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On θ-Continuity And Strong θ-Continuity *

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Abstract

P. E. Long and D. A. Carnahan in [6] and Noiri in [10] studied several properties of $a.c.S, a.c.H$ and weak continuity. In this paper it is shown that results similar to those in the above mentioned papers still hold for θ-continuity and strong θ-continuity. Furthermore, several decomposition theorems of θ-continuity and strong θ-continuity are obtained.

1 Introduction

The concepts of δ-closure, θ-closure, δ-interior and θ-interior operators were first introduced by Veličko. These operators have since been studied intensively by many authors. The collection of all δ-open sets in a topological space (X, Γ) forms a topology Γ_s on X, called the semiregularization topology of Γ, weaker than Γ and the class of all regular open sets in Γ forms an open basis for Γ_s. Similarly, the collection of all θ-open sets in a topological space (X, Γ) forms a topology Γ_θ on X, weaker than Γ_s. So far, numerous applications of such operators have been found in studying different types of continuous like maps, separation of axioms, and above all, to many important types of compact like properties. In 1961, [5] introduced the concept of weak continuity (wθ-continuity in the sense of Fomin [4]) as a generalization of continuity, later in 1966, Husain introduced almost continuity as another generalization, and Andrew and Whittlesy [2], the concept of closure continuity (θ-continuity in the sense of Fomin) which is stronger than weak continuity. In 1968, Singal and Singal [19] introduced a new almost continuity which is different from that of Husain. A few years later, P. E. Long and Carnahan [6] studied similarities and dissimilarities between the two concepts of almost continuity. The purpose of this paper is to further the study of the concepts of closure, strong continuity, faint, and quasi-θ-continuity. We get similar results to those in [6], [7], [8], [10], [11], [12], [15], [16], [17] applied to θ-continuity, strong θ-continuity, faint, and quasi-θ-continuity . Among other results we prove that the graph mapping of a function f is θ-continuous iff f is θ-continuous. In Theorem 2.5, we show that the graph mapping of a function f is strongly θ-continuous iff f is strongly θ-continuous and its domain is regular. Theorem 2.23 is a stronger result of Theorem 5 in [10]. Theorem 2.11 shows that a strong retraction of a Hausdorff space is θ-closed. Several

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decomposition theorems of θ-continuity and strong θ-continuity are given in this paper. Example 2.6 shows that [7, Corollary to Theorem 6] is not true.

For a set A in a space X, let us denote by $Int(A)$ and $\overline{A}$ for the interior and the closure of A in X, respectively. Following Veličko, a point x of a space X is called a θ-adherent point of a subset A of X iff $\overline{U} \cap A \neq \emptyset$, for every open set U containing x. The set of all θ-adherent points of A is called the θ-closure of A, denoted by $cls_\theta A$. A subset A of a space X is called θ-closed iff $A = cls_\theta A$. The complement of a θ-closed set is called θ-open. Similarly, the θ-interior of a set A in X, written $Int_\theta A$, consists of those points x of A such that for some open set U containing x, $\overline{U} \subseteq A$. A set A is θ-open iff $A = Int_\theta A$, or equivalently, $X - A$ is θ-closed.

A function $f : X \to Y$ is weakly continuous at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq \overline{V}$. If this condition is satisfied at each $x \in X$, then f is said to be weakly continuous. A function $f : X \to Y$ is closure continuous (θ-continuous) at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}) \subseteq \overline{V}$. If this condition is satisfied at each $x \in X$, then f is said to be closure continuous (θ-continuous). A function $f : X \to Y$ is strongly continuous (strongly θ-continuous) at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}) \subseteq V$. If this condition is satisfied at each $x \in X$, then f is said to be strongly continuous (strongly θ-continuous). A function $f : X \to Y$ is said to be almost continuous in the sense of Singal and Singal (briefly $a.c.S$) if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq Int(\overline{V})$. A function $f : X \to Y$ is said to be almost continuous in the sense of Husain (briefly $a.c.H$) if for each $x \in X$ and each open set V in Y containing $f(x)$, $\overline{f^{-1}(V)}$ is a neighborhood of x in X. A space X is called Urysohn if for every $x \neq y \in X$, there exist an open set U containing x and an open set V containing y such that $\overline{U} \cap \overline{V} = \emptyset$.

2 On θ-Continuity and Strong θ-Continuity

We start this section with the following useful lemmas.

LEMMA 2.1 [7, Theorem 5]. Let $f : X \to Y$ be strongly θ-continuous and let $g : Y \to Z$ be continuous. Then $g \circ f$ is strongly θ-continuous.

LEMMA 2.2. Let $f : X \to Y$ be θ-continuous and let $g : Y \to Z$ be θ-continuous. Then $g \circ f$ is θ-continuous.

LEMMA 2.3. Let $f : X \to Y$ be θ-continuous and let $g : Y \to Z$ be strongly θ-continuous. Then $g \circ f$ is strongly θ-continuous.

In [10] it is shown that a function f is weakly continuous iff its graph mapping g is weakly continuous. This is still true for the case of θ-continuity as it is shown in the next theorem. Also, it is claimed in [7] that this is true for strong θ-continuity which is not the case as it is shown in Example 2.6.

THEOREM 2.4. Let $f : X \to Y$ be a mapping and let $g : X \to X \times Y$ be the graph mapping of f given by $g(x) = (x, f(x))$ for every point $x \in X$. Then $g : X \to X \times Y$ is θ-continuous iff $f : X \to Y$ is θ-continuous.

PROOF. If g is θ-continuous. Since the projection map is continuous and every continuous map is θ-continuous, it follows from Lemma 2.2 that f is θ-continuous. Conversely, assume f is θ-continuous. Let $x \in X$ and let W be an open set in $X \times Y$ containing $g(x)$. Then there exist an open set A in X and an open set V in Y such that $g(x) = (x, f(x)) \in A \times V \subseteq W$. Since f is θ-continuous there exists an open set U containing x such that $U \subseteq A$ and $f(\overline{U}) \subseteq \overline{V}$. Therefore, $g(\overline{U}) \subseteq \overline{A} \times \overline{V} = \overline{A \times V} \subseteq \overline{W}$, proving that g is θ-continuous.

THEOREM 2.5. Let $f : X \to Y$ be a mapping and let $g : X \to X \times Y$ be the graph mapping of f given by $g(x) = (x, f(x))$ for every point $x \in X$. Then $g : X \to X \times Y$ is strongly θ-continuous iff $f : X \to Y$ is strongly θ-continuous and X is regular.

PROOF. Lemma 2.1 implies that f is strongly θ-continuous if the graph mapping g is strongly θ-continuous, and it follows easily that X is regular. Conversely, assume f is strongly θ-continuous. Let $x \in X$ and let W be an open set in $X \times Y$ containing $g(x)$. Then there exist an open set A in X and an open set V in Y such that $g(x) = (x, f(x)) \in A \times V \subseteq W$. By the regularity of X, there exists an open set B containing x such that $\overline{B} \subseteq A$. Since f is strongly θ-continuous there exists an open set U containing x such that $U \subseteq A$ and $f(\overline{U}) \subseteq V$. Let $C = U \cap B$. Then $g(\overline{C}) \subseteq \overline{C} \times V \subseteq A \times V \subseteq W$, proving that g is strongly θ-continuous.

In [7, Corollary to Theorem 6] it is claimed that the graph mapping g is strongly θ-continuous iff the mapping f is strongly θ-continuous which is not true as it is shown in the next example.

EXAMPLE 2.6. Let $X = Y = \{1, 2, 3\}$ with topologies $\Im_X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$, $\Im_Y = \{\emptyset, \{3\}, Y\}$, $f(x) = 3$, for all $x \in X$. Then f is strongly θ-continuous but the graph mapping g of the function f, where $g(x) = (x, f(x))$, is not strongly θ-continuous at 1 and 2.

By a θ-retraction we mean a θ-continuous function $f : X \to A$ where $A \subseteq X$ and $f|_A$ is the identity function on A. In this case, A is said to be a θ-retraction of X.

THEOREM 2.7. Let $A \subseteq X$ and let $f : X \to A$ be a θ-retraction of X onto A. If X is a Urysohn space, then A is a θ-closed subset of X.

PROOF. Suppose not, then there exists a point $x \in cls_\theta A \backslash A$. Since f is a θ-retraction we have $f(x) \neq x$. Since X is Urysohn, there exist open sets U and V of x and $f(x)$ respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Now let W be any open set in X containing x. Then $U \cap W$ is an open set containing x and hence $\overline{(U \cap W)} \cap A \neq \emptyset$ since $x \in cls_\theta A$. Therefore, there exists a point $y \in \overline{U \cap W} \cap A$. Since $y \in A$, $f(y) = y \in \overline{U}$ and hence $f(y) \notin \overline{V}$. This shows that $f(\overline{W})$ is not contained in $\overline{V}$. This contradicts the hypothesis that f is θ-continuous. Thus A is θ-closed as claimed.

THEOREM 2.8 [13, Theorem 4]. Let $f : X \to Y$ be a θ-continuous and injective function. If Y is Urysohn, then X is Urysohn.

COROLLARY 2.9. Let $A \subseteq X$ and let $f : X \to A$ be a bijective θ-continuous function. If A is Urysohn, then A is a θ-closed subset of X.

THEOREM 2.10 [13, Theorem 5]. Let f, g be θ-continuous from a space X into a Urysohn space Y. Then the set $A = \{x \in X : f(x) = g(x)\}$ is a θ-closed set.

By a strong retraction we mean a strongly θ-continuous function $f : X \to A$ where $A \subseteq X$ and $f|_A$ is the identity function on A. In this case, A is said to be a strong retraction of X.

The proofs for strong θ-continuity are similar to those for θ-continuity and thus will be omitted.

THEOREM 2.11. Let $A \subseteq X$ and let $f : X \to A$ be a strong retraction of X onto A. If X is Hausdorff, then A is a θ-closed subset of X.

THEOREM 2.12 [7, Theorem 4]. Let $f : X \to Y$ be a strongly θ-continuous and injective function. If Y is a T_1-space, then X is Hausdorff.

COROLLARY 2.13. Let $A \subseteq X$ and let $f : X \to A$ be a bijective strongly θ-continuous function. If A is a T_1-space, then A is a θ-closed subset of X.

THEOREM 2.14 [10]. Let f, g be weakly continuous from a space X into a Urysohn space Y. Then the set $A = \{x \in X : f(x) = g(x)\}$ is a closed set.

THEOREM 2.15 [7, Theorem 2]. Let f, g be strongly θ-continuous from a space X into a Hausdorff space Y. Then the set $A = \{x \in X : f(x) = g(x)\}$ is a θ-closed set.

Definition 2.16. A subset A of a space X is said to be θ-dense if its θ-closure equals X.

The next corollaries are generalizations to a well-known principle of extension of identities.

COROLLARY 2.17. Let f, g be θ-continuous from a space X into a Urysohn space Y. If f, g agree on a θ-dense subset of X. Then $f = g$ everywhere.

COROLLARY 2.18. Let f, g be weakly continuous from a space X into a Urysohn space Y. If f, g agree on a dense subset of X. Then $f = g$ everywhere.

COROLLARY 2.19. Let f, g be strongly θ-continuous from a space X into a Hausdorff space Y. If f, g agree on a θ-dense subset of X. Then $f = g$ everywhere.

We conclude this section with some decomposition theorems of θ-continuity and strong θ-continuity that some of them are contained in [15]. First we need some lemmas from [6], [8], [10].

LEMMA 2.20 [10, Theorem 4]. Let $f : X \to Y$ be a weakly continuous function. Then $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$, for every open set V in Y.

LEMMA 2.21 [6, Lemma to Theorem 4]. Let $f : X \to Y$ be an open function. Then $f^{-1}(\overline{V}) \subseteq \overline{f^{-1}(V)}$, for every open set V in Y.

LEMMA 2.22 [19, Theorem 4]. An open function $f : X \to Y$ is weakly continuous iff it is $a.c.S.$

The following results are some decomposition theorems for different forms of continuity which are similar to those in [6] and [10]. The next result is a stronger result of Theorem 5 in [10].

THEOREM 2.23 [15, Theorem 12], [14, Theorem 1]. Let $f : X \to Y$ be $a.c.H.$ and $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$ for every open set V in Y. Then f is θ-continuous.

COROLLARY 2.24 [15, Corollary 8], [14, Corollary 1]. An $a.c.H.$ function $f : X \to Y$ is θ-continuous iff $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$ for every open set V in Y.

COROLLARY 2.25. [15, Corollary 9], [14, Remark 4(i)]. A weakly continuous function and $a.c.H$ is θ-continuous.

THEOREM 2.26. An open $a.c.H$ function $f : X \to Y$ is θ-continuous iff $\overline{f^{-1}(V)} = f^{-1}(\overline{V})$ for every open set V in Y.

PROOF. Let f be θ-continuous. Lemma 2.22 implies that f is $a.c.S$. Thus by Corollary to [6, Theorem 7], it follows that $\overline{f^{-1}(V)} = f^{-1}(\overline{V})$, for every open set V in Y. Conversely, let $x \in X$ and let V be an open neighborhood of $f(x)$. Since f is $a.c.H.$, there exists an open set U containing x such that $\overline{U} \subseteq \overline{f^{-1}(V)} = f^{-1}(\overline{V})$. Thus $f(\overline{U}) \subseteq \overline{V}$, proving that f is θ-continuous.

THEOREM 2.27. Let $f : X \to Y$ be an open and weakly continuous. Then f is $a.c.H$.

PROOF. Let $x \in X$, and let V be an open set containing $f(x)$ in Y. Since f is weakly continuous, there exists an open set U containing x such that $f(U) \subseteq \overline{V}$. Thus $U \subseteq f^{-1}(\overline{V})$. Since f is open, Lemma 2.21 implies that $f^{-1}(\overline{V}) \subseteq \overline{f^{-1}(V)}$ and thus $U \subseteq \overline{f^{-1}(V)}$, proving that f is $a.c.H$.

THEOREM 2.28. Let $f : X \to Y$ be $a.c.H.$ and $\overline{f^{-1}(V)} = f^{-1}(V)$ for every open set V in Y. Then f is strongly θ-continuous.

PROOF. Let $x \in X$ and let V be an open neighborhood of $f(x)$. Since f is $a.c.H$ and by our hypothesis, $\overline{f^{-1}(V)}$ is a neighborhood of x and thus there exists an open set U in X containing x such that $\overline{U} \subseteq \overline{f^{-1}(V)} = f^{-1}(V)$. Therefore, $f(\overline{U}) \subseteq V$, proving that f is strongly θ-continuous.

3 On Faint and Quasi θ-Continuity.

Definition 3.1. A function $f : X \to Y$ is said to be faintly continuous (f.c.) [11](resp., quasi-θ-continuous ($q.\theta.c.$)[8]) if the inverse image of every θ-open set is open (θ-open).

LEMMA 3.2. Let $f : X \to Y$ be weakly continuous (resp., θ-continuous). Then the inverse image of every θ-open set is open (θ-open).

COROLLARY 3.3. Every weakly continuous (resp., θ-continuous) function is faintly continuous (resp., quasi-θ-continuous).

COROLLARY 3.4. Let $f : X \to Y$ be faintly continuous (resp., quasi-θ-continuous) where Y is a Hausdorff space. Then f has closed (θ-closed) point inverses.

As a consequence of Corollary 3.4, we get Theorem 6 in [3]. A quasi-θ-continuous need not be weakly continuous as it is shown in the next example.

EXAMPLE 3.5. Let $X = R$ with the cocountable topology $\Im_c$, $Y = \{0, 1, 2\}$ with $\Im = \{\emptyset, \{0\}, \{1\}, \{0, 1\}, Y\}$. Define $f : X \to Y$ as $f(x) = 0$ if x is rational, and $f(x) = 1$ if x is irrational. Then f is quasi-θ-continuous but not weakly continuous.

The proofs of the following results follow easily from the definitions.

LEMMA 3.6. Let $f : X \to Y$ be faintly continuous and let $g : Y \to Z$ be quasi-θ-continuous (resp., strongly θ-continuous). Then $g \circ f : X \to Z$ is faintly continuous (resp., continuous).

LEMMA 3.7. Let $f : X \to Y$ be continuous and let $g : Y \to Z$ be faintly continuous. Then $gof : X \to Z$ is faintly continuous.

LEMMA 3.8. Let $f : X \to Y$ be a quasi-θ-continuous and let $g : Y \to Z$ be quasi-θ-continuous (resp., strongly θ-continuous). Then $g \circ f : X \to Z$ is quasi-θ-continuous (resp., strongly θ-continuous).

During the Bethlehem Conference in August of 2000, the following questions were raised: (1) is the composite of weakly continuous (resp., $f.c.$) function weakly continuous (resp., $f.c.$). (2) Does there exist an $f.c.$ function which is not $q.\theta.c.$. The next example shows that the continuity of f in Lemma 3.7 can not be weakened into θ-continuity, and thus it shows that the composite of weakly continuous functions need not be weakly continuous which answers the first question, but still we do not know an answer of the second question.

EXAMPLE 3 .9. Let $X = \{x, y, z, w\}$ with topology $\{\emptyset, \{x, y, z\}, \{z\}, \{z, w\}, X\}$ and let $Y = \{a, b, c, d\}$ with topology $\{\emptyset, \{a, b\}, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, Y\}$. Define $g : X \to Y$ by $g(x) = a, g(y) = b, g(z) = c, g(w) = d$. Then g is weakly continuous but not θ-continuous. Define $f : (R, U) \to X$, where U is the usual topology on R by $f(x) = y$ if x is rational, and $f(x) = w$ if x is irrational. Then f is θ-continuous but not continuous, and $g \circ f$ is $q.\theta.c.$ but not weakly continuous.

Notice that the spaces in Example 3.9 are not Hausdorff spaces, so we restate the above questions as follows:

Question 3.10. (1) is the composite of weakly continuous (resp., $f.c.$)functions over Hausdorff spaces weakly continuous (resp., $f.c.$). (2) Does there exist an $f.c.$ function which is not $q.\theta.c.$.

The proof of the next theorem is similar to those for θ-continuity, and strong θ-continuity given in Theorem 2.5, 2.7 and thus will be omitted.

THEOREM 3.11. Let $f : X \to Y$ be a mapping and let $g : X \to X \times Y$ be the graph mapping of a function f given by $g(x) = (x, f(x))$ for every point $x \in X$. Then $g : X \to X \times Y$ is $f.c.$(resp., $q.\theta.c.$) iff $f : X \to Y$ is $f.c.$(resp., $q.\theta.c.$).

Similar to δ-continuity, θ-continuity, and strong θ-continuity [11, Theorems 3.3, 3.4] and following similar arguments as in [7, Theorems 6,7], we get the following results.

THEOREM 3.12. Let $f : X \to \prod_{\alpha\in I} X_\alpha$ be given. Then f is $q.\theta.c.$ (resp., $f.c.$) iff the composition with each projection π_α is $q.\theta.c.$(resp., $f.c.$).

THEOREM 3.13. Define $\prod_{\alpha\in I} f_\alpha : \prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha$ by $\{x_\alpha\} \to \{f_\alpha(x_\alpha)\}$. Then $\prod f_\alpha$ is $q.\theta.c.$(resp., $f.c.$) iff each $f_\alpha : X_\alpha \to Y_\alpha$ is $q.\theta.c.$(resp., $f.c.$).

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