Existence Of Solutions For Nonconvex Second Order Differential Inclusions *

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Abstract

In this paper we prove an existence result for a second order differential inclusion

\[ x'' \in F(x, x') + f(t, x, x'), \quad x(0) = x_0, \quad x'(0) = y_0, \]

where \( F \) is an upper semicontinuous, compact valued multifunction, such that \( F(x, y) \subset \partial V(y) \), for some convex proper lower semicontinuous function \( V \), and \( f \) is a Carathéodory function.

1 Introduction

For the Cauchy problem

\[ x' \in F(x), \quad x(0) = \xi, \]

where \( F \) is an upper semicontinuous, cyclically monotone, compact valued multifunction, the existence of local solutions was obtained by Bressan, et al. \[4\]. For some extensions of this results we refer to \[1\], \[7\], \[12\] and \[13\]. On the other hand, for second order differential inclusions

\[ x'' \in F(x, x'), \quad x(0) = x_0, \quad x'(0) = y_0, \]

existence results were obtained by many authors (we refer to \[3\], \[8\], \[9\], \[11\], \[13\]). The case when \( F \) is an upper semicontinuous, compact valued multifunction, such that \( F(x, y) \subset \partial V(y) \), for some convex proper lower semicontinuous function \( V \), was considered in \[10\].

In this paper we prove an existence result for a second order differential inclusion

\[ x'' \in F(x, x') + f(t, x, x'), \quad x(0) = x_0, \quad x'(0) = y_0, \]

where \( F \) is an upper semicontinuous, compact valued multifunction, such that \( F(x, y) \subset \partial V(y) \), for some convex proper lower semicontinuous function \( V \), and \( f \) is a Carathéodory function.

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2 Statement of Result

Let $\mathbb{R}^m$ be the $m$-dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For $x \in \mathbb{R}^m$ and $\varepsilon > 0$ let

$$B_\varepsilon (x) = \{ y \in \mathbb{R}^m : \| x - y \| < \varepsilon \}$$

be the open ball centered at $x$ with radius $\varepsilon$, and let $\overline{B}_\varepsilon (x)$ be its closure. For $x \in \mathbb{R}^m$ and for a closed subsets $A \subset \mathbb{R}^m$ we denote by $d(x, A)$ the distance from $x$ to $A$ given by

$$d(x, A) = \inf \{ \| x - y \| : y \in A \}.$$

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a proper lower semicontinuous convex function. The multifunction $\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ defined by

$$\partial V (x) = \{ \xi \in \mathbb{R}^m : V(y) - V(x) \geq \langle \xi, y - x \rangle, \forall y \in \mathbb{R}^m \}$$

is called the subdifferential (in the sense of convex analysis) of the function $V$.

We say that a multifunction $F : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is upper semicontinuous if for every $x \in \mathbb{R}^m$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(y) \subset F(x) + B_\varepsilon (0), \forall y \in B_\delta (x).$$

For a multifunction $F : \Omega \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$ and for any $(x_0, y_0) \in \Omega$ we consider the Cauchy problem

$$x'' \in F(x, x'), x(0) = x_0, x'(0) = y_0,$$

under the following assumptions:

(H1) $\Omega \subset \mathbb{R}^{2m}$ is an open set and $F : \Omega \to 2^{\mathbb{R}^m}$ is an upper semicontinuous compact valued multifunction.

(H2) There exists a proper convex and lower semicontinuous function $V : \mathbb{R}^m \to \mathbb{R}$ such that

$$F(x, y) \subset \partial V(y), \forall (x, y) \in \Omega. \quad (2)$$

(H3) $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function, i.e. for every $x, y \in \mathbb{R}^m$, $t \mapsto f(t, x, y)$ is measurable, for $t \in \mathbb{R}$, $(x, y) \mapsto f(t, x, y)$ is continuous and there exists $m(.) \in L^2(\mathbb{R}_+^1)$ such that:

$$\| f(t, x, y) \| \leq m(t), \forall (x, y) \in \mathbb{R}^m \times \mathbb{R}^m, \text{a.e. } t \in \mathbb{R}. \quad (3)$$

By a solution of the problem (1) we mean any absolutely continuous function $x : [0, T] \to \mathbb{R}^m$ with absolutely continuous derivative $x'$ such that $x(0) = x_0, x(0) = y_0$, and

$$x''(t) \in F(x(t), x'(t)) + f(t, x(t), x'(t)), \text{a.e. on } [0, T].$$

Our main result is the following:

THEOREM 1. If $F : \Omega \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$, $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and $V : \mathbb{R}^m \to \mathbb{R}$ satisfy assumptions (H1), (H2) and (H3) then for every $(x_0, y_0) \in \Omega$ there exist $T > 0$ and a solution $x : [0, T] \to \mathbb{R}^m$ of the problem (1).
3 Proof of Our Result

Let \((x_0, y_0) \in \Omega\). Since \(\Omega\) is open, there exists \(r > 0\) such that the compact set \(K := \overline{B_r} (x_0, y_0)\) is contained in \(\Omega\). Moreover, by the upper semicontinuity of \(F\) in \((H_1)\) and by Proposition 1.1.3 in [2], the set

\[
F (K) := \bigcup_{(x,y) \in K} F (x,y)
\]

is compact, hence there exists \(M > 0\) such that

\[
\sup \{ \| v \| : v \in F (x,y), (x,y) \in K \} \leq M.
\]

Set

\[
T' := \min \left\{ \frac{r}{M} \sqrt{\frac{r}{M^2 - 2 (\| y_0 \| + 1)}} \right\}.
\]

By \((H_3)\) there exists \(T'' > 0\) such that

\[
\int_0^{T''} (m(t) + M) \, dt < r.
\]

We shall prove the existence of a solution of the problem (1) defined on the interval \([0, T]\), where \(0 < T \leq \min \{ T', T'' \}\).

For each integer \(n \geq 1\) and for \(1 \leq j \leq n\) we set \(t_n^j := \frac{j T'}{n}\), \(I_n^j = [t_n^{j-1}, t_n^j]\) and for \(t \in I_n^j\) we define

\[
x_n (t) = x_n^0 + (t - t_n^j) y_n^j + \frac{1}{2} (t - t_n^j)^2 v_n^j + \int_{t_n^j}^t (s - t) f (s, x_n^j, y_n^j) \, ds,
\]

where \(x_n^0 = x_0, y_n^0 = y_0\), and, for \(0 \leq j \leq n - 1\), \(v_n^j \in F (x_n^j, y_n^j)\),

\[
\begin{align*}
x_n^{j+1} &= x_n^j + \frac{T}{n} y_n^j + \frac{1}{2} \left( \frac{T}{n} \right)^2 v_n^j, \\
y_n^{j+1} &= y_n^j + \frac{T}{n} v_n^j.
\end{align*}
\]

Set, for \(t \in (t_n^{j-1}, t_n^j), j \in \{ 1, 2, ..., n \}\), \(f_n (t) := f (s, x_n^j, y_n^j)\).

We claim that \((x_n^j, y_n^j) \in K\) for each \(j \in \{ 1, 2, ..., n \}\). By the choice of \(T\) one has

\[
\| x_n^1 - x_0 \| \leq \frac{T}{n} \| y_0 \| + \frac{1}{2} \left( \frac{T}{n} \right)^2 \| v_0 \| < T \| y_0 \| + \frac{1}{2} M T^2 < r
\]

and

\[
\| y_n^1 - y_0 \| \leq T \| v_0 \| < r,
\]

hence the claim is true for \(j = 1\).

We claim that for each \(j > 1\) one has

\[
x_n^j = x_n^0 + j \frac{T}{n} y_n^0 + \frac{1}{2} \left( \frac{T}{n} \right)^2 [(2j - 1) v_n^0 + (2j - 3) v_n^1 + ... + v_n^{j-1}]
\]

\[
y_n^j = y_n^0 + \frac{T}{n} [v_n^0 + v_n^1 + ... + v_n^{j-1}].
\]
The statement holds true for \( j = 0 \). Assume it holds for \( j \), with \( 1 \leq j < n \). Then by (5) one obtains that
\[
x^{j+1}_n = x^j_n + \frac{T}{n}y^j_n + \frac{1}{2}\left(\frac{T}{n}\right)^2 v^j_n
\]
\[
= x^0_n + \frac{jT}{n}y^0_n + \frac{1}{2}\left(\frac{T}{n}\right)^2 [(2j - 1)v^0_n + (2j - 1)v^1_n + ... + v^{j-1}_n] + 
\]
\[
+ \frac{T}{n}y^0_n + \frac{T}{n}[(2j + 1)v^0_n + (2j - 1)v^1_n + ... + v^{j+1}_n] + \frac{1}{2}\left(\frac{T}{n}\right)^2 v^{j+1}_n
\]
\[
= x^0_n + (j + 1)\frac{T}{n}y^0_n + \frac{1}{2}\left(\frac{T}{n}\right)^2 [(2j + 1)v^0_n + (2j - 1)v^1_n + ... + v^{j+1}_n],
\]
and
\[
y^{j+1}_n = y^j_n + \frac{T}{n}v^j_n = y^0_n + \frac{T}{n}[v^0_n + v^1_n + ... + v^j_n].
\]
Therefore the relations in (6) are satisfied for each \( j \), with \( 1 \leq j \leq n \) and our claim was proved.

Now, by (6) it follows easily that
\[
\|x^j_n - x_0\| \leq \frac{jT}{n}\|y_0\| + \frac{1}{2}\left(\frac{T}{n}\right)^2 [(2j - 1) + (2j - 3) + ... + 3 + 1] M
\]
\[
= \frac{jT}{n}\|y_0\| + \frac{1}{2}\left(\frac{jT}{n}\right)^2 < T\|y_0\| + \frac{1}{2}MT^2 < r.
\]
and
\[
\|y^j_n - y_0\| \leq \frac{jT}{n} M < r,
\]
proving that \((x^j_n, y^j_n) \in K := B_r (x_0, y_0), \) for each \( j \), with \( 1 \leq j \leq n \).

By (4) we have that
\[
x^j_n (t) = y^j_n + (t - t^j_n)v^j_n + \int_{t^j_n}^t f_n (s) ds,
\]
\[
x^{j+1}_n (t) = v^j_n + f_n (t), \forall t \in I^j_n,
\]
hence
\[
\|x^{j+1}_n (t)\| \leq M + m (t), \forall t \in [0, T],
\]
\[
\|x^j_n (t)\| \leq \|y_0\| + 2r, \forall t \in [0, T]
\]
(7)
\[
\|x_n (t)\| \leq \|x_0\| + 2r (T + 1), \forall t \in [0, T]
\]
Moreover, for all \( t \in [0, T] \) we have
\[
d ((x_n(t), x'_n(t), x''_n(t) - f_n(t), \text{graph}(F)) \leq \frac{2r(T + 1)}{n}. \)
(8)

Then, by (7), we have
\[
\int_0^T \|x''_n(t)\|^2 dt \leq \int_0^T (M + m(t))^2 dt
\]
and therefore the sequence \((x_n')_n\) is bounded in \(L^2 ([0, T], \mathbb{R}^m)\).

For all \(\tau, t \in [0, T]\), we have that

\[
\|x' (t) - x' (\tau)\| \leq \left| \int_{\tau}^{t} \|x'' (s)\| \, ds \right| \leq \left| \int_{\tau}^{t} (M + m(s))^2 \, ds \right|
\]

so that the sequence \((x_n')_n\) is equiuniformly continuous. Moreover, by (7) we see that \((x_n)_n\) is equi-Lipschitzian, hence equiuniformly continuous.

Therefore, \((x_n')_n\) is bounded in \(L^2 ([0, T], \mathbb{R}^m)\), \((x_n')_n\) and \((x_n)_n\) are bounded in \(C ([0, T], \mathbb{R}^m)\) and equiuniformly continuous, hence, by Theorem 0.3.4 in [2] there exist a subsequence, still denoted by \((x_n)_n\), and an absolutely continuous function \(x : [0, T] \to \mathbb{R}^m\) such that

(i) \((x_n)_n\) converges uniformly to \(x\);
(ii) \((x_n')_n\) converges uniformly to \(x'\);
(iii) \((x_n'')_n\) converges weakly in \(L^2 ([0, T], \mathbb{R}^m)\) to \(x''\).

Since \((f_n (\cdot))_n\) converges to \(f (\cdot, \cdot)\) in \(L^2 ([0, T], \mathbb{R}^m)\), then, by \((H_2)\), (8) and Theorem 1.4.1 in [2] we obtain

\[
x'' (t) - f (t, x(t), x' (t)) \in \text{co} \mathcal{F} (x (t), x' (t)) \subset \partial V (x' (t)), \text{ a.e., } t \in [0, T], \quad (9)
\]

where \(\text{co}\) stands for the closed convex hull.

By (9) and Lemma 3.3 in [5] we obtain that

\[
\frac{d}{dt} V (x' (t)) = \langle x'' (t), x'' (t) - f (t, x(t), x' (t)) \rangle, \text{ a.e., } t \in [0, T],
\]

hence,

\[
V (x' (T)) - V (x' (0)) = \int_{0}^{T} \|x'' (t)\|^2 \, dt - \int_{0}^{T} \langle x'' (t), f (t, x(t), x' (t)) \rangle \, dt. \quad (10)
\]

On the other hand, since

\[
x_n'' (t) - f_n (t) = v_n^j \in F (x_n (t), y_n^j) \subset \partial V (x_n (t)), \forall t \in I_n^j,
\]

and so from the properties of the subdifferential of a convex function, it follows that

\[
V (x_n (t_n) + 1)) - V (x_n (t_n)) \geq \langle x'' (t) - f_n (t), x_n (t_n) + 1) - x_n (t_n) \rangle = \langle x'' (t) - f_n (t), \int_{t_n}^{t_n + 1} x_n (s) \, ds \rangle = \int_{t_n}^{t_n + 1} \|x'' (t)\|^2 \, dt - \int_{t_n}^{t_n + 1} \langle f_n (t), x_n (t) \rangle \, dt.
\]

By adding the \(n\) inequalities from above, we get

\[
V (x_m (T)) - V (y_0) \geq \int_{0}^{T} \|x'' (t)\|^2 \, dt - \int_{0}^{T} \langle f_n (t), x_n (t) \rangle \, dt. \quad (11)
\]
The convergence of \((f_n(\cdot))_n\) in \(L^2\)-norm and of \((x''_n(\cdot))_n\) in the weak topology of \(L^2\) implies that
\[
\lim_{n \to \infty} \int_0^T \langle f_n(t), x''_n(t) \rangle dt = \int_0^T \langle f(t, x(t), x'(t)), x''(t) \rangle dt.
\]

By passing to the limit as \(n \to \infty\) in (11) and using the continuity of \(V\) we see that
\[
V(x'(T)) - V(y_0) \geq \limsup_{n \to \infty} \int_0^T \|x''_n(t)\|^2 dt - \int_0^T \langle f(t, x(t), x'(t)), x''(t) \rangle dt,
\]
and, by (10) and (12), we obtain
\[
\|x''(t)\|_{L^2} \geq \limsup_{n \to \infty} \|x''_n(t)\|_{L^2}.
\]

Since, by the weak lower semicontinuity of the norm,
\[
\|x''(t)\|_{L^2} \leq \liminf_{n \to \infty} \|x''_n(t)\|_{L^2},
\]
we have that \(\|x''(t)\|_{L^2}^2 = \lim_{n \to \infty} \|x''_n(t)\|_{L^2}^2\) i.e. \((x''_n)_n\) converge to \(x''\) strongly in \(L^2([0, T], \mathbb{R}^m)\) (Proposition III.30 in [6]). Hence a subsequence again denoted by \((x''_n)_n\) converge pointwise to \(x''\).

Since by \((H_1)\) the graph of \(F\) is closed and, by (8),
\[
\lim_{n \to \infty} d((x_n(t), x'_n(t), x''_n(t) - f_n(t)), \text{graph}(F)) = 0,
\]
we obtain that
\[
x''(t) \in F(x(t), x'(t)) + f(t, x(t), x'(t)), \text{ a.e., } t \in [0, T].
\]

Since \(x\) obviously satisfies the initial conditions, it is a solution of the problem (1).

4 An Example

For \(D \subset \mathbb{R}^n\) and \(x \in D\), denote by \(T_D(x)\) the Bouligand’s contingent cone of \(D\) at \(x\), defined by
\[
T_D(x) = \left\{ v \in \mathbb{R}^m; \liminf_{h \to 0^+} \frac{d(x + hv, D)}{h} = 0 \right\}.
\]

Also, \(N_D(x)\) is the normal cone of \(D\) at \(x\), defined by
\[
N_D(x) = \{ v \in \mathbb{R}^m; \langle y, v \rangle \leq 0, (\forall) v \in T_D(x) \}.
\]

In what follows we consider \(D\) as a closed subset such that \(\theta \in D\) and \(\theta \notin \text{Int}(D)\), where \(\theta\) is the zero element of \(\mathbb{R}^m\).

We set \(K = T_D(\theta), Q = \text{Int}(N_D(\theta)), \Omega = B_1(\theta) \times Q\) and denote by \(\pi_K(y)\) the projection
\[
\pi_K(y) = \{ u \in K : d(y, u) = d(y, K) \}.
\]
THEOREM 2. Suppose \( \text{Int} \left( N_D(x) \right) \neq \emptyset \) and \( f : R \times R^m \times R^m \to R^m \) satisfies the assumption \((H_3)\). Then there exist \( T > 0 \) and a solution \( x : [0, T] \to R^m \) for the following Cauchy problem

\[
x'' \in (1 - \|x\|) \pi_K (x') + f(t, x, x'), \quad (x(0), x'(0)) = (x_0, y_0).
\]

PROOF. By Proposition 2 in [4] there exists a convex function \( V \) such that

\[
\pi_K (y) \subset \partial V(y), \quad (\forall) y \in Q.
\]

We recall (see [4]) that the function \( V \) is defined by

\[
V(y) = \sup \{ \varphi_u(y); u \in K \},
\]

where

\[
\varphi_u(y) = \langle u, y \rangle - \frac{1}{2} \|u\|^2, \quad y \in Q.
\]

Also, we observe that the following assertions are equivalent:

\[
\begin{cases}
(i) \; u \in \pi_K (y); \\
(ii) \; \|y - u\| \leq \|y - v\|, \quad (\forall) v \in K; \\
(iii) \; \varphi_u(y) \geq \varphi_v(y), \quad (\forall) v \in K.
\end{cases} \quad (13)
\]

Let \( (x, y) \in \Omega \) be and let \( z \in F(x, y) \). Then there exists \( u \in \pi_K (y) \) such that \( z = (1 - \|x\|) u \). We have that

\[
\varphi_{(1-\|x\|)}(y) = \langle (1 - \|x\|) u, y \rangle - \frac{1}{2} (1 - \|x\|)^2 \|u\|^2 \\
\geq \langle (1 - \|x\|) u, y \rangle - \frac{1}{2} (1 - \|x\|) \|u\|^2 \\
= \langle u, y \rangle - \frac{1}{2} \|u\|^2 - \|x\| \langle \langle u, y \rangle - \frac{1}{2} \|u\|^2 \rangle \\
= (1 - \|x\|) \varphi_u(y),
\]

hence

\[
\varphi_{(1-\|x\|)}(y) \geq (1 - \|x\|) \varphi_u(y). \quad (14)
\]

Since \( u \in \pi_K (y) \), then \( \varphi_u(y) \geq \varphi_v(y) \), for every \( v \in K \), and by (14) it follows that

\[
\varphi_{(1-\|x\|)}(y) - \varphi_v(y) \geq (1 - \|x\|) \varphi_u(y) - \varphi_v(y) \\
\geq (1 - \|x\|) \varphi_v(y) - \varphi_v(y) = -\|x\| \varphi_v(y),
\]

hence

\[
\varphi_{(1-\|x\|)}(y) - \varphi_v(y) \geq -\|x\| \varphi_v(y) \quad (15)
\]

for every \( v \in K \).

Since \( y \in Q = \text{Int} \left( N_D(\theta) \right) \) we have that

\[
\langle y, v \rangle \leq 0, \quad \text{for every } v \in K = T_D(\theta),
\]

\[
\varphi_{(1-\|x\|)}(y) - \varphi_v(y) \geq -\|x\| \varphi_v(y) \quad (15)
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for every \( v \in K \).

Since \( y \in Q = \text{Int} \left( N_D(\theta) \right) \) we have that

\[
\langle y, v \rangle \leq 0, \quad \text{for every } v \in K = T_D(\theta),
\]
hence
\[ \varphi_v(y) = (y, v) - \frac{1}{2} \| v \|^2 \leq 0, \text{ for every } v \in K. \] \hfill (16)

From (15) and (16), it follows that
\[ \varphi((1-\|x\|)u) \geq \varphi_v(y), \quad v \in K. \] \hfill (17)

Then (17) and the equivalent assertions in (13) imply that
\[ z = (1-\|x\|)u \in \pi_K(y) \subset \partial V(y). \]

If we define the multifunction \( F : \Omega \to 2^{\mathbb{R}^m} \) by
\[ F(x, y) = (1-\|x\|) \pi_K(y), \]
then \( F \) is with compact valued and upper semicontinuous and there exists a convex function \( V : \mathbb{R}^m \to \mathbb{R} \) such that
\[ F(x, y) \subset \partial V(y), \quad (\forall) (x, y) \in \Omega. \]

Therefore, \( F \) and \( f \) satisfies assumptions \((H_1), (H_2), (H_3)\) and our proof is complete.

References


