Applied Mathematics E-Notes, 3(2003), 124-129 © ISSN 1607-2510
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Generalized Pólya Condition For Birkhoff Interpolation With Lacunary Polynomials *

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Received 3 February 2003

Abstract

Lacunary polynomials are polynomials of the form $\sum_{i=1}^{n} a_i x^{k_i}$. We introduce the problem of Birkhoff interpolation with lacunary polynomials and formulate a generalized Pólya condition. In the framework of Birkhoff interpolation with lacunary polynomials, the generalized Pólya condition characterizes conditionally regular interpolation matrices. The generalized Pólya condition is simpler than the Pólya condition for algebraic Birkhoff interpolation and it is equivalent to the Pólya condition when $k_j = j - 1$.

1 Introduction

An *algebraic Birkhoff interpolation problem* is defined by $(X, E, C, \mathcal{P}_{n-1})$, where $X = (x_1, \dots, x_m)$ consists of m distinct real points (called nodes), $E = (e_{i,j})_{i=1,j=0}^{m,\ q}$ is an *interpolation matrix* with $e_{i,j} \in \{0,1\}$ and with exactly n ones, C is a vector formed by n real values $(c_{i,j} : e_{i,j} = 1)$, and $\mathcal{P}_{n-1}$ is the space of real polynomials of degree at most $n-1$. The objective of the algebraic Birkhoff interpolation problem is to find a polynomial $p(x)$ in $\mathcal{P}_{n-1}$ that satisfies the n conditions:

$$D^{(j)} p(x_i) = c_{i,j}, \quad e_{i,j} = 1. \tag{1}$$

If we write the polynomial $p(x) \in \mathcal{P}_{n-1}$ in the form

$$p(x) = \sum_{i=0}^{n-1} a_i \frac{x^i}{i!},$$

then the n conditions (1) determine a system of n linear equations with n unknowns. The interpolation problem $(X, E, C, \mathcal{P}_{n-1})$ has a unique solution if and only if the *generalized Vandermonde determinant*

$$D(E, X) = \det \left[\frac{x_i^{-j}}{(-j)!}, \dots, \frac{x_i^{n-j-1}}{(n-j-1)!};\ e_{i,j} = 1 \right]$$

*Mathematics Subject Classifications: 41A05; 41A10; 26C05.

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does not vanish ($1/k!$ is defined 0 if $k < 0$). An interpolation matrix E is called *conditionally regular* if $D(E, X) \neq 0$ for some system of nodes X.

Let $|E|$ denote the number of ones in E. An interpolation matrix is called a *normal matrix* if it has exactly $|E|$ columns. If the highest derivative specified in E do not exceed $|E|-1$, we always can normalize E by adding a suitable number of zero columns. Obviously, matrices which have elements $e_{i,j} = 1$ with $j \geq |E|$ cannot be normalized. Nevertheless, those matrices are irrelevant for algebraic interpolation since, in this case, we have $D(E, X) = 0$ for all system of nodes X. So, in algebraic interpolation, we can suppose without loss of generality that E is a normal matrix.

For a normal matrix $E = (e_{i,j})_{i=1,j=0}^{m,\,|E|-1}$, the *Pólya constants* are the numbers:

$$M_j = \sum_{s=0}^{j} \sum_{i=1}^{m} e_{i,s}, \quad j = 0, \ldots, |E| - 1.$$

Note that some of the Pólya constants are undefined if E has less than $|E|-1$ columns. In [1], Ferguson proved that conditionally regular normal matrices can be characterized by the *Pólya condition*:

$$M_j \geq j + 1, \quad j = 0, \ldots, |E| - 1. \tag{2}$$

THEOREM 1. A normal matrix E is conditionally regular if and only if it satisfies the Pólya condition.

A review on algebraic Birkhoff interpolation can be found in the book of Lorentz, Jetter and Riemenschneider [2, pp. 1-12].

In this paper, a *lacunary polynomial* is a polynomial of the form

$$p(x) = \sum_{j=1}^{n} a_j x^{k_j},$$

where $0 \leq k_1 < \cdots < k_n$ are integers, in general, no consecutive. These polynomials are a restricted case of Müntz polynomials (where the numbers $k_1, \ldots, k_n$ can take real values). A complete study on the regularity for the one-node Birkhoff interpolation problem with lacunary polynomials can be found in [3]. To the authors' knowledge, there exist no more general results on Birkhoff interpolation with lacunary polynomials. The aim of this work is to characterize conditionally regular interpolation matrices for Birkhoff interpolation with lacunary polynomials, this is carried out by means of a simple and elegant generalization of the Pólya condition.

2 Lacunary-polynomial interpolation

Given an increasing sequence of nonnegative integers $K = (k_1, \ldots, k_n)$, we denote by $\mathcal{P}_K$ the real linear space of lacunary polynomials spanned by the powers $x^{k_1}, \ldots, x^{k_n}$. We will refer to K as the *degree sequence* of the space $\mathcal{P}_K$. The system $(X, E, C, \mathcal{P}_K)$ defines a *lacunary-polynomial Birkhoff interpolation problem*, whose objective is to determine a polynomial $p(x)$ in $\mathcal{P}_K$ that satisfies the $n = |E|$ conditions (1). Note that the algebraic Birkhoff interpolation problem is obtained as a particular case of lacunary-polynomial interpolation when the degree sequence is $K = (0, \dots n-1)$. If we express the polynomial $p(x) \in \mathcal{P}_K$ in the form

$$p(x) = \sum_{j=1}^{n} a_j \frac{x^{k_j}}{k_j!},$$

we obtain the generalized Vandermonde determinant

$$D(E, X, K) = \det \left[\frac{x_i^{k_1-j}}{(k_1-j)!}, \dots, \frac{x_i^{k_n-j}}{(k_n-j)!};\ e_{i,j} = 1 \right]. \tag{3}$$

As with the algebraic case, the interpolation problem $(X, E, C, \mathcal{P}_K)$ has a unique solution if and only if $D(E, X, K) \neq 0$.

DEFINITION 1. Given a degree sequence K, we say that an interpolation matrix E is conditionally K-regular if $D(E, X, K) \neq 0$ for some system of nodes X.

3 Generalized Pólya condition

In an interpolation matrix E, elements $e_{i,j} = 1$ correspond to the derivative orders that are prescribed in the interpolation conditions. We define the *derivative sequence* of an interpolation matrix as the nondecreasing sequence $Q(E) = (q_1, \dots, q_n)$ whose elements are the derivative orders specified in E, that is, the column index corresponding to nonzero entries in E. For the interpolation matrix:

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4}$$

the derivative sequence is $Q(E) = (0, 0, 1, 6, 6, 7)$. Note that columns in interpolation matrices are indexed from 0.

DEFINITION 2. Let $K = (k_1, \dots, k_n)$ be a degree sequence, E an interpolation matrix with n ones and $Q(E) = (q_1, \dots, q_n)$ the derivative sequence of E. We say that E satisfies the Pólya condition with respect to K, (or more briefly, the Pólya K-condition) if

$$q_i \leq k_i \quad \text{for} \quad i = 1, \dots, n. \tag{5}$$

In lacunary-polynomial interpolation, we must consider non-normal matrices. For instance, if we take the degree sequence $K = (0, 1, 2, 6, 7, 8)$ and the interpolation matrix (4), we have $D(E, X, K) \neq 0$ for every system of nodes X. We can see that the Pólya K-condition is well defined for every interpolation matrix with $|E| = n$.

The following proposition shows that the Pólya K-condition actually generalizes the Pólya condition. Here, we say that a normalizable matrix ($e_{i,j} = 1$ only for $j < |E|$) satisfies the Pólya condition if its normal form does; we also agree on that non-normalizable matrices ($e_{i,j} = 1$ for some $j \geq |E|$) do not satisfy the Pólya condition.

PROPOSITION 1. An interpolation matrix E satisfies the Pólya condition if and only if it satisfies the Pólya condition with respect to the degree system $K = (0, 1, \ldots, n-1)$.

PROOF. Let $Q(E) = (q_1, \ldots, q_n)$ be the derivative sequence of E. If E is non-normalizable, then E does not satisfy the Pólya condition. As $k_n = n-1$ and for some $j \geq n$ is $e_{i,j} = 1$, we have $q_n \geq n > k_n$; therefore, E does not satisfy the Pólya K-condition either. If E is a normalizable matrix, we can suppose without loss of generality that E is, in fact, normal. If the normal matrix E satisfies de Pólya condition (2), then, for every j with $0 \leq j \leq n-1$, there are in E at least $j+1$ ones in columns 0 to j. Hence, we have

$$q_{j+1} \leq j = k_{j+1} \quad \text{for} \quad j = 0, \ldots, n-1.$$

If we set $j+1 = i$, then condition (5) results. Conversely, suppose that $q_j \leq k_j$ for $j = 1, \ldots, n$. Then $q_1 \leq \ldots \leq q_j \leq j-1$ holds, and we obtain $M_{j-1} \geq j$ for $j = 1, \ldots n$. Making the change of index $s = j-1$, we get $M_s \geq s+1$ for $s = 0, \ldots, n-1$.

Before dealing with the regularity problem, we will prove a technical lemma. We use the notation $Q \leq K$ to indicate $q_j \leq k_j$ for $j = 1, \ldots, n$.

LEMMA 1. Let $K = (k_1, \ldots, k_n)$ and $Q = (q_1, \ldots, q_n)$ be nondecreasing sequences and let K', Q', represent the nondecreasing sequences obtained from K and Q after inserting, in the right place, a new element r. If $Q \leq K$, then $Q' \leq K'$.

PROOF. Let $K' = \left(k'_1, \ldots, k'_{n+1}\right)$ and $Q' = (q'_1, \ldots, q'_{n+1})$ be the extended sequences. If $r < q_1$, then we have

$$Q' = (r, q_1, \ldots, q_n), \quad K' = (r, k_1, \ldots, k_n)$$

and $Q' \leq K'$ holds immediately.

Now, suppose that $q_1 \leq r < k_1$ and define $i_0 = \max\{i : q_i \leq r\}$, then the extended sequences take the form

$$Q' = (q_1, \ldots q_{i_0}, r, q_{i_0+1}, \ldots, q_n), \; K' = \left(r, k_1, \ldots, k_{i_0}, k_{i_{0+1}}, \ldots, k_n\right).$$

The relation

$$q'_1 \leq \cdots \leq q'_{i_0+1} = r = k'_1 \leq \cdots \leq k'_{i_0+1}$$

holds for the first i_{0+1} elements in Q' and K'. If $i_0 < n$, then

$$q'_i = q_{i-1} \leq k_{i-1} = k'_i \quad \text{for} \quad i = i_0+2, \ldots, n+1.$$

Finally, if $k_1 < r$, we also define $j_0 = \max\{j : k_j \leq r\}$. First, we observe that it must be $j_0 \leq i_0$, for otherwise, under the hypothesis $j_0 > i_0$, one obtain the contradictory inequality $k_{j_0} \leq r < q_{j_0}$. If equality holds, that is, if $j_0 = i_0$, then the extended sequences Q' and K' have the following structure

$$Q' = (q_1, \ldots, q_{i_0}, r, q_{i_0+1}, \ldots, q_n), \; K' = (k_1, \ldots, k_{i_0}, r, k_{i_0+1}, \ldots, k_n),$$

and $Q' \leq K'$ is straightforward. When $j_0 < i_0$, we get the following structure for Q' and K'

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{j_0} & \multicolumn{1}{c}{j_0+1} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{i_0} & \multicolumn{1}{c}{i_0+1} & \multicolumn{1}{c}{i_0+2} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{n+1} \\
\hline
\multicolumn{1}{|c|}{k_1} & \dots & k_{j_0} & r & \dots & k_{i_0-1} & k_{i_0} & k_{i_0+1} & \dots & k_n \\
\hline
\multicolumn{1}{|c|}{q_1} & \dots & q_{j_0} & q_{j_0+1} & \dots & q_{i_0} & r & q_{i_0+1} & \dots & q_n \\
\hline
\end{array}\ .$$

For $i = j_0 + 1, \ldots, i_0 + 1$, we obtain the inequality $q'_i \leq r \leq k'_i$. For other values of i, the relation $q'_i \leq k'_i$ is immediate.

Now, we can prove the main result.

THEOREM 2. The interpolation matrix E is regularly K-conditional if and only if it satisfies the Pólya K-condition.

PROOF. Let $K = (k_1, \ldots, k_n)$ be the degree sequence and $Q = (q_1, \ldots, q_n)$ the derivative sequence of E. First, we shall prove that the Pólya K-condition is necessary for regularity. Suppose that E does not satisfy the Pólya K-condition, then there exists an i_0, with $1 \leq i_0 \leq n$, such that $q_{i_0} > k_{i_0}$. Let us organize the elements $e_{i,j} = 1$ in the lexicographical order of the pairs (i, j) with prevalence of the second coordinate, that is, (i, j) precedes (i', j') if and only if $j < j'$ or $j = j'$ and $i < i'$. As $k_1 < \cdots < k_{i_0} < q_{i_0}$, the determinant (3) takes the following structure:

$$\begin{array}{r}
\\ \\ \\ e_{i,q_{i_0}} \\ \\ \\ \\
\end{array}
\left|\begin{array}{ccc|c|ccc}
 & & & * & * & \cdots & * \\
 & M_1 & & \vdots & \vdots & \ddots & \vdots \\
 & & & * & * & \cdots & * \\
\hline
0 & \cdots & 0 & 0 & * & \cdots & * \\
\hline
0 & \cdots & 0 & 0 & & & \\
\vdots & \ddots & \vdots & \vdots & & M_2 & \\
0 & \cdots & 0 & 0 & & & \\
\end{array}\right|,$$

where M_1 is a square matrix with order $i_0 - 1$ and M_2 is a square matrix with order $n - i_0$. Thus, $D(E, X, K) = 0$ for every system of nodes X and the Pólya K-condition is necessary for E to be conditionally K-regular.

Suppose now that E satisfies the Pólya K-condition. If $k_n = n - 1$, then $\mathcal{P}_K$ coincides with $\mathcal{P}_{n-1}$. From Proposition 1 and Theorem 1, E is conditionally K-regular.

In the case $k_n > n - 1$, let $K^* = (k_1^*, \ldots, k_s^*)$ be the degree sequence consisting of the elements of $K_{k_n} = (0, 1, \ldots, k_n)$ which are not in K. The sequence K^* has $s = k_n - n + 1$ distinct elements. We construct a new interpolation matrix E^* adding a new first row to E, this new first row has ones in places $k_1^*, \ldots, k_s^*$, and zeros elsewhere. More precisely, if m is the number of rows of E, then $E^* = (e^*_{i,j})$ is a matrix with $m+1$ rows and $k_n + 1$ columns, whose elements verify $e^*_{1,k_j^*} = 1$, for $j = 1, \ldots, s$; $e^*_{i+1,j} = 1$ if $e_{i,j} = 1$ and $e^*_{i,j} = 0$ otherwise.

Let Q^* and Q be, respectively, the derivative systems of E^* and E. One can form Q^* inserting, in the right place, the s numbers $k_1^*, \ldots, k_s^*$, in Q. Analogously, adding the elements of K^* to K, we can obtain K_{k_n}.

As E satisfies the Pólya K-condition, we have $Q \leq K$ and, applying Lemma 1 s times, it results that E^* satisfies the Pólya condition with respect to K_{k_n}. By

Theorem 1, E^* is conditionally regular and there exists a system of nodes $X^* = \left(x_1^*, \ldots, x_{m+1}^*\right)$ such that $D(E^*, X^*) \neq 0$. We now recall that the determinant $D(E^*, X^*)$ is invariant under translations of the nodes, that is,

$$D\left(E^*, \left(x_1^*, \ldots, x_{m+1}^*\right)\right) = D\left(E^*, \left(x_1^* + \alpha, \ldots, x_{m+1}^* + \alpha\right)\right)$$

(see [2, p. 5]). Consequently, we can set $x_1^* = 0$.

Next, we organize the elements $e_{i,j}^* = 1$ in the lexicographic order of the pairs (i, j), this time with prevalence of the first coordinate, we also rearrange the basis of $\mathcal{P}_{k_n}$ in the form

$$\frac{x^{k_1^*}}{k_1^*!}, \ldots, \frac{x^{k_s^*}}{k_s^*!}, \frac{x^{k_1}}{k_1!}, \ldots, \frac{x^{k_n}}{k_n!}.$$

The generalized Vandermonde matrix

$$V(E^*, X^*) = \left[\frac{x_i^{k_1^*-j}}{(k_1^*-j)!}, \ldots, \frac{x_i^{k_s^*-j}}{(k_s^*-j)!}, \frac{x_i^{k_1-j}}{(k_1-j)!}, \ldots, \frac{x_i^{k_n-j}}{(k_n-j)!};\ e_{i,j}^* = 1\right]$$

will have the following structure

$$V(E^*, X^*) = \left(\begin{array}{c|c} \mathbf{I}_s & \mathbf{O}_{s\times n} \\ \hline M & V(E, X, K) \end{array}\right),$$

were $\mathbf{I}_s$ is the $s \times s$ unity matrix, $\mathbf{O}_{s\times n}$ is the $s \times n$ zero matrix, M is an $n \times s$ matrix and

$$V(E, X, K) = \left[\frac{x_i^{k_1-j}}{(k_1-j)!}, \ldots, \frac{x_i^{k_n-j}}{(k_n-j)!};\ e_{i,j} = 1\right]$$

is a generalized Vandermonde matrix corresponding to the problem $(E, X, \mathcal{P}_K, C)$ whose system of nodes is $X = \left(x_2^*, \ldots, x_{m+1}^*\right)$. It follows that

$$D\left(E, X, K\right) = D\left(E^*, X^*\right)$$

and, consequently, E is conditionally K-regular.

References

[1] D. R. Ferguson, The question of uniqueness for G. D. Birkhoff interpolation problems, J. Approx. Theory, 2(1969), 1–28.

[2] G. G. Lorentz and K. Jetter, Riemenschneider, Birkhoff Interpolation, Addison-Wesley Pub., 1984.

[3] F. Palacios and P. Rubió, Matrices associated to the generalized Taylor interpolation problem (in Spanish), Meeting on Matrix Analysis and Applications EAMA-97. pp. 350-357. Sevilla (1997).