the convergence under Cauchy's principal value. Feng [41] gave a brief survey about the study of this problem.

All the theorems above assume that the parametric equations of the homoclinic or heteroclinic cycles are all known. But in many cases, it is impossible to give the parametric equations of the homoclinic cycle or heteroclinic cycle. In [8], [9] and [58], the authors studied some special systems. When Zhang [58] proved the stability of a heteroclinic cycle, this problem is changed into the uniqueness of the heteroclinic cycle, i.e. to prove that there does not exist a limit cycle in the neighborhood of the heteroclinic cycle, and then the stability of the singular point can determine the stability of the heteroclinic cycle.

Another problem is the stability of an infinity separatrix cycle. An infinity separatrix cycle is a solution path s_∞ both of its ends tend to infinity as $t \to \infty$ (See Figure 4).

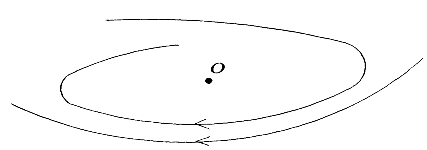

Figure 4

The stability of s_∞ is taken to mean that the solution paths near s_∞ approach it when $t \to \infty$. In [26], [27] and [28], a quadratic system is considered, and its infinity separatrix cycle whose interior has a focus point is studied. Feng [13] studied the infinity separatrix cycle whose interior has a center. In [39], Feng also studied the Melnikov functions of an infinity separatrix cycle. And finally, in 1995, Feng [70] obtained the criterion for determining the stability of a separatrix tending to infinity as follows.

THEOREM 6. Consider the system

$$\begin{aligned} \dot{x} &= P_0(x, y), \\ \dot{y} &= yQ_0(x, y), \end{aligned} \tag{5}$$

where $y = 0$ is the separatrix tending to s_∞. (Remark: The system (2) can be turned into this form by means of a transformation) Assume that (i) a return map can be defined in the neighborhood of the upside of $y = 0$, (ii) in system (5), $P_0 = (P_0 - P_\infty) + P_\infty$, $Q_0 = (Q_0 - Q_\infty) + Q_\infty$, where P_∞, Q_∞ has the properties (H_1) yQ_∞/P_∞ is the homogeneous function of x, y, (H_2) $(P_0 - P_\infty)/P_\infty \to 0$ and $(Q_0 - Q_\infty)/Q_\infty \to 0$ as $x^2 + y^2 \to 0$, (iii) the integral $I_1 = \int_{-\infty}^{+\infty} Q_0(x, 0)/P_0(x, 0)dx$, interpreted as Cauchy's principal value, is convergent, (iv) the integral $I_2 = \int_{-\infty}^{+\infty} \frac{(1-c)xQ_\infty + cP_\infty}{x(xQ_\infty - P_\infty)}(x, 1)dx$, interpreted as Cauchy's principal value, is convergent, where $c = Q_\infty(1, 0)/(Q_\infty(1, 0) - P_\infty(1, 0))$. Then the separatrix tending to s_∞ is stable (unstable) upside if $\sigma_\infty = I_1 + I_2/(1-c) < 0$ (respectively > 0).

A concept which is important in discussing the stability of a space separatrix cycle is given by Feng [71] in 1996.

DEFINITION 1. Let A be a 3×3 matrix. If A has three real eigenvalues, which can be denoted as $\lambda^- < 0 < \lambda^+$ and λ^* such that the tangent vectors b^+ and b^- of $S^{(1)}$ at the saddle O are the characteristic vectors corresponding to λ^- and λ^+ respectively, whereas λ^* is the remaining third eigenvalue. If A has a pair of complex conjugate eigenvalues and one real eigenvalue, then the unique real eigenvalue is denoted as λ^*.

In a similar way we can define the concept of λ_i^-, λ_i^+ and λ_i^*.

A stability criterion is given below for a homoclinic cycle (and a corresponding criterion for a heteroclinic cycle is similar to it).

THEOREM 7. Consider the system

$$\dot{X} = F(X), X \in R^3, F \in C^2, \tag{6}$$

which admits a homoclinic cycle $S^{(1)}$ and the saddle point is the origin O. Assume that (i) the eigenvalues of $A = D_X F(O)$ are all real, (ii) for some $\varepsilon_0 > 0$, $S^{(1)}$ has a positive direction invariant ε_0-part neighborhood, (iii) $\lambda^* < 0$, $\lambda = -\lambda^-/\lambda^+ > 1$ (< 1), then, $S^{(1)}$ is positively asymptotic stable (respectively unstable).

In the above result ,we say that U is a positive invariant ϵ_0-neighborhood of a solution $\varphi(X_0, t)$ of (6) passing through X_0 if $\{\varphi(X_0, t) \mid X_0 \in U\} \subset \overline{U}$ for any t and $d(\overline{U}, S^{(1)}) < \epsilon_0$ for any $X_0 \in U$.

We remark that it is an interesting problem to determine the stability of $S^{(1)}$ when $\lambda = 0$.

By extending Theorem 7 to high dimensional case, we give an application in biology. The results we give extend those obtained by May and Leonard in [109] and by Hofbauer and Sigmund in [110].

All the problems above have relations with the normalized form of the system in the neighborhood of the saddle point (see [85]). For example, in [12], [14] and [23], the C^1-Hartman Theorem, invariant manifold orthogonalizing theorem and Joyal normalized form theorem are employed for obtaining their results.

Normalized form was studied for a long time and it has relations with the smoothness of the system, the property of the eigenvalue and the smoothness of the transformation. For example, a system which cannot be linearized by means of an analytic transformation may be linearized by means of a C^1 transformation. In [12] and [14], the systems are assumed to be C^2, and in [23], the systems C^3. Although there were many profound results about the normalized form, there are some unsettled problems. For instance, the famous Poincare-Sternber Theorem says that if the eigenvalues of a planar system are nonresonant, then this system can be C^∞ linearized. But there also exist some systems which do not satisfy the nonresonant conditions, yet can be linearized by means of analytic transformations. For example, the system

$$\begin{aligned}\dot{x} &= -y + xy \\ \dot{y} &= x + y^2\end{aligned}$$

can be linearized by means of the transformation

$$\begin{aligned}u &= \frac{x}{1-x}, \\ v &= \frac{y}{1-x}.\end{aligned}$$

So, the conditions of linearization are worthy of studying. For related studies, the reader may consult [4], [5], [12], [14], [16],[23], [33], [45] ,[60]-[63] and [94].

4 The Mutual Position Between the Stable and the Unstable Manifold of a Saddle Point

In 1963, Melnikov [11] first obtained an important result which provides an analytic criterion for determining the mutual position of the separatrixes.

This method has been applied widely and is called the Melnikov method. The determining function in this method is called Melnikov function. A brief sketch of this method is as follows.

Consider the system

$$\begin{aligned}\dot{x} = P(x,y) + \varepsilon P_1(x,y,\epsilon) \\ \dot{y} = Q(x,y) + \varepsilon Q_1(x,y,\epsilon).\end{aligned} \tag{7}$$

Let us denote the system (7) by $I(\varepsilon)$ to indicate its dependence on ε. There exists separatrix cycle L_0 in system $I(0)$. Furthermore, when the system is perturbed, L_0 "breaks" and becomes the stable manifold $L_{\varepsilon s}$ and the unstable manifold $L_{\varepsilon u}$ (See Figure 5). The problem is to determine the mutual position of $L_{\varepsilon s}$ and $L_{\varepsilon u}$. Take the points M_t^+ on $L_{\varepsilon s}$ and M_t^- on $L_{\varepsilon u}$, then, we get a vector $M_t^- M_t^+$ depending on the time t and the parameter ε. It is clear that the direction of this vector can determine the mutual position of $L_{\varepsilon s}$ and $L_{\varepsilon u}$. The information about the system $I(0)$ is known, but it does not contain the information about $M_t^- M_t^+$, so it is almost impossible to study the $M_t^- M_t^+$ itself to get the mutual position of $L_{\varepsilon s}$ and $L_{\varepsilon u}$. Melnikov's method is to take the exterior normal vector $n(t)$ corresponding to the time t on the separatrix cycle of the unperturbed system $I(0)$, then by comparing the directions of $M_t^- M_t^+$ and $n(t)$, we can determine the mutual position of $L_{\varepsilon s}$ and $L_{\varepsilon u}$, i.e. when $M_t^- M_t^+$ and $n(t)$ have the same directions, $L_{\varepsilon s}$ is on the outside of $L_{\varepsilon u}$, otherwise, $L_{\varepsilon s}$ is on the inside of $L_{\varepsilon u}$. Therefore, it is natural to consider the inner product of $M_t^- M_t^+$ and $n(t)$, i.e. to consider the following function due to Melnikov

$$\Delta_\varepsilon(t) = -n(t) \cdot M_t^- M_t^+.$$

The function Δ cannot be computed. For any vectors $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ in $\Re^2$, define $\alpha \wedge \beta = (-\alpha_1, \alpha_2) \cdot (\beta_1, \beta_2)^\top$. The next step is to prove the expansion

$$\Delta_\varepsilon(t) = \Delta_1(t)\varepsilon + \Delta_2(t)\varepsilon^2 + \cdots,$$

and to prove that the differential equation below is satisfied by $\Delta_1(t)$

$$\tfrac{d\Delta_1(t)}{dt} = \sigma(t)\Delta_1 - D(t), \tag{8}$$

where

$$\sigma(t) = \operatorname{div}(P, Q)|_{x=\varphi(t), y=\psi(t)},$$

and

$$D(t) = \left[\begin{pmatrix} P \\ Q \end{pmatrix} \wedge \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \right]_{x=\varphi(t), y=\psi(t), \varepsilon=0},$$

here $x = \varphi(t)$ and $y = \psi(t)$ are the parametric equations of L_0.

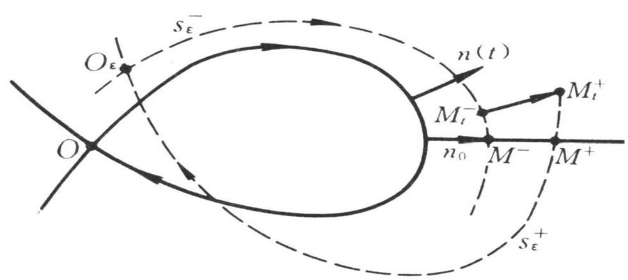

Figure 5

Thus, when ε is sufficiently small, we can get the sign of Δ_ε through the sign of Δ_1, where Δ_1 can be computed. In view of (8), Melnikov asserts that

$$\Delta_1 = \int_{-\infty}^{+\infty} [D(t)e^{-\int_0^t \sigma(\xi)d\xi}]dt.$$

This is the so-called Melnikov criterion. However, Feng and Qian in [12] found that (8) can only yield

$$\Delta_1(0) = \Delta_1^+(T^+)e^{-\int_0^{T^+}\sigma(t)dt} - \Delta_1^-(T^-)e^{-\int_0^{T^-}\sigma(t)dt} + \int_{T^-}^{T^+}[D(t)e^{-\int_0^t \sigma(\xi)d\xi}]dt. \quad (9)$$

As a consequence, Melnikov function can be used as the first order approximation only when the first two terms on the right hand side (called the remainder terms) in (9) tend to zero as $T^+ \to +\infty$ and $T^- \to -\infty$. In [12], it is proved that when the saddle point is elementary (when the two eigenvalues are both nonzero), the remainder terms tend to zero as $T^+ \to +\infty$ and $T^- \to -\infty$. We believe that Melnikov assumed that the saddle point of the system was elementary, and thus the terms tend to zero. But for the higher order singular point, a counter-example is given in [12] to illustrate that the remainder terms do not tend to zero. This prompts us to consider conditions which are sufficient for the remainder terms to vanish. For instance, in [13], such a condition is given for the system which has the homogeneous main part, i.e.

$$\begin{aligned} \dot{x} &= P + \varepsilon P_1, \\ \dot{y} &= Q + \varepsilon Q_1, \\ P &= R_1 + f_1, \\ Q &= R_2 + f_2, \end{aligned} \qquad (10)$$

where $P = P(x,y), Q = Q(x,y), P_1 = P_1(x,y,\varepsilon), Q_1 = Q_1(x,y,\varepsilon)$, R_1 and R_2 are the m-th order homogeneous polynomials, P_1 and Q_1 have order greater than or equal to m, f_1 and f_2 have order greater than or equal to $m+1$. Another general result is given in [39].

THEOREM 8. Assume that (a) the system (10) when $\varepsilon = 0$ is a Hamiltonian system, or (b) the saddle point of (10) when $\varepsilon = 0$ is hyperbolic, i.e. $\det A < 0$, where A is the linearized matrix of the system at the point O, or (c) $m + 1 - |\pi^*| > 0$, where

$$\pi^{\pm} = \frac{R_{1x} + R_{2y}}{R_1}|_{x=1, y=k^{\pm}},$$

and k^+ or k^- is the rate of slope of the stable manifold (k^+) or the unstable manifold (k^-) at the point O. Then the remainder terms vanish.

Under the conditions of the Theorem above, we can use the Melnikov criterion as the first order approximation of the Melnikov function. At the same time, we can determine the mutual position of separatrix cycle in the perturbed system with $\Delta_1(0)$. Feng and Qian [12] presented the following theorem.

THEOREM 9. Assume that ε is sufficiently small, then if $\varepsilon \cdot \Delta_1(0) < 0$ (> 0), L_ε^+ is on the outside (respectively inside) of L_ε^- (see Figure 6).

We remark that for systems which have nonhomogeneous main parts, the vanishing conditions for the remainder terms have not been studied. We remark further that for systems which have homogeneous main parts, the result of [13] does not include the case $m = 1$, i.e., the hyperbolic situation. Another interesting problem is the computation of the remainder terms when they do not vanish.

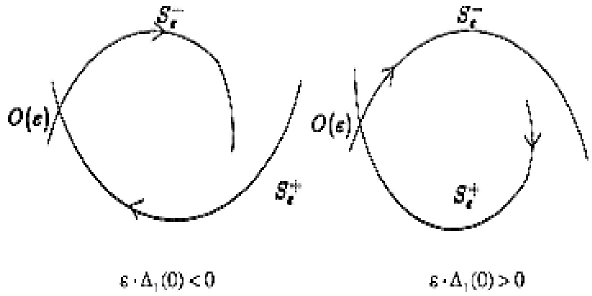

Figure 6

When $\Delta_1(0) = 0$, the first order approximation cannot be used to determine the sign of $\Delta_\varepsilon(0)$, so we must consider the so-called higher order Melnikov function. Yuan [40] and Sun [50] studied this problem, and gave the expressions of the second order Melnikov function. But their expressions are different. Yuan [40] obtained the criterion through computing the right-hand member of the system, and Sun [50] obtained the criterion by the solution of a linear variational equation. No one has offered a comparison of these two methods yet.

All the problems above involve small perturbations, i.e. it is assumed that there exists a separatrix cycle in the unperturbed system. We also want to find out the mutual position of the two separatrixes in the perturbed system. For the "large region" case, there are few results. Even the formulation of the problem is unclear. We have tried to formulate the problem as follows.

Consider the system (2), assume that $O(0,0)$ is the saddle point of the system. (a) Under what conditions will the stable manifold L_s and unstable manifold L_u of O all intersect with some arc without contact of the system? (b) If L_s and L_u all intersect with some arc without contact, what are the conditions for determining their mutual positions.

Dumortier [55] and Zhou [56] have studied this problem. But it is difficult to give a "good" condition such that when the system (2) turns into the system $I(\varepsilon)$, this condition reduces to the Melnikov condition. But there is a question here. In the system $I(\varepsilon)$, the information of the separatrix cycle L_0 of $I(0)$ is known, but in system (2), we cannot take L_s and L_u as the known quantities. So, when the system (2) becomes $I(\varepsilon)$, we know more information than before. However, we cannot express this process. In other words, this problem has not been settled, and many important problems depend on the determination of the mutual position of the separatrixes.

5 The Bifurcation of a Homoclinic or a Heteroclinic Orbit

There are several problems related to the study of bifurcation:

(1) to determine the types of the bifurcation,

(2) to determine the number of limit cycles due to bifurcation,
(3) to determine the stability of bifurcation,
(4) to give the conditions for determining the problems above.

In [3]-[5],[48], [49] and [63], the authors have studied the first aspect thoroughly and considered almost all the possible cases. Recently, the authors of [34], [44], [46], [54] and [55] also did many works. Yet, they all restricted their studies in the possibility of the bifurcation, and did not consider the condition of the bifurcation. But determining the condition of bifurcation may never be settled completely. The author of [49] has pointed out that many bifurcation sets are not algebraic, i.e. it is impossible to give the algebraic determinant. So we see there are many bifurcation conditions which cannot be given. Li et al. [80] also did some related works.

As for the second problem, in 1958, Andronov [3] proved that if $\sigma_0 \neq 0$, the limit cycle which bifurcates from the separatrix cycle has the same type of stability with the original separatrix cycle, so this limit cycle must be unique. In [3], a counterexample is also given to show that when $\sigma_0 = 0$, the separatrix cycle can bifurcate into two limit cycles. So with this result in mind, many people had thought that there was no uniqueness when $\sigma_0 = 0$. To their surprise, Luo and Zhu [22] assumed that $\sigma_0(\varepsilon) \equiv 0$ and proved that when $\sigma_0 = 0$ and $\sigma_1 = \int_L \operatorname{div}(I)dt \neq 0$, the limit cycle has the same stability with the separatrix cycle from which it bifurcates (and hence is unique). In fact, the integrals $\int_\Gamma \operatorname{div}I(\varepsilon)dt$ and $\int_L \operatorname{div}(I)dt$ determine the stability of the limit cycle and the separatrix cycle respectively. While $I(\varepsilon) \to I$ and $\Gamma \to L$ as $\varepsilon \to 0$, so $\int_\Gamma \operatorname{div}I(\varepsilon)dt \to \int_L \operatorname{div}(I)dt$ as $\varepsilon \to 0$. In the same manner, Hu and Feng [72] proved that under appropriate conditions, when $\sigma_0 = \sigma_1 = 0$ and $\sigma_2 \neq 0$, there exist at most 2 limit cycles bifurcating from L in the neighborhood of L. The newest results about the uniqueness of bifurcation can be seen in [91].

For the number of the bifurcation limit cycles, there are some results about Hamiltonian systems. In 1951, Leontovich [30] gave the result below without proof. R. Roussarie [53] gave the same result with proof. Consider the system

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x,y), \\ \dot{y} &= -H_x + \varepsilon g(x,y). \end{aligned} \tag{11}$$

Let $I(\varepsilon)$ denote the system (11). Assume that L_h is the contour line $H(x,y) = h$ of system $I(0)$. L_0 is the separatrix cycle, $H > 0$ denotes the interior of L_0 and

$$I(h) = \int\int_{H \geq h} (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y})dxdy.$$

THEOREM 10. When h is sufficiently small,

$$I(h) = c_1 + c_2 h \ln h + c_3 h + c_4 h^2 \ln h + \cdots,$$

where

$$c_1 = I(0), c_2 = (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y})|_{x=y=0}, c_3 = \frac{\partial}{\partial h} I(h)|_{h=0}.$$

If $I(h) \neq 0$ and $c_1 = c_2 = \cdots = c_n = 0, c_{n+1} \neq 0$, then there exists a perturbed system of $I(\varepsilon)$, and there exist n limit cycles near L_0 in this system. Furthermore, there exist at most n limit cycles in the perturbation of $I(\varepsilon)$.

W. G. Li proved the convergence of the above expansion in [94].

Based on the theorem above, the authors of [20] and [21] proved that there exist at most 3 limit cycles bifurcating from the quadratic system. Rousseau [64] first showed the existence of two limit cycles bifurcating from the separatrix cycle in the quadratic system, and Liu [43] three limit cycles bifurcating from the separatrix cycle in the quadratic system (see also [78], [83] and [84]).

In Theorem 10, $c_1, c_2, \cdots$ are called separatrix quantities. Notice that $\mathrm{div}I(0) = 0$ for Hamiltonian systems, we have

$$\begin{aligned} c_1 &= \int\int_{h\geq 0}(\mathrm{div}I(\varepsilon))dxdy, \\ c_2 &= (\mathrm{div}I(\varepsilon))|_{x=0,y=0} = \sigma_0(I(\varepsilon)), \\ c_3 &= \int(\mathrm{div}I(\varepsilon))dt = \sigma_1(\varepsilon). \end{aligned}$$

It is easy to see that c_2, c_3 are just the criterion for determining the stability of the separatrix cycle of system $I(\varepsilon)$. The reason for this phenomenon is not clear. We remark further that [30] is not only concerned with Hamiltonian systems, and the methods presented in it are useful for further study.

Some of the separatrix quantities are related to the saddle point itself, some to the separatrix cycle. Many people studied the so-called saddle quantities in [16], [18], [19], [20], [21], [22] and [43] (see also [81]). Kokubu [43] pointed out that for some special systems, the saddle quantities can determine the number of the bifurcation. The authors of [63] showed that sometimes the saddle quantities can determine the stability of the separatrix cycle. Luo and Zhu [22] stated that generally the saddle quantities cannot be used to determine the stability of the separatrix cycle. Han [66] also did some studies. Finally, Hu and Feng [72] and Han and Zhu [73] proved that the saddle quantities whose order are less than two can determine the stability of the separatrix cycle under suitable conditions. But for the higher order cases, there are no better result. So it is interesting to study the role of the saddle quantities in the stability and bifurcation of a separatrix cycle. Are the saddle quantities and focus quantities the antithetic variables? Are there relations between the saddle quantities and the separatrix quantities? All these are worthy of studying.

As for problem (4), from the above discussions, we see that it is difficult in general to give the conditions for every possible bifurcation. But for some special cases, it is possible. We have known that there are some limit cycles bifurcating from separatrix cycle for a long time, but, until 1985, Feng and Qian [12] gave the analytic condition for bifurcation, and Feng [17] gave the criterion for determining the limit cycle bifurcating from heteroclinic orbit. For Hamiltonian systems, Li [37] weakened the condition in [17], and gave the condition for the bifurcation of homoclinic cycle and heteroclinic cycle. Feng and Xiao [38] gave the condition under which the homoclinic cycle and heteroclinic cycle bifurcated from a heteroclinic cycle which extends the result in [37].

Recently, the attention of many researchers have turned to the homoclinic cycles or heteroclinic cycles in R^n and some new problems appear. For example, the homoclinic cycle in higher dimension system may have some bifurcations which are impossible for planar system. The most remarkable case is the so-called homoclinic doubling bifurcation, 2-impulse limit cycle and n-impulse homoclinic cycle and limit cycle (see Figure 7).

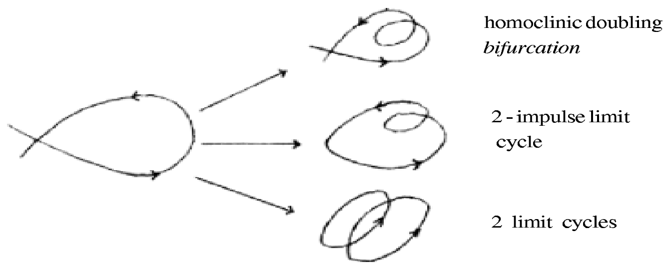

Figure 7

Deng obtained some conditions for homoclinic doubling bifurcation in [90], he gave a homoclinic doubling bifurcation Theorem there. From the discussions mentioned above, we should not only trace the information of the separatrix cycle itself as we do in the planar system, but also study the invariant manifold near the separatrix cycle. For instance, in order to determine whether there are homoclinic doubling bifurcations, we must know whether the homoclinic cycle in the unperturbed system is twisted in the sense that the two-dimensional surface formed by the stable manifold in a neighborhood of the cycle is a Möbius band, but it is impossible to get the twisted property from the homoclinic cycle itself. The required information must be obtained from the invariant manifold (global not local) near the homoclinic cycle.

Recently, Sanstede [88] did some useful research works in this area, and he studied the so-called hyperbolic center manifold, i.e. the invariant manifolds near the space homoclinic cycle. [82], [86] and [89] are additional references in this area.

As for the invariant manifolds near the space homoclinic cycle, there remains a lot of important studies to be done. For example, is it possible to get the analytic criterion for twisted and non-twisted? In addition, it is clear that the space homoclinic cycle contains more detailed information than the planar system does. For example, the three kinds of bifurcations mentioned above (homoclinic doubling bifurcation, 2-impulse limit cycle, and two limit cycles which are very closer) all tends to the same homoclinic cycle as the perturbation parameter ε tends to zero. This fact means that under certain conditions, there are at least 3 bifurcation directions for the space homoclinic cycle, and there are at least 3 kinds of corresponding analytic criterion. What are these criterion? What are the differences between them? These are interesting and unsettled problems.

Figure 8

Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \epsilon f \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{12}$$

$$\omega = \alpha + i\beta.$$

The following result is in [74] and [75].

THEOREM 11. If $|\alpha| < \lambda, \beta \neq 0$ and when $\epsilon = 0$, there exists a homoclinic orbit γ (passing through $O(0,0,0)$), then there exists a perturbed system of (12) in which there is a homoclinic orbit γ' near γ, and the return map of γ' includes horseshoes which are countable (see Figure 9).

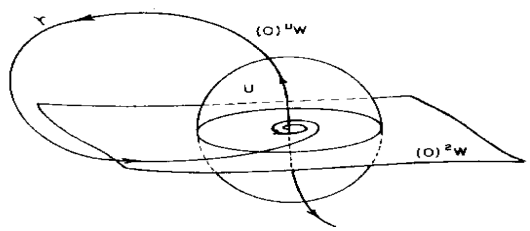

Figure 9

The homoclinic cycle mentioned above is called the Silnikov homoclinic cycle. The eigenvalues of the system are $\lambda^u > 0$ and a pair of complex conjugate values λ^s and $\bar{\lambda}^s$ with $\mathrm{Im}\lambda^s \neq 0$ satisfying $\lambda^u > |\mathrm{Re}\lambda^s|$. In 1965, the authors of [74] and [75] showed the following.

THEOREM 12. Assume that D is an appropriately defined domain of return map π. For any positive integer $m \geq 2$, there exists a Π-invariant subset $V_m \subset D$ such that $\Pi|_? V_m$ is topologically conjugate to the shift dynamics

$$\sigma : S_m = \{1, \cdots, m\}^Z \to S_m; (s_i)_{i=-\infty}^{+\infty} \mapsto (s_{i+1})_{i=-\infty}^{+\infty}.$$

In other words, it admits a homeomorphism $h : S_m \to V_m$ satisfying

$$\Pi|_{V_m} \circ h = h \circ \sigma.$$

In [69], a proof for the above theorem is given. In 1990, Deng [76] gave a stronger result. Deng [33] did further studies on the Silnikov problem. Sun [42] generalized the result of [12] for the three dimensional case, and studied the bifurcation of the three dimensional homoclinic orbit. Other results can be found in [95]. The results of [67] showed that although it is difficult to study the higher dimensional cases, many results are extensions of the two dimensional case. So many methods for the two dimensional case are still useful for studying the higher dimensional cases.

Acknowledgment. The author is grateful to the reviewers, who make many helpful suggestions, correct some of our mistakes and indicate several useful references [92]–[107]. The author is also grateful to Professor Sui Sun Cheng. It is impossible to complete this paper without his editorial help.

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