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Existence Of Solutions For Nonconvex Second Order Differential Inclusions *

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Abstract

In this paper we prove an existence result for a second order differential inclusion

$$x'' \in F\left(x, x'\right) + f\left(t, x, x'\right),\ x(0) = x_0,\ x'(0) = y_0,$$

where F is an upper semicontinuous, compact valued multifunction, such that $F(x, y) \subset \partial V(y)$, for some convex proper lower semicontinuous function V, and f is a Carathéodory function.

1 Introduction

For the Cauchy problem

$$x' \in F(x),\ x(0) = \xi,$$

where F is an upper semicontinuous, cyclically monotone, compact values multifunction, the existence of local solutions was obtained by Bressan, et al. [4]. For some extensions of this results we refer to [1], [7], [12] and [13]. On the other hand, for second order differential inclusions

$$x'' \in F\left(x, x'\right), x(0) = x_0,\ x'(0) = y_0,$$

existence results were obtained by many authors (we refer to [3], [8], [9], [11], [13]). The case when F is an upper semicontinuous, compact valued multifunction, such that $F(x, y) \subset \partial V(y)$, for some convex proper lower semicontinuous function V, was considered in [10].

In this paper we prove an existence result for a second order differential inclusion

$$x'' \in F\left(x, x'\right) + f\left(t, x, x'\right),\ x(0) = x_0,\ x'(0) = y_0,$$

where F is an upper semicontinuous, compact valued multifunction, such that $F(x, y) \subset \partial V(y)$, for some convex proper lower semicontinuous function V, and f is a Carathéodory function.

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2 Statement of Result

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with scalar product $\langle .,. \rangle$ and norm $\|.\|$. For $x \in \mathbb{R}^m$ and $\varepsilon > 0$ let

$$B_\varepsilon(x) = \{y \in \mathbb{R}^m : \|x - y\| < \varepsilon\}$$

be the open ball centered at x with radius ε, and let $\overline{B}_\varepsilon(x)$ be its closure. For $x \in \mathbb{R}^m$ and for a closed subsets $A \subset \mathbb{R}^m$ we denote by $d(x, A)$ the distance from x to A given by

$$d\,(x, A) = \inf\{\|x - y\| : y \in A\}\,.$$

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a proper lower semicontinuous convex function. The multifunction $\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ defined by

$$\partial V(x) = \{\xi \in \mathbb{R}^m : V(y) - V(x) \geqslant \langle \xi, y - x\rangle, \forall y \in \mathbb{R}^m\}$$

is called the *subdifferential* (in the sense of convex analysis) of the function V.

We say that a multifunction $F : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is upper semicontinuous if for every $x \in \mathbb{R}^m$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F\,(y) \subset F\,(x) + B_\varepsilon\,(0)\,,\ \forall y \in B_\delta\,(x)\,.$$

For a multifunction $F : \Omega \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$ and for any $(x_0, y_0) \in \Omega$ we consider the Cauchy problem

$$x'' \in F\,(x, x') + f\,(t, x, x')\,,\ x\,(0) = x_0,\ x'\,(0) = y_0, \tag{1}$$

under the following assumptions:

(H_1) $\Omega \subset \mathbb{R}^{2m}$ is an open set and $F : \Omega \to 2^{\mathbb{R}^m}$ is an upper semicontinuous compact valued multifunction.

(H_2) There exists a proper convex and lower semicontinuous function $V : \mathbb{R}^m \to \mathbb{R}$ such that

$$F\,(x, y) \subset \partial V\,(y)\,, \forall\,(x, y) \in \Omega. \tag{2}$$

(H_3) $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function, i.e. for every $x, y \in \mathbb{R}^m$, $t \mapsto f\,(t, x, y)$ is measurable, for $t \in \mathbb{R}$, $(x, y) \mapsto f\,(t, x, y)$ is continuous and there exists $m\,(.) \in L^2\left(\mathbb{R}_+^*\right)$ such that:

$$\|f\,(t, x, y)\| \leq m\,(t)\,,\ (\forall)\,(x, y) \in \mathbb{R}^m \times \mathbb{R}^m,\ a.e.\ t \in \mathbb{R}. \tag{3}$$

By a solution of the problem (1) we mean any absolutely continuous function $x : [0, T] \to \mathbb{R}^m$ with absolutely continuous derivative x' such that $x(0) = x_0$, $x(0) = y_0$, and

$$x''(t) \in F(x(t), x'(t)) + f\,(t, x\,(t)\,, x'\,(t))\,,\ a.e. \text{ on } [0, T].$$

Our main result is the following:

THEOREM 1. If $F : \Omega \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$, $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and $V : \mathbb{R}^m \to \mathbb{R}$ satisfy assumptions (H_1), (H_2) and (H_3) then for every $(x_0, y_0) \in \Omega$ there exist $T > 0$ and a solution $x : [0, T] \to \mathbb{R}^m$ of the problem (1).

3 Proof of Our Result

Let $(x_0, y_0) \in \Omega$. Since Ω is open, there exists $r > 0$ such that the compact set $K := \overline{B}_r(x_0, y_0)$ is contained in Ω. Moreover, by the upper semicontinuity of F in (H_1) and by Proposition 1.1.3 in [2], the set

$$F(K) := \bigcup_{(x,y)\in K} F(x,y)$$

is compact, hence there exists $M > 0$ such that

$$\sup\{\|v\| : v \in F(x,y),\ (x,y) \in K\} \leq M.$$

Set

$$T' := \min\left\{\frac{r}{M}, \sqrt{\frac{r}{M}}, \frac{r}{2(\|y_0\|+1)}\right\}.$$

By (H_3) there exists $T'' > 0$ such that

$$\int_0^{T''} (m(t) + M)\, dt < r.$$

We shall prove the existence of a solution of the problem (1) defined on the interval $[0, T]$, where $0 < T \leq \min\{T', T''\}$.

For each integer $n \geq 1$ and for $1 \leq j \leq n$ we set $t_n^j := \frac{jT}{n}$, $I_n^j = [t_n^{j-1}, t_n^j]$ and for $t \in I_n^j$ we define

$$x_n(t) = x_n^j + (t - t_n^j)y_n^j + \frac{1}{2}(t - t_n^j)^2 v_n^j + \int_{j\frac{T}{n}}^t (s-t)\, f(s, x_n^j, y_n^j)ds, \tag{4}$$

where $x_n^0 = x_0$, $y_n^0 = y_0$, and, for $0 \leq j \leq n-1$, $v_n^j \in F(x_n^j, y_n^j)$,

$$\begin{cases} x_n^{j+1} = x_n^j + \frac{T}{n}y_n^j + \frac{1}{2}\left(\frac{T}{n}\right)^2 v_n^j \\ \\ y_n^{j+1} = y_n^j + \frac{T}{n}v_n^j. \end{cases} \tag{5}$$

Set, for $t \in (t_n^{j-1}, t_n^j)$, $j \in \{1, 2, ..., n\}$, $f_n(t) := f(s, x_n^j, y_n^j)$.

We claim that $(x_n^j, y_n^j) \in K$ for each $j \in \{1, 2, ..., n\}$. By the choice of T one has

$$\left\|x_n^1 - x_0\right\| \leq \frac{T}{n}\|y_0\| + \frac{1}{2}(\frac{T}{n})^2\|v_0\| < T\|y_0\| + \frac{1}{2}MT^2 < r$$

and

$$\left\|y_n^1 - y_0\right\| \leq T\|v_0\| < r,$$

hence the claim is true for $j = 1$.

We claim that for each $j > 1$ one has

$$\begin{cases} x_n^j = x_n^0 + j\frac{T}{n}y_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 [(2j-1)\,v_n^0 + (2j-3)\,v_n^1 + ... + v_n^{j-1}] \\ \\ y_n^j = y_n^0 + \frac{T}{n}[v_n^0 + v_n^1 + ... + v_n^{j-1}]. \end{cases} \tag{6}$$

The statement holds true for $j = 0$. Assume it holds for j, with $1 \leq j < n$. Then by (5) one obtains that

$$\begin{aligned}
x_n^{j+1} &= x_n^j + \frac{T}{n} y_n^j + \frac{1}{2}(\frac{T}{n})^2 v_n^j \\
&= x_n^0 + \frac{jT}{n} y_n^0 + \frac{1}{2}(\frac{T}{n})^2 [(2j-1)\, v_n^0 + (2j-1)\, v_n^1 + ... + v_n^{j-1}] + \\
&\quad + \frac{T}{n} y_n^0 + (\frac{T}{n})^2 [v_n^0 + v_n^1 + ... + v_n^{j-1}] v_n^j + \frac{1}{2}(\frac{T}{n})^2 v_n^j \\
&= x_n^0 + (j+1)\, \frac{T}{n} y_n^0 + \frac{1}{2}(\frac{T}{n})^2 [(2j+1)\, v_n^0 + (2j-1)\, v_n^1 + ... + v_n^j],
\end{aligned}$$

and

$$y_n^{j+1} = y_n^j + \frac{T}{n} v_n^j = y_n^0 + \frac{T}{n}[v_n^0 + v_n^1 + ... + v_n^j].$$

Therefore the relations in (6) are satisfied for each j, with $1 \leq j \leq n$ and our claim was proved.

Now, by (6) it follows easily that

$$\begin{aligned}
\|x_n^j - x_0\| &\leq \frac{jT}{n} \|y_0\| + \frac{1}{2}(\frac{T}{n})^2 \, [(2j-1) + (2j-3) + ... + 3 + 1]\, M \\
&= \frac{jT}{n} \|y_0\| + \frac{1}{2} M (\frac{jT}{n})^2 < T \|y_0\| + \frac{1}{2} M T^2 < r.
\end{aligned}$$

and

$$\|y_n^j - y_0\| \leq \frac{jT}{n} M < r,$$

proving that $(x_n^j, y_n^j) \in K := B_r\,(x_0, y_0)$, for each j, with $1 \leq j \leq n$.

By (4) we have that

$$\begin{aligned}
x_n'\,(t) &= y_n^j + (t - t_n^j) v_n^j + \int_{j\frac{T}{n}}^t f_n\,(s)\, ds, \\
x_n''\,(t) &= v_n^j + f_n\,(t)\,, \forall t \in I_n^j,
\end{aligned}$$

hence

$$\begin{cases}
\|x_n''\,(t)\,\| \leq M + m\,(t)\,, \forall t \in [0, T]\,, \\
\|x_n'\,(t)\| \leq \|y_0\| + 2r, \forall t \in [0, T] \\
\|x_n\,(t)\| \leq \|x_0\| + 2r\,(T+1)\,, \forall t \in [0, T]
\end{cases} \tag{7}$$

Moreover, for all $t \in [0, T]$ we have

$$d\,((x_n(t), x_n'(t), x_n''(t) - f_n\,(t)\,, \mathrm{graph}(F)) \leq \frac{2r(T+1)}{n}. \tag{8}$$

Then, by (7), we have

$$\int_0^T \|x_n''\,(t)\|^2\, dt \leq \int_0^T (M + m\,(t))^2\, dt$$

and therefore the sequence $(x_n'')_n$ is bounded in $L^2([0,T],\mathbb{R}^m)$.

For all τ, $t \in [0,T]$, we have that

$$\|x'(t) - x_n'(\tau)\| \leq \left| \int_\tau^t \|x_n''(s)\| \, ds \right| \leq \left| \int_\tau^t (M + m(s))^2 \, ds \right|$$

so that the sequence $(x_n')_n$ is equiuniformly continuous. Moreover, by (7) we see that $(x_n)_n$ is equi-Lipschitzian, hence equiuniformly continuous.

Therefore, $(x_n'')_n$ is bounded in $L^2([0,T],\mathbb{R}^m)$, $(x_n')_n$ and $(x_n)_n$ are bounded in $C([0,T],\mathbb{R}^m)$ and equiuniformly continuous, hence, by Theorem 0.3.4 in [2] there exist a subsequence, still denoted by $(x_n)_n$, and an absolutely continuous function $x:[0,T] \to \mathbb{R}^m$ such that

(i) $(x_n)_n$ converges uniformly to x;

(ii) $(x_n')_n$ converges uniformly to x';

(iii) $(x_n'')_n$ converges weakly in $L^2([0,T],\mathbb{R}^m)$ to x''.

Since $(f_n(.))_n$ converges to $f(.,(.))$ in $L^2([0,T],\mathbb{R}^m)$, then, by (H_2), (8) and Theorem 1.4.1 in [2] we obtain

$$x''(t) - f(t, x(t), x'(t)) \in coF(x(t), x'(t)) \subset \partial V(x'(t)), \ a.e., t \in [0,T], \tag{9}$$

where *co* stands for the closed convex hull.

By (9) and Lemma 3.3 in [5] we obtain that

$$\frac{d}{dt} V(x'(t)) = \langle x''(t), x''(t) - f(t, x(t), x'(t)) \rangle, \ a.e., t \in [0,T],$$

hence,

$$V(x'(T)) - V(x'(0)) = \int_0^T \|x''(t)\|^2 \, dt - \int_0^T \langle x''(t), \ f(t, x(t), x'(t)) \rangle dt. \tag{10}$$

On the other hand, since

$$x_n''(t) - f_n(t) = v_n^j \in F(x_n^j, y_n^j) \subset \partial V(x_n^{'}(t_n^j)), \forall t \in I_n^j,$$

and so from the properties of the subdifferential of a convex function, it follows that

$$\begin{aligned} V(x_n'(t_n^{j+1})) - V(x_n'(t_n^j)) &\geq \langle x_n''(t) - f_n(t), x_n'(t_n^{j+1}) - x_n'(t_n^j) \rangle \\ &= \langle x_n''(t) - f_n(t), \int_{t_n^j}^{t_n^{j+1}} x_n^{''}(s) \, ds \rangle \\ &= \int_{t_n^j}^{t_n^{j+1}} \|x''(t)\|^2 dt - \int_{t_n^j}^{t_n^{j+1}} \langle f_n(t), x_n''(t) \rangle \, dt. \end{aligned}$$

By adding the n inequalities from above, we get

$$V(x_m'(T)) - V(y_0) \geq \int_0^T \|x_n''(t)\|^2 \, dt - \int_0^T \langle f_n(t), x_n''(t) \rangle \, dt. \tag{11}$$

The convergence of $(f_n(.))_n$ in L^2-norm and of $(x_n''(.))_n$ in the weak topology of L^2 implies that

$$\lim_{n\to\infty}\int_0^T \langle f_n(t), x_n''(t)\rangle\, dt = \int_0^T \langle f(t,x(t),x'(t)), x''(t)\rangle\, dt.$$

By passing to the limit as $n\to\infty$ in (11) and using the continuity of V we see that

$$V(x'(T)) - V(y_0) \geq \limsup_{n\to\infty}\int_0^T \|x_n''(t)\|^2\, dt - \int_0^T \langle f(t,x(t),x'(t)), x''(t)\rangle\, dt, \quad (12)$$

hence, by (10) and (12), we obtain

$$\|x''(t)\|_{L^2}^2 \geq \limsup_{n\to\infty}\|x_n''(t)\|_{L^2}^2 .$$

Since, by the weak lower semicontinuity of the norm,

$$\|x''(t)\|_{L^2}^2 \leq \liminf_{n\to\infty}\|x_n''(t)\|_{L^2}^2 ,$$

we have that $\|x''(t)\|_{L^2}^2 = \lim_{n\to\infty}\|x_n''(t)\|_{L^2}^2$ i.e. $(x_n'')_n$ converge to x'' strongly in $L^2([0,T],\mathbb{R}^m)$ (Proposition III.30 in [6]). Hence a subsequence again denoted by $(x_n'')_n$ converge pointwise to x''.

Since by (H_1) the graph of F is closed and, by (8),

$$\lim_{n\to\infty} d\left((x_n(t), x_n'(t), x_n''(t) - f_n(t)), \mathrm{graph}(F)\right) = 0,$$

we obtain that

$$x''(t) \in F(x(t), x'(t)) + f(t,x(t),x'(t)),\ a.e., t\in[0,T].$$

Since x obviously satisfies the initial conditions, it is a solution of the problem (1).

4 An Example

For $D\subset R^n$ and $x\in D$, denote by $T_D(x)$ the Bouligand's contingent cone of D at x, defined by

$$T_D(x) = \left\{ v\in R^m; \liminf_{h\to 0^+}\frac{d(x+hv,D)}{h} = 0\right\}.$$

Also, $N_D(x)$ is the normal cone of D at x, defined by

$$N_D(x) = \{v\in R^m; \langle y,v\rangle \leq 0, (\forall)\, v\in T_D(x)\}.$$

In what follows we consider D as a closed subset such that $\theta\in D$ and $\theta\notin \mathrm{Int}(D)$, where θ is the zero element of R^m.

We set $K = T_D(\theta)$, $Q = \mathrm{Int}(N_D(\theta))$, $\Omega = B_1(\theta)\times Q$ and denote by $\pi_K(y)$ the projection

$$\pi_K(y) = \{u\in K : d(y,u) = d(y,K)\}.$$

THEOREM 2. Suppose $\mathrm{Int}\,(N_D\,(x)) \neq \emptyset$ and $f : R \times R^m \times R^m \to R^m$ satisfies the assumption (H_3). Then there exist $T > 0$ and a solution $x : [0, T] \to R^m$ for the following Cauchy problem

$$x'' \in (1 - \|x\|)\,\pi_K\,(x') + f(t, x, x'),\;\; (x(0), x'(0)) = (x_0, y_0)\,.$$

PROOF. By Proposition 2 in [4] there exists a convex function V such that

$$\pi_K\,(y) \subset \partial V(y),\;\; (\forall)\, y \in Q.$$

We recall (see [4]) that the function V is defined by

$$V(y) = \sup\{\varphi_u(y);\; u \in K\},$$

where

$$\varphi_u(y) = \langle u, y\rangle - \frac{1}{2}\|u\|^2,\;\; y \in Q.$$

Also, we observe that the following assertions are equivalent:

$$\begin{cases} (i)\; u \in \pi_K\,(y)\,; \\ (ii)\; \|y - u\| \leq \|y - v\|,\; (\forall)\, v \in K; \\ (iii)\; \varphi_u(y) \geq \varphi_v(y),\; (\forall)\, v \in K. \end{cases} \tag{13}$$

Let $(x, y) \in \Omega$ be and let $z \in F(x, y)$. Then there exists $u \in \pi_K\,(y)$ such that $z = (1 - \|x\|)\,u$. We have that

$$\begin{aligned} \varphi_{(1-\|x\|)u}(y) &= \langle (1 - \|x\|)\,u, y\rangle - \frac{1}{2}\,(1 - \|x\|)^2\,\|u\|^2 \\ &\geq \langle (1 - \|x\|)\,u, y\rangle - \frac{1}{2}\,(1 - \|x\|)\,\|u\|^2 \\ &= \langle u, y\rangle - \frac{1}{2}\|u\|^2 - \|x\|(\langle u, y\rangle - \frac{1}{2}\|u\|^2) \\ &= (1 - \|x\|)\,\varphi_u(y), \end{aligned}$$

hence

$$\varphi_{(1-\|x\|)u}(y) \geq (1 - \|x\|)\,\varphi_u(y). \tag{14}$$

Since $u \in \pi_K\,(y)$, then $\varphi_u(y) \geq \varphi_v(y)$, for every $v \in K$, and by (14) it follows that

$$\begin{aligned} \varphi_{(1-\|x\|)u}(y) - \varphi_v(y) &\geq (1 - \|x\|)\,\varphi_u(y) - \varphi_v(y) \\ &\geq (1 - \|x\|)\,\varphi_v(y) - \varphi_v(y) = -\|x\|\varphi_v(y), \end{aligned}$$

hence

$$\varphi_{(1-\|x\|)u}(y) - \varphi_v(y) \geq -\|x\|\varphi_v(y) \tag{15}$$

for every $v \in K$.

Since $y \in Q = \mathrm{Int}\,(N_D\,(\theta))$ we have that

$$\langle y, v\rangle \leq 0,\;\; \text{for every } v \in K = T_D\,(\theta)\,,$$

hence

$$\varphi_v(y) = \langle y, v\rangle - \frac{1}{2}\|v\|^2 \leq 0, \text{ for every } v \in K. \tag{16}$$

From (15) and (16), it follows that

$$\varphi_{(1-\|x\|)u}(y) \geq \varphi_v(y),\ v \in K. \tag{17}$$

Then (17) and the equivalent assertions in (13) imply that

$$z = (1 - \|x\|)\, u \in \pi_K\, (y) \subset \partial V(y).$$

If we define the multifunction $F : \Omega \to 2^{R^m}$ by

$$F(x, y) = (1 - \|x\|)\, \pi_K\, (y)\,,$$

then F is with compact valued and upper semicontinuous and there exists a convex function $V : R^m \to R$ such that

$$F(x, y) \subset \partial V(y),\ (\forall)\,(x, y) \in \Omega.$$

Therefore, F and f satisfies assumptions (H_1) , (H_2) (H_3) and our proof is complete.

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