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On The Method Of Successive Approximations For The J. L. Synge Electromagnetic Two-Body Problem *

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Abstract

Two-body problem of classical electrodynamics is considered. The equations of motion form a nonlinear neutral system with delays depending on the unknown trajectories. It is shown that a sequence of successive approximations for a global solution cannot be constructed.

1 Introduction

In 1940, J. L. Synge [1] formulates for the first time two-body problem of classical electrodynamics using retarded Lineard-Wiechert potentials and the relativistic form of the Lorenz force derived by Pauli [2]. In the same paper [1] Synge proposes a method of successive approximations for solving the system of equations of motion. His method is only heuristic because there is no proof of convergence. The main difficulty is caused by the presence of delays which are not defined explicitly in [1]. In view of the particular type of the system Synge constructs a sequence of successive approximations in such a way that on every step one has to solve a system of ordinary differential equations. On every next step this solution is replaced in another group of equations so that one obtains again an ordinary differential system. Synge assumes implicitly that the system possesses a global solution on $(-\infty, \infty)$. In the present paper we show that even as early as the second step (which is a Kepler problem) one cannot obtain a solution of the system existing on $(-\infty, t_0]$, where t_0 is the initial point. Therefore, one cannot define the next approximation. Thus, we conclude that not only a convergence theorem cannot be proved, but even a sequence of successive approximations does not exist. An immediate consequence is that a correct formulation of the initial value problem for electromagnetic two-body problem is given in [3] for 1-dimensional case and later for 3-dimensional case in [4], [5].

2 Synge's Equations of Motion

As in [1] we denote by $x^{(p)} = (x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t), x_4^{(p)}(t) = ict)$, $p = 1, 2$, where $i^2 = -1$, the space-time coordinates of the moving particles, by m_p their proper masses,

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by e_p their charges and c the speed of light. The coordinates of the velocity vectors are $u^{(p)} = (u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t))$, $p = 1, 2$. The coordinates of the unit tangent vectors to the world-lines are (cf. [2], [3]):

$$\lambda_\alpha^{(p)} = \frac{\gamma_p u_\alpha^{(p)}(t)}{c} = \frac{u_\alpha^{(p)}(t)}{\Delta_p},\ \alpha = 1,2,3; \lambda_4^{(p)} = i\gamma_p = \frac{ic}{\Delta_p} \tag{1}$$

where

$$\gamma_p = \left(1 - \frac{1}{c^2}\sum_{\alpha=1}^{3}[u_\alpha^{(p)}(t)]^2\right)^{-\frac{1}{2}}, \Delta_p = \left(c^2 - \sum_{\alpha=1}^{3}[u_\alpha^{(p)}(t)]^2\right)^{\frac{1}{2}}.$$

It follows $\gamma_p = c/\Delta_p$.

By $\langle .,. \rangle_4$ we denote the scalar product in the Minkowski space, while by $\langle .,. \rangle$ the scalar product in 3-dimensional Euclidean subspace. The equations of motion modelling the interaction of two moving charged particles are the following (cf. [1], [6]):

$$m_p \frac{d\lambda_r^{(p)}}{ds_p} = \frac{e_p}{c^2} F_{rn}^{(p)} \lambda_n^{(p)},\ r = 1,2,3,4, \tag{2}$$

where the elements of proper time are $ds_p = \frac{c}{\gamma_p}dt = \Delta_p dt$, $p = 1, 2$. Recall that in (2) there is a summation in n for $n = 1, 2, 3$. The elements $F_{rn}^{(p)}$ of the electromagnetic tensors are derived by the retarded Lienard-Wiechert potentials $A_r^{(p)} = -\frac{e_p \lambda_r^{(p)}}{\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}$, $r = 1, 2, 3, 4$, that is, $F_{rn}^{(p)} = \frac{\partial A_n^{(p)}}{\partial x_r^{(p)}} - \frac{\partial A_r^{(p)}}{\partial x_n^{(p)}}$. By $\xi^{(pq)}$ we denote the isotropic vectors (cf. [4], [5])

$$\xi^{(pq)} = (x_1^{(p)}(t) - x_1^{(q)}(t - \tau_{pq}(t)), x_2^{(p)}(t) - x_2^{(q)}(t - \tau_{pq}(t)), x_3^{(p)}(t) - x_3^{(q)}(t - \tau_{pq}(t)), ic\tau_{pq}(t))$$

where $\langle \xi^{(p,q)}, \xi^{(p,q)} \rangle_4 = 0$ or

$$\tau_{pq}(t) = \frac{1}{c}\left(\sum_{\beta=1}^{3}[x_\beta^{(p)}(t) - x_\beta^{(q)}(t - \tau_{pq}(t))]^2\right)^{\frac{1}{2}},\ (pq) = (12), (21). \tag{3}$$

Calculating $F_{rn}^{(p)}$ as in [1] and [4] we write equations from (2) in the form:

$$\begin{aligned} \frac{d\lambda_\alpha^{(p)}}{ds_p} &= \frac{Q_p}{c^2}\left\{ \frac{\xi_\alpha^{(pq)}\langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_\alpha^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3}\left[1 + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4\right] + \right. \\ &\left. + \frac{1}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^2}\left[\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 \frac{d\lambda_\alpha^{(q)}}{ds_q} - \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_\alpha^{(pq)}\right]\right\}, \end{aligned} \tag{4}$$

for $\alpha = 1, 2, 3$ and

$$\begin{aligned} \frac{d\lambda_4^{(p)}}{ds_p} &= \frac{Q_p}{c^2}\left\{ \frac{\xi_4^{(pq)}\langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_4^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3}\left[1 + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4\right] + \right. \\ &\left. + \frac{1}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^2}\left[\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 \frac{d\lambda_4^{(q)}}{ds_q} - \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_4^{(pq)}\right]\right\} \end{aligned} \tag{5}$$

where $Q_p = e_1e_2/m_p$ for $p = 1, 2$. Further on we denote $u^{(q)} \equiv u^{(q)}(t - \tau_{pq})$,

$$\lambda^{(q)} = (\gamma_{pq}u_1^{(q)}/c, \gamma_{pq}u_2^{(q)}/c, \gamma_{pq}u_3^{(q)}/c, i\gamma_{pq}) = (u_1^{(q)}/\Delta_{pq}, u_2^{(q)}/\Delta_{pq}, u_3^{(q)}/\Delta_{pq}, ic/\Delta_{pq})$$

where

$$\gamma_{pq} = \left(1 - \frac{1}{c^2}\sum_{\alpha=1}^{3}[u_\alpha^{(q)}(t - \tau_{pq}(t)]^2\right)^{-\frac{1}{2}},$$

$$\Delta_{pq} = \left(c^2 - \sum_{\alpha=1}^{3}[u_\alpha^{(q)}(t - \tau_{pq}(t))]^2\right)^{\frac{1}{2}}$$

and

$$\frac{d\lambda_\alpha^{(p)}}{ds_p} = \frac{d(\frac{\gamma_p}{c}u_\alpha^{(p)})}{\frac{c}{\gamma_p}dt} = \frac{d(\frac{u_\alpha^{(p)}}{\Delta_p})}{\Delta_p dt} = \frac{1}{\Delta_p^2}\dot{u}_\alpha^{(p)} + \frac{u_\alpha^{(p)}}{\Delta_p^4}\langle u^{(p)}, \dot{u}^{(p)}\rangle,\ \alpha = 1, 2, 3,$$

$$\frac{d\lambda_4^{(p)}}{ds_p} = \frac{d(i\gamma_p)}{\frac{c}{\gamma_p}dt} = \frac{icd(\frac{1}{\Delta_p})}{\Delta_p dt} = \frac{ic}{\Delta_p^4}\langle u^{(p)}, \dot{u}^{(p)}\rangle,$$

where the dot means a differentiation in t.

In order to calculate $\frac{d\lambda_\alpha}{ds_q}$ we need the derivative $\frac{dt}{dt_{pq}} \equiv D_{pq}$ which should be calculated from the relation

$$t - t_{pq} = \frac{1}{c}\left(\sum_{\alpha=1}^{3}[x_\alpha^{(p)}(t) - x_\alpha^{(q)}(t_{pq})]^2\right)^{\frac{1}{2}}$$

where $t_{pq} < t; t - \tau_{pq}(t) = t_{pq}$ by assumption. So we have

$$\frac{dt}{dt_{pq}} - 1 = \frac{\sum_{\alpha=1}^{3}[x_\alpha^{(p)}(t) - x_\alpha^{(q)}(t_{pq})][u_\alpha^{(p)}(t)\frac{dt}{dt_{pq}} - u_\alpha^{(q)}(t_{pq})]}{c\left(\sum_{\alpha=1}^{3}[x_\alpha^{(p)}(t) - x_\alpha^{(q)}(t_{pq})]^2\right)^{\frac{1}{2}}}.$$

Since (3) has a unique solution (cf. [4], [5]) we can solve the above equation with respect to D_{pq}: $D_{pq} = \frac{c^2\tau_{pq} - \langle\xi^{pq}, u^{(q)}\rangle}{c^2\tau_{pq} - \langle\xi^{pq}, u^{(p)}\rangle}$. We have also $\frac{d}{ds_p} = \frac{d}{\Delta_p dt}$. Then $\frac{d}{ds_q} = \frac{1}{\Delta_{pq}}\frac{d}{dt_{pq}} = \frac{1}{\Delta_{pq}}\frac{d}{dt_{pq}}\frac{d}{dt} = \frac{D_{pq}}{\Delta_{pq}}\frac{d}{dt}$;

$$\begin{aligned}\frac{d\lambda_\alpha^{(p)}}{ds_q} &= \frac{d(\frac{\gamma_{pq}}{c}u_\alpha^{(q)})}{\frac{c}{\gamma_{pq}}dt_{pq}} = \frac{d\left(\frac{u_\alpha^{(q)}}{\Delta_{pq}}\right)}{\Delta_{pq}dt_{pq}} = D_{pq}\frac{d\left(\frac{u_\alpha^{(q)}}{\Delta_{pq}}\right)}{\Delta_{pq}dt_{pq}} \\ &= D_{pq}\left[\dot{u}_\alpha^{(q)}\frac{1}{\Delta_{pq}^2} + \frac{u_\alpha^{(q)}}{\Delta_{pq}^4}\langle u^{(q)}, \dot{u}^{(q)}\rangle\right],\ \alpha = 1, 2, 3;\end{aligned}$$

$$\frac{d\lambda_4^{(q)}}{ds_q} = \frac{icD_{pq}}{\Delta_{pq}^4}\langle u^{(q)}, \dot{u}^{(q)}\rangle; \langle\lambda^{(p)}\lambda^{(q)}\rangle_4 = \frac{\langle u^{(p)}, u^{(q)}\rangle - c^2}{\Delta_p\Delta_{pq}};$$

$$\langle\lambda^{(p)},\xi^{(pq)}\rangle_4=\frac{\langle u^{(p)},\xi^{(pq)}\rangle-c^2\tau_{pq}}{\Delta_p};\langle\lambda^{(q)},\xi^{(pq)}\rangle_4=\frac{\langle u^{(q)},\xi^{(pq)}\rangle-c^2\tau_{pq}}{\Delta_{pq}};$$

$$\langle\xi^{(pq)},\frac{d\lambda^q}{ds_q}\rangle_4=D_{pq}\left[\frac{1}{\Delta_{pq}^2}\langle\xi^{(pq)},\dot u^{(q)}\rangle+\frac{\langle\xi^{(pq)},u^q\rangle-c^2\tau_{pq}}{\Delta_{pq}^4}\langle u^{(q)},\dot u^{(q)}\rangle\right];$$

$$\langle\lambda^{(p)},\frac{d\lambda^q}{ds_q}\rangle_4=\frac{D_{pq}}{\Delta_p\Delta_{pq}^2}\left[\langle u^{(p)},\dot u^{(q)}\rangle+\frac{\langle u^{(p)},u^q\rangle-c^2}{\Delta_{pq}^2}\langle u^{(q)},\dot u^{(q)}\rangle\right].$$

We note that in the above expressions $\xi^{(pq)}$ is 4-dimensional vector in the left-hand sides, while in the right-hand sides $\xi^{(pq)}$ is 3-dimensional part of the first three coordinates.

Replacing the above expressions in (4) and (5) and performing some obvious transformations we obtain for $(pq)=(12),(21)$, and $\alpha=1,2,3:$

$$\begin{aligned}&\frac{1}{\Delta_p}\dot u_\alpha^{(p)}+\frac{u_\alpha^{(p)}}{\Delta_p^3}\langle u^{(p)},\dot u^{(p)}\rangle\\ =\;&\frac{Q_p}{c^2}\left\{\frac{[c^2-\langle u^{(p)},u^{(q)}\rangle]\xi_\alpha^{(pq)}-[c^2\tau_{pq}-\langle u^{(p)},\xi^{(pq)}\rangle]u_\alpha^q}{[c^2\tau_{pq}-\langle u^{(q)},\xi^{(pq)}\rangle]^3}\right.\\ &\times\frac{\Delta_{pq}^4+D_{pq}\left[\Delta_{pq}^2\langle\xi^{(pq)},\dot u^{(q)}\rangle+(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq})\langle u^{(q)},\dot u^{(q)}\rangle\right]}{\Delta_{pq}^2}\\ &+D_{pq}\frac{[\langle u^{(p)},\xi^{(pq)}\rangle-c^2\tau_{pq}][\dot u_\alpha^{(q)}+u_\alpha^q\langle u^{(q)},\dot u^{(q)}\rangle/\Delta_{pq}^2]}{[c^2\tau_{pq}-\langle u^{(q)},\xi^{(pq)}\rangle]^2}\\ &\left.-D_{pq}\frac{[\langle u^{(p)},\dot u^{(q)}\rangle+(\langle u^{(p)},u^{(q)}\rangle-c^2)/\Delta_{pq}^2]\langle u^{(q)},\dot u^{(q)}\rangle\xi_\alpha^{(pq)}}{[c^2\tau_{pq}-\langle u^{(q)},\xi^{(pq)}\rangle]^2}\right\},\end{aligned}\tag{6}$$

$$\begin{aligned}&\frac{1}{\Delta_p^3}\langle u^{(p)},\dot u^{(p)}\rangle\\ =\;&\frac{Q_p}{c^2}\left\{\frac{\langle u^{(p)},\xi^{(pq)}\rangle-\tau_{pq}\langle u^{(p)},u^{(q)}\rangle}{[c^2\tau_{pq}-\langle u^{(q)},\xi^{(pq)}\rangle]^3}\right.\\ &\times\left[\Delta_{pq}^2+D_{pq}(\langle\xi^{(pq)},\dot u^{(q)}\rangle+(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq})\langle u^{(q)},\dot u^{(q)}\rangle/\Delta_{pq}^2)\right]+\\ &\left.+D_{pq}\frac{\frac{\langle u^{(p)},\xi^{(pq)}\rangle\langle u^{(q)},\dot u^{(q)}\rangle}{\Delta_{pq}^2}-\frac{\tau_{pq}\langle u^{(p)},\dot u^{(q)}\rangle-\tau_{pq}\langle u^{(p)},u^{(q)}\rangle\langle u^{(q)},\dot u^{(q)}\rangle}{\Delta_{pq}^2}}{[c^2\tau_{pq}-\langle u^{(q)},\xi^{(pq)}\rangle]^2}\right\}\end{aligned}\tag{7}$$

One can prove (as in [5]) that (7) is a consequence of (6). Indeed, multiplying (6) by $u_\alpha^{(p)}$, summing up in α and dividing into c^2 we obtain (7). Therefore we can consider a system consisting of the $1^{st},2^{nd},3^{rd},5^{th},6^{th}$ and 7^{th} equations. The last equations form a nonlinear functional differential system of neutral type (cf. [7], [8]) with respect to the unknown velocities. The delays τ_{pq} depend on the unknown trajectories by the relations (3).

Let us formulate the initial value problem for (6) in the following way: to find unknown velocities $u_\alpha^{(p)}(t)$, $p=1,2;\alpha=1,2,3$, for $t\geq 0$ satisfying equations (8), (9) of motion (written in details below):

$$\begin{aligned}
&\frac{1}{\Delta_1}\dot{u}_\alpha^{(1)}+\frac{u_\alpha^{(1)}}{\Delta_1^3}\langle u^{(1)},\dot{u}^{(1)}\rangle\\
=\;&\frac{Q_1}{c^2}\left\{\frac{[c^2-\langle u^{(1)},u^{(2)}\rangle]\xi_\alpha^{(12)}-[c^2\tau_{12}-\langle u^{(1)},\xi^{(12)}\rangle]u_\alpha^{(2)}}{[c^2\tau_{12}-\langle u^{(2)},\xi^{(12)}\rangle]^3}\right.\\
&\times[\Delta_{12}^4+D_{12}\Delta_{12}^2\langle\xi^{(12)},\dot{u}^{(2)}\rangle+(\langle\xi^{(12)},u^{(2)}\rangle-c^2\tau_{12})\langle u^{(2)},\dot{u}^{(2)}\rangle]/\Delta_{12}^2+\\
&+D_{12}\frac{(\langle u^{(1)},\xi^{(12)}\rangle-c^2\tau_{12})\dot{u}_\alpha^{(2)}-\langle u^{(1)},\dot{u}^{(2)}\rangle\xi_\alpha^{(12)}+\frac{(\langle u^{(1)},\xi^{(12)}\rangle-c^2\tau_{12})u_\alpha^{(2)}\langle u^{(2)},\dot{u}^{(2)}\rangle}{\Delta_{12}^2}}{[c^2\tau_{12}-\langle u^{(2)},\xi^{(12)}\rangle]^2}\\
&\left.+D_{12}\frac{(c^2-\langle u^{(1)},u^{(2)}\rangle)\xi_\alpha^{(12)}\langle u^{(2)},\dot{u}^{(2)}\rangle/\Delta_{12}^2}{[c^2\tau_{12}-\langle u^{(2)},\xi^{(12)}\rangle]^2}\right\}. \qquad (8)
\end{aligned}$$

Recall that in the above equations $u^{(1)}=u^{(1)}(t),u^{(2)}=u^{(2)}(t-\tau_{12})$. We also have

$$\begin{aligned}
&\frac{1}{\Delta_2}\dot{u}_\alpha^{(2)}+\frac{u_\alpha^{(2)}}{\Delta_2^3}\langle u^2,\dot{u}^{(2)}\rangle\\
=\;&\frac{Q_2}{c^2}\left\{\frac{[c^2-\langle u^2,u^{(1)}\rangle]\xi_\alpha^{(21)}-[c^2\tau_{21}-\langle u^{(2)},\xi^{(21)}\rangle]u_\alpha^{(1)}}{[c^2\tau_{21}-\langle u^{(1)},\xi^{(21)}\rangle]^3}\right.\\
&\times\langle[\Delta_{21}^4+D_{21}\Delta_{21}^2\langle\xi^{(21)},\dot{u}^{(1)}\rangle+(\langle\xi^{(21)},u^{(1)}\rangle-c^2\tau_{21})\langle u^{(1)},\dot{u}^{(1)}\rangle]/\Delta_{21}^2\\
&+D_{21}\frac{(\langle u^{(2)},\xi^{(21)}\rangle-c^2\tau_{21})\dot{u}_\alpha^{(1)}-\langle u^{(2)},\dot{u}^{(1)}\rangle\xi_\alpha^{(21)}+\frac{(\langle u^{(2)},\xi^{(21)}\rangle-c^2\tau_{21})u_\alpha^{(1)}\langle u^{(1)},\dot{u}^{(1)}\rangle}{\Delta_{21}^2}}{[c^2\tau_{21}-\langle u^{(1)},\xi^{(21)}\rangle]^2}\\
&\left.+D_{21}\frac{(c^2-\langle u^{(2)},u^{(1)}\rangle)\xi_\alpha^{(21)}\langle u^{(1)},\dot{u}^{(1)}\rangle/\Delta_{21}^2}{[c^2\tau_{21}-\langle u^{(1)},\xi^{(21)}\rangle]^2}\right\}. \qquad (9)
\end{aligned}$$

Recall that in the above equations $u^{(2)}=u^{(2)}(t)$, $u^{(1)}=u^{(1)}(t-\tau_{21})$. We note the delay functions $\tau_{pq}(t)$ satisfy functional equations (3) for $t\in R^1$. For $t\leq 0$, $u_\alpha^{(p)}(t)$ are prescribed functions $\overline{u}_\alpha^{(p)}(t)$, i.e.

$$u_\alpha^{(p)}(t)=\overline{u}_\alpha^{(p)}(t),\; t\leq 0,$$

where

$$\overline{u}_\alpha^{(p)}(t)=\frac{d\overline{x}_\alpha^{(p)}(t)}{dt},\; t\leq 0. \qquad (10)$$

This means that for prescribed trajectories $(\overline{x}_1^{(1)}(t),\overline{x}_2^{(1)}(t),\overline{x}_3^{(1)}(t))$, $(\overline{x}_1^{(2)}(t),\overline{x}_2^{(2)}(t),$ $\overline{x}_3^{(2)}(t))$ for $t\leq 0$ one has to find trajectories, satisfying the above system of equations for $t>0$. (We recall, $x_\alpha^{(p)}(t)=x_{\alpha 0}^{(p)}+\int_0^t u_\alpha^{(p)}(s)ds$ where $x_{\alpha 0}^{(p)}$ are the coordinates of the initial positions).

In what follows we consider the Kepler problem for equations (8), (9), (10), $p = 1, 2$; $\alpha = 1, 2, 3; (pq) = (12), (21)$. We suppose that the first particle P_1 is fixed at the origin $O(0, 0, 0)$, that is,

$$P_1 : \begin{cases} x_1^{(1)}(t) = 0, \\ x_2^{(1)}(t) = 0, \ t \in (-\infty, \infty), \\ x_3^{(1)}(t) = 0. \end{cases}$$

It follows by necessity

$$\begin{cases} \overline{x}_1^{(1)}(t) = 0 \\ \overline{x}_2^{(1)}(t) = 0 \\ \overline{x}_3^{(1)}(t) = 0 \end{cases} . \tag{11}$$

For the velocities and accelerations of the particles we obtain

$$\begin{cases} u_1^{(1)}(t) = 0 \\ u_2^{(1)}(t) = 0 \\ u_3^{(1)}(t) = 0 \end{cases} , \begin{cases} w_1^{(1)}(t) = 0 \\ w_2^{(1)}(t) = 0 \\ w_3^{(1)}(t) = 0 \end{cases} . \tag{12}$$

Replacing coordinates velocities and acceleration from (11) and (12) in (8) we obtain a system of three equations containing $x_\alpha^{(2)}(t - \tau_{12})$ (respectively $u_\alpha^{(2)}(t - \tau_{12})$ and $w_\alpha^{(2)}(t - \tau_{12})$). In other words the unknown functions have arguments $t - \tau_{12}(t)$. After the same replacing of (11) and (12) in (9) we obtain again a system of 3-equations but the unknown functions $x_\alpha^{(2)}(t)$ (respectively $u_\alpha^{(2)}, w_\alpha^{(2)}(t); \alpha = 1, 2, 3$) are taken at the instant t, i.e. their argument is t. Let us fix arbitrarily an initial point $t_0 \in (-\infty, \infty)$. Since we have already assumed that there is no collision between moving particles for $t \le t_0$ one can suppose that $\tau_{12}(t) \ge \tau_0 > 0$ or $-\tau_{12}(t) \le -\tau_0 < 0$. But for every $t \in (-\infty, t_0]$ we have $t - \tau_{12}(t) \le t - \tau_0 \le t_0 - \tau_0$. Therefore $t_s = \sup\{t - \tau_{12}(t) : t \in (-\infty, t_0]\} \le t_0 - \tau_0 < t_0$, i.e. the interval $[t_s, t_0]$ has a non-empty interior. This implies that even if the system (8) possesses any solution on $(-\infty, t_s]$, it has no influence on the solution of (9) on $[t_0, \infty)$ through $[t_s, t_0]$. Consequently one can disregard the system (8) and consider (9) as a "local" system on $[t_0, \infty)$. Saying "local" we mean a system with described initial conditions at the point t_0 (without taking into account the history for $t \le t_0$). Thus to the Kepler problem one can assign the system (9) on $[t_0, \infty)$ and the delays vanish. This can be explained by the fact that a charged particle at rest (at the origin point in our case) does not generate an electromagnetic field. Let us recall that $\tau_{pq}(t)$ is generated by the finite velocity of propagation of the electromagnetic field (cf.[2], [1], [6], [3]).

In order to discuss the method of successive approximations heuristically proposed by Synge [1], we have to write down the equations of motion (8), (9) in the following way:

$$\frac{d\vec{u}^{(1)}(t)}{dt} = \frac{e_1 e_2}{m_1} \vec{F} \left(\vec{\xi}^{(12)}, \vec{u}^{(1)}(t), \vec{u}^{(2)}(t - \tau_{12}), \frac{d\vec{u}^{(2)}(t - \tau_{12})}{dt} \right) \tag{13}$$

$$\frac{d\vec{u}^{(2)}(t)}{dt} = \frac{e_1 e_2}{m_2} \vec{F} \left(\vec{\xi}^{(21)}, \vec{u}^{(2)}(t), \vec{u}^{(1)}(t - \tau_{21}), \frac{d\vec{u}^{(1)}(t - \tau_{21})}{dt} \right) \tag{14}$$

where

$$\vec{u}^{(p)}(t) = \left[u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t)\right],\ p = 1, 2,$$

$$\vec{\xi}^{(pq)} = \left[\xi_1^{(pq)}, \xi_2^{(pq)}, \xi_3^{(pq)}\right],\ (p, q) = (12), (21).$$

The above system (13), (14) consists of six equations because we have already proved in [5] the 4-th and 8-th equations from the original system (4), (5) are consequences of the rest ones (that is, (5) is a consequence of (4)).

Let us briefly recall the original reasonings from [1]. Denote by $L^{(1)}$ and $L^{(2)}$ the world-lines for the two particles. Let $A^{(1)}$ and $A^{(2)}$ be any two events in space-time and let $(\lambda_r^{(1)})_{A^{(1)}}, (\lambda_r^{(2)})_{A^{(2)}}$ be time-like unit tangent vectors arbitrarily assigned at these events. The problem is to find world-lines $L^{(1)}$ and $L^{(2)}$ such that (13) and (14) are satisfied and $L^{(1)}$ has to pass through $A^{(1)}$ with the direction $(\lambda_r^{(1)})_{A^{(1)}}$ and $L^{(2)}$ - through $A^{(2)}$ with the direction $(\lambda_r^{(2)})_{A^{(2)}}$. The condition, $L^{(p)}$, $p = 1, 2$, to pass through $A^{(p)}$ with the direction $(\lambda_r^{(p)})_{A^{(p)}}$, plays the role of an initial condition.

As a basic approximation for $L^{(1)}$, Synge chooses $L_0^{(1)}$ - the world-line which passes through $A^{(1)}$ in the direction $(\lambda_r^{(1)})_{A^{(1)}}$ and satisfies (13) with the right-hand side replaced by zero, i.e. $L_0^{(1)}$ is a geodesic.

Then he finds $L_0^{(2)}$ to pass through $A^{(2)}$ with the direction $(\lambda_r^{(2)})_{A^{(2)}}$ and to satisfy (14), in which the field is taken to be that due to $L_0^{(1)}$. Obviously $L_0^{(2)}$ is the orbit of the Kepler problem discussed above.

Further on, he finds $L_1^{(1)}$ to pass through $A^{(1)}$ with the direction $(\lambda_r^{(1)})_{A^{(1)}}$ and to satisfy (13) and so on. Thus he gets a sequence of world-lines

$$L_0^{(1)} \to L_0^{(2)} \to L_1^{(1)} \to L_1^{(2)} \to L_2^{(1)} \to L_2^{(2)} \to \ldots \to L_n^{(1)} \to L_n^{(2)} \to \ldots$$

The idea becomes more clear if we notice that the system (13)-(14) is a particular case of a general neutral system of functional differential equations (cf. [7], [8]). Indeed (13) does not contain $\frac{d\vec{u}^{(1)}(t-\tau_{12})}{dt}$ while (14) does not contain $\frac{d\vec{u}^{(2)}(t-\tau_{21})}{dt}$. Therefore following the Synge method [1] we replace in (14) the solution of (13) with $\vec{F} \equiv \vec{0}$ and thus we obtain an ordinary differential system

$$\frac{d\vec{u}^{(2)}(t)}{dt} = \frac{e_1 e_2}{m_2}\vec{F}\left(\vec{\xi}^{(21)}, \vec{u}^{(2)}(t), \vec{u}^{(1)}(t-\tau_{21}), \frac{d\vec{u}^{(1)}(t-\tau_{21})}{dt}\right),$$

because $\vec{u}^{(1)}(t-\tau_{21})$ and $\frac{d\vec{u}^{(1)}(t-\tau_{21})}{dt}$ taking a part in the right-hand sides of the last system are already known functions. Solving the last system we can replace $\vec{u}^{(2)}(t-\tau_{12})$ and $\frac{d\vec{u}^{(2)}(t-\tau_{12})}{dt}$ in (13) and again (13) becomes an ordinary differential system without delays and so on.

Our main goal is already obvious. The process just describing cannot be continued beyond $L_0^{(2)}$. Indeed since $L_0^{(2)}$ is the Kepler problem we have already established that it can be solved only on an interval of the type $[t_0, \infty)$, that is, we do not know what happens to the left from t_0 and therefore we cannot obtain (replacing in (13)) an ordinary differential system. The last reasonings warrant the correctness of the considerations in [3]-[5].

As a final remark, in the original paper [1] the system (13), (14) is presented in the form $\frac{d\vec{u}^{(1)}(t)}{dt} = -\mu kF$ and $\frac{d\vec{u}^{(2)}(t)}{dt} = -kF$, where $\mu = \frac{m_2}{m_1}, k = -\frac{e_1 e_2}{m_2 c^2}$ (μ is a small parameter). In view of the above exposition this fact does not play an essential role.

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