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Integration Of The Exponential Function Of A Complex Quadratic Form *

Michele Elia and Giorgio Taricco[†]

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Abstract

Necessary and sufficient conditions for the existence of an improper integral of the exponential function of a complex quadratic form and its value are given.

We consider the evaluation of the improper integral of the exponential function of a complex quadratic form

$$\varphi(A) = \int_{C^n} \exp(-\pi x^\dagger A x) dx \tag{1}$$

where A is a complex matrix.

Following well-established conventions, we denote the Hermitian transpose of a matrix by $(\cdot)^\dagger$. Moreover, we use $\int_C f(z)dz$ as a short-hand notation for $\int_{R^2} f(x,y)dxdy$ with $z = x + \iota y$ and $\iota = \sqrt{-1}$. Here, R and C denote the sets of real and complex numbers, respectively.

This kind of integrals arise sometimes in relationship with saddle-point integration in the complex domain.

The result is well known when the matrix A is Hermitian. As in this case the integrand reduces to the exponential function of a real quadratic form, $\varphi(A) = \det A^{-1}$ *iff* A is positive definite. However, to the authors' knowledge, the case of non-Hermitian A has not yet appeared in the literature. This property is also basic to the development of several theoretical physics results related to the replica method analysis of spin glasses [2].

The main result of this work is the following proposition.

THEOREM 1. Given a square matrix $A \in C^{n\times n}$, the improper integral (1) exists, in the Lebesgue sense if, and only if, all the eigenvalues of A have positive real parts. In that case, $\varphi(A) = \det A^{-1}$.

PROOF. The proof is divided in two parts (i) and (ii).

(i) First, we prove that if all the eigenvalues of A have positive real parts, then the improper integral (1) exists in the Lebesgue sense and its value is $\varphi(A) = \det A^{-1}$. This proof is articulated in the following steps.

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[†]Politecnico di Torino, Dip. Elettronica, Torino, Italy

Using Schur's unitary triangularization theorem [1, p. 79], we can factor the matrix A as $A = U^\dagger T U$ where U is a unitary matrix and T is an upper triangular matrix whose diagonal contains the eigenvalues $\lambda_1, \dots, \lambda_n$ of A in any prescribed order.

Then, we can write

$$x^\dagger A x = x^\dagger U^\dagger T U x = y^\dagger T y \tag{2}$$

where we set $y = Ux$.

Integration on C^n can be interpreted as integration on R^{2n} and the mapping $y = Ux$ can be equivalently written as

$$\begin{pmatrix} \Re y \\ \Im y \end{pmatrix} = \begin{pmatrix} \Re U & -\Im U \\ \Im U & \Re U \end{pmatrix} \begin{pmatrix} \Re x \\ \Im x \end{pmatrix}$$

Since the Jacobian of this transformation is given by

$$J(y \to x) = \det \begin{pmatrix} \Re U & -\Im U \\ \Im U & \Re U \end{pmatrix} = |\det U|^2 = 1$$

we can write (1) as follows:

$$\varphi(A) = \int_{C^n} \exp\left(-\pi y^\dagger T y\right) dy \tag{3}$$

Changing variables $y_i = \rho_i \exp(\iota\theta_i)$ for $i = 1, \dots, n$ in (3) and limiting the integration domain to the Cartesian product of the domains $D_R \triangleq \{y : \ |y| < R\}$ for a given $R > 0$ yields:

$$\begin{aligned} \varphi_R(A) &= \int_{D_R^n} \exp\left(-\pi y^\dagger T y\right) dy \\ &= \prod_{i=1}^{n} \int_0^R e^{-\pi\lambda_i\rho_i^2} \rho_i d\rho_i \\ &\quad \times \int_{(0,2\pi)^n} \left\{ \prod_{i=1}^{n} \exp\left(-\pi e^{-\iota\theta_i} \rho_i \sum_{j=i+1}^{n} (T)_{ij} \rho_j e^{\iota\theta_j} \right) \right\} d\theta_1 ... d\theta_n. \end{aligned} \tag{4}$$

Since

$$\int_0^{2\pi} \exp\left(\alpha e^{\iota\theta}\right) d\theta = \oint_{|z|=1} \exp(\alpha z) \frac{dz}{\iota z} = 2\pi \tag{5}$$

is independent of α, we can calculate the inner integrals of (4) in the $\theta_1, \dots, \theta_n$. They all turn out to be equal to the constant 2π so that the overall result can be obtained as

$$\varphi_R(A) = \prod_{i=1}^{n} 2\pi \int_0^R e^{-\pi\lambda_i\rho_i^2} \rho_i d\rho_i = \prod_{i=1}^{n} \frac{1 - \exp(-\pi\lambda_i R^2)}{\lambda_i}. \tag{6}$$

Finally, we observe that the limit of $\varphi_R(A)$ as $R \to \infty$ exists if $\Re\lambda_i > 0$ for $i = 1, \dots, n$. In that case,

$$\varphi(A) = \lim_{R \to \infty} \varphi_R(A) = \prod_{i=1}^{n} \lambda_i^{-1} = \det A^{-1}. \tag{7}$$

(ii) Next, we consider the *only if* part of our statement and show that if one eigenvalue has non positive real part then the integral (1) does not exist in the Lebesgue sense. The proof is articulated in the following steps.

Again, we use Schur's unitary triangularization theorem to factor the matrix A as $A = U^\dagger T U$ and assume that the last eigenvalue λ_n of A has non-positive real part. Then, we partition the matrix T and the vector y respectively as

$$T = \begin{pmatrix} T_{11} & t_{12} \\ 0 & \lambda_n \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_n \end{pmatrix}.$$

We can rewrite (1) as

$$\varphi(A) = \int_{C^{n-1}} \exp(-\pi y_1^\dagger T_{11} y_1) dy_1 \int_C \exp(-\pi(\lambda_n |y_n|^2 + y_1^\dagger t_{12} y_n)) dy_n$$

From Fubini's theorem [3], a necessary condition for the existence of the integral (in the Lebesgue sense) is that the inner integral exists (in the Lebesgue sense) as a function of y_1. We show that the latter condition does not hold.

Let us write $\lambda_n = \alpha + \iota\beta$ with $\alpha \leq 0$, $y_1^\dagger t_{12} = \gamma + \iota\delta$, and $y_n = u + \iota v$. The inner integral can be written as

$$\begin{aligned} & \int_R \int_R \exp\left(-\pi\left((\alpha+\iota\beta)(u^2+v^2) + (\gamma u - \delta v) + \iota(\gamma v + \delta u)\right)\right) du dv \\ = & \int_R \exp\left(-\pi\left((\alpha+\iota\beta)u^2 + \gamma u + \iota\delta u\right)\right) du \\ & \times \int_R \exp\left(-\pi\left((\alpha+\iota\beta)v^2 + \iota\gamma v - \delta v\right)\right) dv \end{aligned} \tag{8}$$

The integral exists (in the Lebesgue sense) only if the integral of the magnitude of the integrand exists, i.e., if

$$\int_R \exp\left(-\pi(\alpha u^2 + \gamma u)\right) du$$

and

$$\int_R \exp\left(-\pi(\alpha v^2 - \delta v)\right) dv$$

exist. The existence of the above integrals requires strictly that $\alpha > 0$ which is what has to be shown.

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