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Exact Linearization Of Stochastic Dynamical Systems By State Space Coordinate Transformation And Feedback II — $g\sigma$-Linearization. *

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Abstract

Given a dynamical system, the task of exact feedback linearization by coordinate transformation of the state vector is to look for a combination of coordinate transformation and feedback which will make the system linear and controllable.

This paper studies linearization methods for stochastic SISO affine dynamical systems represented by vectorfield triplets in Euclidean space.

The paper is divided into two self-contained parts. In this part, we study the problem of $g\sigma$-linearization in detail and the results are illustrated by means of a numerical example solved with the help of symbolic algebra.

1 Introduction

This is the second part of the article devoted to the exact linearization of stochastic dynamical systems. In the first part [4] we defined the problem, discussed the properties of the correcting term, and studied the case of g-linearization. In the second part we are about to analyze the case of $g\sigma$-linearization. Finally, the results are illustrated with a numerical example, namely, the control of a crane under influence of noise.

2 Stratonovich $g\sigma$-linearization

The Stratonovich problems are not complicated by the second order Itô term. The transformation laws for Stratonovich systems are the same as the deterministic transformation laws, therefore many results of the deterministic linearization theory can be used.

2.1 Canonical Form

Recall that we require g-controllability of the resulting system. Since this is a Stratonovich problem, the transformed vector fields $\tilde{f}$ and $\tilde{g}$ do not depend on the dispersion vector

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field σ. Therefore, the *control part* and the *dispersion part* can be studied independently.

Any g-linear system can be transformed into integrator chain by a combination of a linear coordinate transformation and linear feedback. Therefore, if we set $\sigma = 0$, the canonical form is the integrator chain. In general, the dispersion vector field $\tilde{\sigma}$ is assumed to be arbitrary constant vector field $\tilde{\sigma}(x)_i = s_i$, $1 \leq i \leq n$ (see [4, Definition 4]) and this form is preserved by arbitrary linear transformations. Therefore the canonical form can be written as:

$$\begin{aligned}
\tilde{f}_i(x) &= x_{i+1},\ 1 \leq i \leq n-1 && (1)\\
\tilde{f}_n(x) &= 0 && (2)\\
\tilde{g}_i(x) &= 0,\ 1 \leq i \leq n-1 && (3)\\
\tilde{g}_n(x) &= 1 && (4)\\
\tilde{\sigma}_i(x) &= s_i,\ 1 \leq i \leq n. && (5)
\end{aligned}$$

We can compare this canonical form with the equations which define the transformed system $\tilde{\Theta}$.

PROPOSITION 1. There is a $g\sigma$-linearizing transformation $\mathcal{J}_{T,\alpha,\beta}$ of the SISO Stratonovich system $\Theta_S = (f(x), g(x), \sigma(x), U, x_0) \in X_S(n,1,1)$ into a g-controllable linear system if, and only if, there is a solution $\lambda : \mathrm{R}^n \to \mathrm{R}$ of the set of partial differential equations:

$$\begin{aligned}
\left\langle d\lambda, \mathrm{ad}_f^i g \right\rangle &= 0,\ 0 \leq i \leq n-2 && (6)\\
\left\langle d\lambda, \mathrm{ad}_f^{n-1} g \right\rangle &\neq 0 && (7)\\
\left\langle d\lambda, \mathrm{ad}_f^i \sigma \right\rangle &= s'_{i+1},\ 0 \leq i \leq n-1 && (8)
\end{aligned}$$

such that $s'_i \in \mathrm{R}$ are constants on U for $1 \leq i \leq n$. Then the linearizing transformation is given by:

$$\begin{aligned}
T_i &= \mathcal{L}_f^{i-1}\lambda,\ 1 \leq i \leq n && (9)\\
\alpha &= \frac{-\mathcal{L}_f^n \lambda}{\mathcal{L}_g \mathcal{L}_f^{n-1}\lambda},\ \beta = \frac{1}{\mathcal{L}_g \mathcal{L}_f^{n-1}\lambda}. && (10)
\end{aligned}$$

PROOF. Assume that Θ_S is transformed by $\mathcal{J}_{T,\alpha,\beta}$ into $\tilde{\Theta} := \left(\tilde{f}, \tilde{g}, \tilde{\sigma}, T(U), T(x_0)\right)$ where the i-th component of f,g, and σ can be expressed as: $\tilde{f}_i = \mathcal{L}_f T_i$, $\tilde{g}_i = \mathcal{L}_g T_i$, $\tilde{\sigma}_i = \mathcal{L}_\sigma T_i$. Moreover, the feedback is defined by $u = \alpha + \beta v$. The equations of Θ can be compared to the equation of the canonical form (1)

$$\begin{aligned}
\mathcal{L}_f T_i &= T_{i+1},\ 1 \leq i \leq n-1 && (11)\\
\mathcal{L}_g T_i &= 0,\ 1 \leq i \leq n-1 && (12)\\
\mathcal{L}_g T_n &= 1/\beta \neq 0 && (13)\\
\mathcal{L}_f T_n &= -\alpha/\beta. && (14)
\end{aligned}$$

The relations (6), (7) and (8) can be obtained from (11) using the recursive Leibniz rule. See e.g. [5, Theorem 7.4.16]. The relation (8) can be verified in a similar way: $\mathcal{L}_\sigma T_i = s_i$ for $1 \le i \le n$, thus by (11), $\mathcal{L}_\sigma \mathcal{L}_f T_i = s_{i+1}$ for $1 \le i \le n-1$ and by the Leibniz rule, $\mathcal{L}_\sigma \mathcal{L}_f T_i = \mathcal{L}_f \mathcal{L}_\sigma T_i - \mathcal{L}_{[f,\sigma]} T_i$ for $1 \le i \le n-1$. Moreover the Lie derivative of a constant is zero: $\mathcal{L}_f \mathcal{L}_\sigma T_i = \mathcal{L}_f s_i = 0$ and $s_{i+1} := \mathcal{L}_\sigma \mathcal{L}_f T_i = -\mathcal{L}_{[f,\sigma]} T_i$ for $1 \le i \le n-1$. The equations (8) is obtained by successive application of this relation. The symbols s_i are equal to s'_i except for the signs.

2.2 Necessary Conditions for the Control Part

The necessary conditions for linearizability of the *control part* of Θ (i.e. the system $(f, g, 0, U, x_0)$) can be expressed in geometrical form. These conditions are necessary but not sufficient since (8) must also be satisfied.

PROPOSITION 2. The $g\sigma$-linearizing transformation $\mathcal{J}_{T,\alpha,\beta}$ of the Stratonovich system Θ_S into a $g\sigma$-controllable system linear system exists only if the distribution $\{\mathrm{ad}_f^i g, 0 \le i \le n-2\}$ is involutive and the distribution $\{\mathrm{ad}_f^i g, 0 \le i \le n-1\}$ is n-dimensional.

For proof, see [3, Corollary 6.17], [5, Theorem 7.4.16] and [2, Theorem 4.2.3].

2.3 Conditions for the Dispersion Part

The conditions of Proposition 2 can be written in matrix form. We are looking for $T_1 = \lambda : \mathrm{R}^n \to \mathrm{R}$ such that

$$\begin{bmatrix} \mathrm{ad}_f^0 g \\ \mathrm{ad}_f^1 g \\ \vdots \\ \mathrm{ad}_f^{n-2} g \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda}{\partial x_1} \\ \frac{\partial \lambda}{\partial x_2} \\ \vdots \\ \frac{\partial \lambda}{\partial x_n} \end{bmatrix} = [0] \,. \tag{15}$$

The vectors $\mathrm{ad}_f^0 g, ..., \mathrm{ad}_f^{n-2} g$ are written in coordinates as $1 \times n$ rows. The first matrix is of dimension $(n-1) \times n$. Moreover it is required that

$$\left\langle d\lambda, \mathrm{ad}_f^{n-1} g \right\rangle = 0. \tag{16}$$

We will use the algorithm for feedback deterministic linearization to find such a transformation λ. Then we will verify if the conditions for linearity of the dispersion part of the system (8) are also valid. There are n additional linearity conditions (s_i are constants):

$$\begin{bmatrix} \mathrm{ad}_f^0 g \\ \mathrm{ad}_f^1 g \\ \vdots \\ \mathrm{ad}_f^{n-2} g \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda}{\partial x_1} \\ \frac{\partial \lambda}{\partial x_2} \\ \vdots \\ \frac{\partial \lambda}{\partial x_n} \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \tag{17}$$

In the deterministic case we were satisfied with *arbitrary* solution λ to the equations (15) and (16). In the stochastic case we must find the class of *all* solutions and then check if this class contains the solution for the σ part (17). Details depend on the methods used for solving the set of PDEs.

This result is summarized in the following algorithm:

- Step 1. Find $\Delta_k := \mathrm{ad}_f^i g$ for $0 \le i \le k-1$.
- Step 2. Verify that $\dim(\Delta_n)$ is n.
- Step 3. Verify that Δ_{n-1} is involutive (see [3, the remark following Definition 2.39]), otherwise no linearizing transformation exists.
- Step 4. Find all λ satisfying (18) by solving PDEs (18); denote C the set of all such functions.
- Step 5. Verify that there is a $\lambda_1 \in C$ such that the conditions (17) are satisfied, otherwise no linearizing transformation exists. Compute T,α,β from (11)-(14).

Now, we can illustrate one possible practical approach which worked for several simple problems solved by us (see the example in Section 4).

First we can compute the kernel of the matrix M_g to find the form $\omega = [\omega_1, \omega_2, ..., \omega_n]^T$ which satisfies $M_g \omega = 0$, i.e.. ω is perpendicular to M_g. In modern computer algebra systems there is a single command for this.

Proposition 2 assumes that n vector fields $\Delta_n := \{\mathrm{ad}_f^i g, 0 \le i \le n-1\}$ form an n dimensional space. The vector fields $\Delta_{n-1} := \{\mathrm{ad}_f^i g, 0 \le i \le n-2\}$ are chosen from them and consequently must form an $(n-1)$-dimensional space. Thus their kernel $d\lambda$ is exactly one dimensional and arbitrary $\omega' = c(x)\omega(x)$ also belongs to the kernel ($c(x)$ is a scalar).

But not every ω' that is perpendicular to M_g is a solution to the original linearization problem. The function ω' must be an exact one-form, i.e., there must be a scalar function λ such that $d\lambda = c(x)\omega(x)$. The Frobenius theorem guarantees that if Δ_{n-1} is involutive, then there is always $c(x) \in \mathrm{R}$ such that $c(x)\omega(x)$ is the exact one-form.

A necessary condition for a one-form $\omega = \sum_{i=1}^{n} \omega_i$ to be exact is $\partial\omega_i/\partial x_j = \partial\omega_j/\partial x_i$ for $1 \le i, j \le n$. Hence for every $1 \le i, j \le n$, $(\partial/\partial x_j)\,(c(x)\omega_i) = (\partial/\partial x_i)\,(c(x)\omega_j)$, thus for every $1 \le i, j \le n$,

$$\frac{\partial c(x)}{\partial x_i}\omega_j - \frac{\partial c(x)}{\partial x_j}\omega_i + c(x)\left(\frac{\partial \omega_i}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\right) = 0. \tag{18}$$

The later condition is a set of linear PDEs, with unknown $c(x)$, which are guaranteed to have a solution by the involutivness of Δ_{n-1} (the Frobenius theorem).

In our example the equation (18) is in a simple form which allows us to determine all the solutions easily. More complicated cases will require more sophisticated analysis.

3 Itô $g\sigma$-linearization

In the previous section we tried to find $g\sigma$-linearizations for Stratonovich dynamical systems. Once this is done, the correcting mapping can be used to construct Itô $g\sigma$-linearizing transformation. This method works for both the feedback and the 'feedbackless' linearization.

Given an Itô system Θ_I, the corresponding Stratonovich system Θ_S can be obtained using the correcting mapping $\Theta_S = \mathrm{Corr}\,(\Theta_I)$. Afterward, the Stratonovich $g\sigma$-linearization algorithm can be applied giving a linear system Θ_{2S}. Due to linearity of the drift vector field $\tilde{\sigma}$ of Θ_{2S}, the correcting term $\mathrm{corr}_{\tilde{\sigma}}(z)$ of the backward transformation $\mathrm{Corr}_{\tilde{\sigma}}^{-1}$ vanishes.

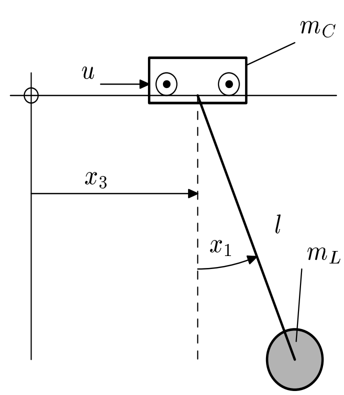
THEOREM 3. The $g\sigma$-linearizing transformation $\mathcal{J}_I$ of the Itô dynamical system $\Theta_I = (f(x), g(x), \sigma(x), U, x_0) \in \mathrm{X}_I(n,m,k)$, $f(x_0) + \mathrm{corr}_\sigma(x_0) = 0$, into a $g\sigma$-controllable linear system exists if, and only if, there is a $g\sigma$-linearizing transformation $\mathcal{J}_S$ of the Stratonovich dynamical system $\Theta_S = \left(\vec{f}(x), g(x), \sigma(x), U, x_0\right) = \mathrm{Corr}_\sigma(\Theta_I)$ where $\vec{f} = f + \mathrm{corr}_\sigma(x)$. Moreover $\mathcal{J}_I = \mathcal{J}_S \circ \mathrm{Corr}_\sigma$.

PROOF. We use the properties of the correcting term (see [4, Section 2.1]). Assume that there is a mapping $\mathcal{J}_S$ which transform Θ_S into a linear g-controllable system $(Ax, B, S, U, 0)$. Then $\mathcal{J}_I = \mathrm{Corr}_{\tilde{\sigma}}^{-1} \circ \mathcal{J}_S \circ \mathrm{Corr}_\sigma$. The backward correcting transformation $\mathrm{Corr}_{\tilde{\sigma}}^{-1}$ is the identity because the correcting term of a linear mapping $\mathrm{corr}_{\tilde{\sigma}}(x)$ is zero. Thus $\mathcal{J}_I = \mathcal{J}_S \circ \mathrm{Corr}_\sigma$ and $\mathcal{J}_I(\Theta_I)$ equals $(Ax, B, S, U, 0)$, which is linear and g-controllable by assumption.

To see the converse, assume that there is the Itô transformation $\mathcal{J}_I$ which linearizes Θ_I and $\Theta_I = {\mathrm{Corr}_\sigma}^{-1}(\Theta_S)$. Construct Stratonovich linearization by $\mathcal{J}_S = \mathcal{J}_I \circ \mathrm{Corr}_\sigma^{-1}$. Hence $\mathcal{J}_I$ linearizes $\mathrm{Corr}^{-1}(\Theta_S)$ and $\mathcal{J}_S$ linearizes Θ_S into the same linear and controllable system as $\mathcal{J}_I$.

4 Example — Crane

In this section the methods of stochastic exact linearization are demonstrated by means of an example — control of a crane under the influence of random disturbances. The description of the plant was adopted from [1] where the model of a crane linearized by approximative methods was studied. Unlike Ackermann, we control the same system using the exact model. Moreover the influence of random disturbances is added.



Crane

Consider the crane of the above figure, which can be used for example for loading containers into a ship. The hook must be automatically placed to a given position.

Feedback control is needed in order to dampen the motion before the hook is lowered into the ship. The input signal is the force u that accelerates the crab. The crab mass is m_C, the mass of the load m_L, the rope length is l, and the gravity acceleration g.

We assume that the driving motor has no nonlinearities, there is no friction or slip, no elasticity of the rope and no damping of the pendulum (e.g. from air drag). We will define four state variables: the rope angle x_1 (in radian), the angular velocity $x_2 = \dot{x}_1$, the position of the crab x_3, and the velocity of the crab $x_4 = \dot{x}_3$. As shown in [1], the plant is described by two second order differential equations:

$$u = (m_L + m_C)\ddot{x}_3 + m_L l(\ddot{x}_1 \cos x_1 - \dot{x}_1^2 \sin x_1) \tag{19}$$

$$0 = m_L \ddot{x}_3 \cos x_1 + m_L l \ddot{x}_1 + m_L g \sin x_1. \tag{20}$$

Additionally, we assume that the load is under influence of random disturbance, which can be modeled as a white noise process. The disturbance (wind) is horizontal, has zero mean and can be described by the Itô differential dw:

$$dx_2 = \frac{F \cos x_1}{m_L l}\, dw, \tag{21}$$

where F is a constant having the physical unit of force.

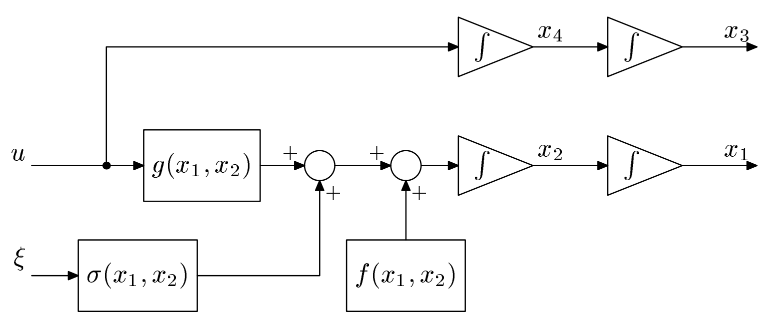
We used the symbolic algebraic system *Mathematica* to handle the computations. The complete *Mathematica* worksheet can be downloaded from the poster section of the web page of Applied Mathematics E-Notes.

Mathematica is used to solve the equations of the system for unknown values $\dot{x}_2$ and $\dot{x}_4$ (angular and positional acceleration). Values of vector fields f, g and σ are derived as follows:

$$f = \left[x_2, -\frac{\sin x_1\,(g(m_L + m_C) + l m_L x_2 \cos x_1)}{l(m_C + m_L - m_L \cos^2 x_1)}, x_4, 0\right]^T \tag{22}$$

$$g = \left[0, -\frac{\cos x_1}{l(m_C + m_L - m_L \cos^2 x_1)}, 0, u\right]^T \tag{23}$$

$$\sigma = \left[0, \frac{F \cos x_1}{m_L l}, 0, 0\right]^T. \tag{24}$$



The State Space Model of Crane

The state space model is shown in the above figure. We can see that the positional state variables x_3 and x_4 are isolated from the angular state variables x_1 and x_2. Later, we will concentrate on the angular variables pretending that the load will be stabilized no matter where the crane is. Consequently, we obtain only two-dimensional system for which the exact linearization techniques can be demonstrated.

Next, consider the random disturbances. Because the correcting term $\mathrm{corr}_\sigma(x)$ is zero, there is no difference in using either the Itô or the Stratonovich integral. In case of more “nonlinear” noise, one of the integrals must be selected. If the Itô model is chosen, Theorem 3 must be applied.

Now we evaluate the conditions of Proposition 2 to check that the system is linearizable. In fact, we need only to evaluate the non-singularity condition because every one-dimensional distribution is involutive, and the integrability is satisfied automatically. To this end, we will compute the null space (kernel) of the matrix $[[f,g],g]$, which is empty and therefore the matrix is nonsingular. We conclude, that the *deterministic* feedback problem is solvable.

Note that the system is already in the integrator chain form and hence $\lambda = x_1$ satisfies this condition. Therefore, the *deterministic* system is linearizable by feedback only, with no state space transformation at all, i.e., $z = T(x) = x$.

This choice of the output function λ is natural but does not cancel the nonlinearity in the dispersion coefficient σ. For this purpose, we must use the algorithm of Section 2.3 to construct another nontrivial coordinate transformation T.

To obtain this transformation, we must find the space of all functions λ satisfying conditions for feedback linearity (6). Observe that $\mathcal{L}_g\lambda$ must be zero hence $(\partial\lambda/\partial x_1)g_1 + (\partial\lambda/\partial x_2)g_2 = 0$. Since $g_1 = 0$ and $g_2 \neq 0$ in a neighborhood of x_0, we see that $\partial\lambda/\partial x_2 = 0$ and $\lambda = c_1(x_1)$ is a function of x_1 only (i.e., without x_2). The coordinate transformation is $T = [\lambda, \mathcal{L}_f\lambda]^T$. We want to select such $c_1(x_1)$ that the dispersion vector field $\tilde{\sigma} := T_*\sigma$ in the new coordinate system $z = T(x)$ will be constant: $(\partial c_1/\partial x_1)(F\cos x_1)/(m_L l)$ is a constant.

We decide to define the constant as $F/(m_L l)$, therefore $(\partial c_1/\partial x_1) = 1/\cos x_1$ and

$$T_1 = \lambda = c_1(x_1) = \int \frac{1}{\cos x_1} dx_1 = -\ln\left(\cos\frac{x_1}{2} - \sin\frac{x_1}{2}\right) + \ln\left(\cos\frac{x_1}{2} + \sin\frac{x_1}{2}\right)$$

$$T_2 = \mathcal{L}_f\lambda = x_2 \sec x_1.$$

Finally, we can compute the feedback from (10) resulting in

$$b = \frac{1}{l\left(m_c + m_l \sin^2(x_1)\right)}$$

and

$$a = \tan(x_1)\left(\sec(x_1)\, x_2^2 - b\, g\,(m_c + m_l) + l\, m_l\, x_2 \cos(x_1)\right).$$

In the *Mathematica* worksheet we validate the results by computing

$$\tilde{\Theta} = \mathcal{J}_{T,\alpha,\beta}(f(x), g(x), \sigma(x), U, x_0).$$

The computation shows that the system $(\hat{f}(x), \hat{g}(x), \hat{\sigma}(x), U, x_0)$ is in the integrator chain form in the z coordinate chart.

5 Conclusion

In this part of the article we showed a method for solving both the Stratonovich and the Itô $g\sigma$-linearization problem. In this case the effect of the Itô term can be reduced to the first order operator and consequently the problem is solvable by differential geometry.

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