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# Spectral Properties of Even Order Derivative Operators *

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**Abstract**

The study of parabolic operators in nonsmooth domains often requires the asymptotic behavior of the eigenvalues and eigenfunctions of some derivative operators. The aim of this work is to determine some spectral properties for an operator of order $2m$.

## 1 Introduction

We have studied the parabolic operators $\partial_t - \partial_x^2$ and $\partial_t + \partial_x^4$ in a nonsmooth (polygonal) domain (Sadallah [7,8,9], Labbas and Sadallah [2] and Labbas *et al.* [3]). The solutions of the associated problems contain some singular parts. Our aim was to determine the optimal regularity of these singular parts in anistropic Sobolev spaces. When we try to generalize this study to the $2m$-parabolic operator $\partial_t + (-1)^m \partial_x^{2m}$ (with $m \in \mathbf{N}$), it is necessary to consider the derivative operator $A$ defined by $Au = (-1)^m u^{(2m)}$ with domain

$$D(A) = H^{2m}(0,1) \cap H_0^m(0,1)$$

where

$$H^m(0,1) = \left\{ u \in L^2(0,1) : u^{(k)} \in L^2(0,1),\ k = 1, ..., m \right\}$$

and

$$H_0^m(0,1) = \left\{ u \in H^m(0,1) : u^{(j)}(0) = u^{(j)}(1) = 0,\ j = 0, ..., m-1 \right\},$$

where $L^2(0,1)$ stands for the usual Lebesgue space.

It is well known that $L^2(0,1)$ admits an orthonormal basis $(\varphi_n)_{n\in\mathbf{N}}$ which is a solution of the spectral problem

$$\begin{cases} A\varphi_n = \lambda_n \varphi_n \\ \varphi_n \in D(A), \end{cases} \tag{1}$$

where $(\lambda_n)_{n\in\mathbf{N}}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \lambda_n = +\infty$ (see Dautry and Lions [1]). The indicated generalization requires some asymptotic estimates about the spectral elements $(\lambda_n)_{n\in\mathbf{N}}$ and $(\varphi_n)_{n\in\mathbf{N}}$. In this work, we prove the following inequalities, where $C_1$ and $C_2$ are two positive constants :

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1. $C_1 n^{2m} \leq \lambda_n \leq C_2 n^{2m}, \forall n \in \mathbf{N}$,
2. $C_1 n^k \leq \left\| \varphi_n^{(k)} \right\|_{L^2(0,1)} \leq C_2 n^k$, $k = 0, ..., 2m, \forall n \in \mathbf{N}$,
3. $C_1 n^k \leq \left| \varphi_n^{(k)}(1) \right| \leq C_2 n^k$, $k = m, 2m-1; \forall n \in \mathbf{N}$,
4. $\left| \varphi_n^{(k)}(1) \right| \leq C_2 n^{k+\frac{1}{2}}$, $k = m+1, ..., 2m-2; \forall n \in \mathbf{N}$.

Observe that $\left| \varphi_n^{(k)}(1) \right|$ can be replaced by $\left| \varphi_n^{(k)}(0) \right|$ in the previous inequalities.

The proof of the first estimate uses the widths of Kolmogorov and Gelfand (see Triebel [10]). The other ones are obtained by classical techniques in Sobolev spaces.

## 2 Estimate of the eigenvalues $\lambda_n$

This short section is devoted to the asymptotic estimate of the eigenvalues $\lambda_n$ of Problem (1).

THEOREM 1. There exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 n^{2m} \leq \lambda_n \leq C_2 n^{2m}, \ \forall n \in \mathbf{N}. \tag{2}$$

In the sequel, the relation (2) will be expressed as $\lambda_n \sim n^{2m}$.

Theorem 1 is a consequence of the two following lemmas. We denote by $N(\lambda) = \sum_{|\lambda_n| \leq \lambda} 1$ where $\lambda$ is a given positive number; this means that $N(\lambda)$ is the number of the eignevalues $\lambda_n$ of Problem (1) such that $|\lambda_n| \leq \lambda$.

LEMMA 1. Let $\tau$ be an arbitrary positive number. Then $N(\lambda) + 1 \sim \lambda^\tau + 1$ if, and only if, $\lambda^\tau + 1 \sim n^{\frac{1}{\tau}}$.

For the proof, see Triebel [10, p. 392].

LEMMA 2. $N(\lambda) + 1 \sim \lambda^{\frac{1}{2m}} + 1$.

The proof of this lemma uses the widths of Kolmogorov and Gelfand (see Pietsch [5]).

Relation (2) of Theorem 1 is obtained from the previous lemmas by putting $\tau = 1/2m$.

REMARK 1. There is another method for the estimate of $(\lambda_n)$ based on some numerical techniques (see Raviard and Thomas [6]).

## 3 Estimate of $\left\| \varphi_n^{(k)} \right\|_{L^2(0,1)}$

The main result of this section is

THEOREM 2. $\left\| \varphi_n^{(k)} \right\|_{L^2(0,1)} \sim n^k$ for $k = 0, 1, ..., 2m$ and $n \in \mathbf{N}$.

PROOF. We shall use some properties of Sobolev spaces $H^s(0,1)$ with $s>0$ and the interpolation inequality

$$\exists C>0, \|u\|_{[D(A),L^2(0,1)]_\theta} \leq C\,\|u\|_{D(A)}^{1-\theta}\,\|u\|_{L^2(0,1)}^{\theta}\,,\ \forall\theta\in[0,1]\,,$$

where $\left[D(A),L^2(0,1)\right]_\theta$ is the interpolation real space between $D(A)$ and $L^2(0,1)$ with index $\theta$ (see Lions and Magenes [4, p.32]). The norm of $D(A)$ is the graph norm, while $\left[D(A),L^2(0,1)\right]_\theta$ is equipped with one of his equivalent norms, for instance the norm of space $H^{2m(1-\theta)}(0,1)$ (see Lions & Magenes [4, p.49]). Then

$$\|u\|_{[D(A),L^2(0,1)]_\theta} = \|u\|_{H^{2m(1-\theta)}(0,1)}\,,\ \forall u\in D(A),\forall\theta\in[0,1]\,.$$

Thus, putting $u=\varphi_n$ and $k/2m=1-\theta$, with $k=0,1,...,2m$, we deduce (note that $\|\varphi_n\|_{L^2(0,1)}=1$)

$$\exists C>0,\ \left\|\varphi_n^{(k)}\right\|_{L^2(0,1)} \leq \|\varphi_n\|_{H^k(0,1)} \leq C\,\|\varphi_n\|_{D(A)}^{\frac{k}{2m}}\,.$$

But

$$\|u\|_{D(A)} \sim \|u\|_{L^2(0,1)} + \left\|u^{(2m)}\right\|_{L^2(0,1)}\,,\ \forall u\in D(A),$$

therefore

$$\exists C>0, \left\|\varphi_n^{(k)}\right\|_{L^2(0,1)} \leq Cn^k. \tag{3}$$

Next, we prove the estimate

$$\exists C>0,\ Cn^k \leq \left\|\varphi_n^{(k)}\right\|_{L^2(0,1)}\,,\ \forall k=0,1,...,2m,\forall n\in\mathbf{N}.$$

Integrating by parts, we obtain for every $k=0,1,...,m$

$$\lambda_n = \left\|\varphi_n^{(m)}\right\|_{L^2(0,1)}^2 = \int_0^1 \left[\varphi_n^{(m)}(x)\right]^2 dx = (-1)^k \int_0^1 \varphi_n^{(m+k)}(x)\varphi_n^{(m-k)}(x)dx.$$

Then, Theorem 1 gives

$$Cn^{2m} \leq \lambda_n \leq \left\|\varphi_n^{(m+k)}\right\|_{L^2(0,1)} \left\|\varphi_n^{(m-k)}\right\|_{L^2(0,1)}\,,\ \forall k=0,1,...,m,\forall n\in\mathbf{N}.$$

On the other hand, in view of (3), it is easy to see that $\exists C>0$,

$$Cn^{m+k} \leq \left\|\varphi_n^{(m+k)}\right\|_{L^2(0,1)}\,,\ \forall k=0,1,...,m,\forall n\in\mathbf{N},$$

$$Cn^{m-k} \leq \left\|\varphi_n^{(m-k)}\right\|_{L^2(0,1)}\,,\ \forall k=0,1,...,m,\forall n\in\mathbf{N}.$$

This means that

$$\exists C>0,\ Cn^k \leq \left\|\varphi_n^{(k)}\right\|_{L^2(0,1)}\,,\ \forall k=0,1,...,2m,\forall n\in\mathbf{N}.$$

This ends the proof of Theorem 2.

# 4 Estimate of $\varphi_n^{(k)}(1)$

This section is devoted to the estimate of the derivatives $\varphi_n^{(k)}(1)$ and $\varphi_n^{(k)}(0)$ for all $k = m, m+1, ..., 2m-1$. Observe that $\varphi_n^{(k)}(1) = \varphi_n^{(k)}(0) = 0$ for all $k = 0, 1, ..., m-1, 2m$, because $\varphi_n \in H_0^m(0,1)$. In the sequel, we shall give the details of the estimate of $\varphi_n^{(k)}(1)$. That of $\varphi_n^{(k)}(0)$ is treated in a similar way.

THEOREM 3. (i) $\varphi_n^{(m)}(1) \sim n^m, \forall n \in \mathbf{N}$, and (ii) $\varphi_n^{(2m-1)}(1) \sim n^{2m-1}, \forall n \in \mathbf{N}$.

PROOF. 1) Thanks to the relation

$$\varphi_n^{(k)}(1) = \varphi_n^{(k)}(0) = 0, \forall k = 0, 1, ..., m-1, \forall n \in \mathbf{N},$$

and integrating by parts, we get,

$$\begin{aligned} \left|\varphi_n^{(m)}(1)\right|^2 &= \int_0^1 \frac{d}{dx}\left[x\left(\varphi_n^{(m)}(x)\right)^2\right]dx \\ &= \left\|\varphi_n^{(m)}\right\|^2_{L^2(0,1)} + 2\sum_{k=1}^{m-1}(-1)^k\int_0^1 \varphi_n^{(m+k)}(x)\varphi_n^{(m-k)}(x)dx + \\ & \quad 2(-1)^{m-1}\int_0^1 x\varphi_n^{(2m)}(x)\varphi_n'(x)dx. \end{aligned}$$

In addition, we have

$$\left\|\varphi_n^{(m)}\right\|^2_{L^2(0,1)} = \lambda_n,$$

$$\int_0^1 \varphi_n^{(m+k)}(x)\varphi_n^{(m-k)}(x)dx = (-1)^k\lambda_n,$$

and

$$(-1)^m\int_0^1 x\varphi_n^{(2m)}(x)\varphi_n'(x)dx = \lambda_n\int_0^1 x\varphi_n(x)\varphi_n'(x)dx = -\frac{\lambda_n}{2},$$

thus

$$\left|\varphi_n^{(m)}(1)\right|^2 = \lambda_n + 2\lambda_n(m-1) + \lambda_n = 2m\lambda_n.$$

So, Theorem 1 implies

$$\varphi_n^{(m)}(1) \sim n^m, \forall n \in \mathbf{N}.$$

2) The same method gives

$$\left|\varphi_n^{(2m-1)}(1)\right|^2$$

$$
\begin{aligned}
&= \int_0^1 \frac{d}{dx}\left[x\left(\varphi_n^{(2m-1)}(x)\right)^2\right]dx \\
&= \left\|\varphi_n^{(2m-1)}\right\|_{L^2(0,1)}^2 + 2(-1)^m\lambda_n \sum_{k=1}^{m-1}\left[(-1)^{k-1}(-1)^{m-k-1}\left\|\varphi_n^{(m-1)}\right\|_{L^2(0,1)}^2\right] \\
&\quad -\lambda_n \left\|\varphi_n^{(m-1)}\right\|_{L^2(0,1)}^2 \\
&= \left\|\varphi_n^{(2m-1)}\right\|_{L^2(0,1)}^2 + (2m-1)\lambda_n\left\|\varphi_n^{(m-1)}\right\|_{L^2(0,1)}^2 .
\end{aligned}
$$

Consequently, Theorem 1 and Theorem 2 lead to

$$
\left\|\varphi_n^{(2m-1)}\right\|_{L^2(0,1)}^2 + (2m-1)\lambda_n\left\|\varphi_n^{(m-1)}\right\|_{L^2(0,1)}^2 \sim n^{2(2m-1)}.
$$

Hence

$$
\varphi_n^{(2m-1)}(1) \sim n^{2m-1}.
$$

This ends the proof of Theorem 3.

REMARK 2. We do not know whether the generalisation of the previous result for $k = m+1, ..., 2m-2$ is possible or not. Indeed, our method do not allow us to prove the relation

$$
\varphi_n^{(k)}(1) \sim n^k, \forall k = m+1, ..., 2m-2, \forall n \in \mathbf{N}. \tag{4}
$$

However, relation (4) is true for $m = 1, 2$ (see Sadallah [7, 8]). Furthermore, the method used here permits us to show relation (4) for $m = 3$.

The following result partially answers the question in the previous Remark 2.

THEOREM 4. $\left|\varphi_n^{(k)}(1)\right| \leq n^{k+1/2}$ for $k = m+1, ..., 2m-2$ and for $n \in \mathbf{N}$.

PROOF. There exists a constant $C > 0$ such that

$$
\begin{aligned}
\left|\varphi_n^{(k)}(1)\right|^2 &= \int_0^1 \frac{d}{dx}\left[x\left(\varphi_n^{(k)}(x)\right)^2\right]dx \\
&= \left\|\varphi_n^{(k)}\right\|_{L^2(0,1)}^2 + 2\int_0^1 x\varphi_n^{(k)}(x)\varphi_n^{(k+1)}(x)dx \\
&\leq \left\|\varphi_n^{(k)}\right\|_{L^2(0,1)}^2 + 2\left\|\varphi_n^{(k)}\right\|\left\|\varphi_n^{(k+1)}\right\|_{L^2(0,1)} \\
&\leq Cn^{2k+1}
\end{aligned}
$$

for $k = m+1, ..., 2m-2$ and for $n \in \mathbf{N}$. This relation leads to the inequality of Theorem 4.

REMARK 3. It is possible to prove the relation

$$
Cn^{k-1} \leq \left\|x\varphi_n^{(k)}\right\|_{L^2(0,1)}, \forall k = 0, 1, ..., 2m, \forall n \in \mathbf{N},
$$

but the converse of this inequality is not sure to hold.

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## References

[1] R. Dautry and J. L. Lions, Analyse Mathématique et Calcul Numérique, 5, Masson, Paris, 1989.

[2] R. Labbas and B. K. Sadallah, Smoothness of the solution of a fourth order parabolic equation in a polygonal domain, Int. J. Appl. Math. 1(1)(1999), 75–90.

[3] R. Labbas, A. Medeghri and B. K. Sadallah, On a parabolic equation in a triangular domain. Appl. Math. Comput. 130(2-3)(2002), 511–523.

[4] J. L. Lions and E. Magenes, Problèmes aux Limites Non Homogènes et Applications, Vol. 1, Dunod, Paris, (1968).

[5] A. Pietsch, $\varepsilon$-Numbers of Operators in Banach Spaces, Studia Math. 54(1974), 201–223.

[6] P. A. Raviard and J. M. Thomas, Introduction à l'Analyse Numérique des Equations aux Dérivées Partielles, Masson, Paris, 1992.

[7] B. K. Sadallah, Smoothness of the singularities for the heat Equation in a polygonal domain (Part 1), Arab Gulf J. Sci. Res. 7(2)(1989), 1–15.

[8] B. K. Sadallah, Smoothness of the singularities for a parabolic equation in a polygonal domain, Arab Gulf J. Sci. Res. 14(2)(1996), 501-518.

[9] B. K. Sadallah, Existence de la solution de l'equation de la chaleur dans un disque, C. R. Acad. Sci. Paris Sér. I Math. 327(9)(1998), 813–816.

[10] H. Triebel, Interpolation Theory: Functions Spaces, Differential Operators, North Holland, Amsterdam, 1978.