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Characterizing Polynomials By Forward Differences *

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Abstract

In this note we study the following question: Under which minimal assumptions on the function f is it true that the condition $(\Delta_h^{N+1} f)(a) = 0$ for all $a \in \mathbf{R}$ and $h \geq 0$ implies that f is a polynomial of degree less than or equal to N?

1 Introduction

The Finite Difference Calculus was developed before Infinitesimal Calculus and was an important tool for its creation by Newton and Leibnitz (see [1, 5]). For example, the computation of tables for some elementary functions such as trigonometric functions, logarithms, etc., was a main problem at that period, since these functions were very useful for their applications in Celestial Mechanics, Navigation, Geography, etc. and such tables were computed by methods based on the Finite Difference Calculus. In particular, Newton studied methods based on the forward differences for the interpolation of functions and the summation of power series.

Nowadays, the Finite Difference Calculus is still an important tool for the study of numerical methods in Science and Engineering [7], since many formulas can be expressed in terms of the forward differences $\Delta_h^k f$, where f is evaluated at a table of equidistant points, $\{x, x+h, x+2h, ..., x+Nh\}$, $(\Delta_h^0 f)(x) = f(x)$, $(\Delta_h^1 f)(x) = f(x+h) - f(x)$ and

$$(\Delta_h^k f)(x) = (\Delta_h^1(\Delta_h^{k-1} f))(x) = (\Delta_h^{k-1} f)(x+h) - (\Delta_h^{k-1} f)(x),\ k = 2, 3, ...$$

On the other hand, many discrete dynamical systems are modeled by the use of finite difference operators [3] and there are many people interested in their numerical simulation, in the study of their stability properties, etc. Another motive why the difference operators are of interest is that they are related with the convexity properties of functions and the approximation by linear operators that preserve convexities ([9, 8]). Thus, there are many reasons that justify the importance of the forward difference operators Δ_h^k.

In the sequel, we denote by Π_N the space of real polynomials of degree less than or equal to N.

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We note that many formulas from Numerical Analysis are based on the use of the interpolation polynomial of a function f at the points $\{a+kh\}_{k=0}^N$. This polynomial satisfies the relation

$$p(t)=f(a)+\frac{\Delta_h^1 f(a)}{1!h}(t-a)+...+\frac{\Delta_h^N f(a)}{N!h^N}(t-a)\cdots(t-(a+(N-1)h)),$$

a formula that is usually interpreted as the analog of Taylor's expansion of a function for the Finite Difference Calculus, although there are some important differences between both expressions. For example, Taylor's formula requires from f to admit the derivatives $\{f'(a),f''(a),...,f^{(N)}(a)\}$ but the formula above requires nothing from f.

It follows from the expression of the error in the Taylor formula (see [5]),

$$R(f)(t)=f(t)-\sum_{k=0}^{N}\frac{f^{(k)}(a)}{k!}(t-a)^k=\frac{f^{(N+1)}(\eta_a)}{(N+1)!}(t-a)^{N+1}$$

that any function $f\in\mathbf{C}^{(N+1)}[a,b]$ which satisfies $f^{(N+1)}(t)=0$ for all $t\in[a,b]$ is a polynomial of degree less than or equal to N. Of course, there are other ways to prove the claim. For example, we can just integrate several times the derivative $f^{(N+1)}(t)=0$ of f, and use in each step the fundamental theorem of Calculus.

Another form to guarantee that a certain function f is a polynomial of degree $\leq N$ is through the use of the divided differences: $f[x_i]=f(x_i)$,

$$f[x_{i_0},x_{i_1},...,x_{i_n}]:=\frac{f[x_{i_0},x_{i_1},...,x_{i_{n-1}}]-f[x_{i_1},x_{i_1},...,x_{i_n}]}{x_{i_0}-x_{i_n}},$$

where $\sigma(k)=i_k$, $k=0,...,n$ is an arbitrary permutation of $\{0,...,n\}$, $n=1,2,...$. It is a well known theorem by Newton (see [5]) that

$$p_N(t)=f[x_0]+f[x_0,x_1](t-x_0)+...+f[x_0,...,x_N](t-x_0)(t-x_1)\cdots(t-x_{N-1})$$

is the unique polynomial that interpolates f at the nodes $\{x_0,...,x_N\}$. Hence

$$f(t)-p_N(t)=f[x_0,...,x_N,t](t-x_0)(t-x_1)\cdots(t-x_N)$$

gives the expression of the error of interpolation when approximating f by $p_N(t)$. Moreover, it follows that f is a polynomial of degree $\leq N$ (on the interval $[a,b]$) if, and only if, $f[x_0,\cdots,x_N,t]=0$ for a certain choice of $\{x_i\}_{i=0}^N\subset[a,b]$ and all $t\in[a,b]\setminus\{x_i\}_{i=0}^N$. We have assumed nothing with respect to the continuity, the differentiability, etc. of the function f. Thus, it is possible to characterize a function to be a polynomial without a priori assumptions about its smoothness, a fact that is somewhat surprising. There are several easy explanations of such a phenomenon. Perhaps the most understandable reason is that there are many ways to determine uniquely a polynomial of degree $\leq N$ since there are many forms to choose a basis of the vector space Π_N. Thus, fixing the basis $\{1,(t-a),(t-a)^2,\cdots,(t-a)^N\}$ of Π_N is only a choice between infinitely many others. When we fix this basis, the expansion of $p(t)$ as a linear combination of the basis is its Taylor expansion

$$p(t)=\sum_{k=0}^{N}\frac{p^{(k)}(a)}{k!}(t-a)^k$$

but, for example, if we fix the basis $\{1, (t-x_0), (t-x_0)(t-x_1), \cdots, (t-x_0)(t-x_1)(t-x_{N-1})\}$ of Π_N, the expansion of $p \in \Pi_N$ as a linear combination of this basis is given by

$$p(t) = p[x_0] + p[x_0, x_1](t-x_0) + \cdots + p[x_0, \cdots, x_N](t-x_0)(t-x_1)\cdots(t-x_{N-1})$$

Now we can ask: what about the basis $\{1, (t-a), (t-a))(t-(a+h)), \cdots, (t-a)\cdots(t-(a+(N-1)h)\}$? In this case we obtain that

$$p(t) = p(a) + \frac{\Delta_h^1 p(a)}{1!h}(t-a) + \cdots + \frac{\Delta_h^N p(a)}{N!h^N}(t-a)\cdots(t-(a+(N-1)h)).$$

The main question we solve in this note is the following: *Under which assumptions on the function f is it true that the condition given by*

$$(\Delta_h^{N+1} f)(a) = 0 \textit{ for all } a \in \mathbf{R} \textit{ and } h \geq 0$$

implies that f is a polynomial of degree less than or equal to N?. We will see that the answer is not so easy as in the divided differences case.

We close this introductory section by recalling that if we write $\Delta_h^1 = \mathbf{1}_d - \mathbf{T}_h$, where $\mathbf{1}_d(f)(x) = f(x)$ and $\mathbf{T}_h(f)(x) = f(x+h)$ and we use the Binomial theorem (since both operators $\mathbf{1}_d$ and $\mathbf{T}_h$ commute), we have that

$$\Delta_h^{k+1} f(x) = (\mathbf{1}_d - \mathbf{T}_h)^{k+1}(f)(x) = \sum_{i=0}^{k+1} \left(\begin{array}{c} k+1 \\ i \end{array} \right) (-1)^{k+1-i} f(x+ih),$$

a formula that will be used several times throughout this note.

2 Main results

Let us start this section with an example of a function $f : \mathbf{R} \to \mathbf{R}$ which is not a polynomial but satisfies $(\Delta_h^2 f)(a) = 0$ for all $a \in \mathbf{R}$ and $h \geq 0$. To this purpose, we use the following well known result from Linear Algebra:

THEOREM 1 ([4, Theorem 2.1]). Suppose that $\mathcal{A}$ is a subset of a vector space $\mathbf{V}$ over the field $\mathbf{K}$ which spans $\mathbf{V}$ and $\mathcal{C}$ is a linearly independent subset of $\mathcal{A}$. Then there exists a basis $\mathcal{B}$ of $\mathbf{V}$ such that $\mathcal{C} \subset \mathcal{B}$.

It follows the existence of a basis $\mathcal{B}$ of $\mathbf{R}$ as a $\mathbf{Q}$-vector space, with $1 \in \mathcal{B}$. Now, we can define the map $\mathcal{L}$ as the unique $\mathbf{Q}$-linear map $\mathcal{L} : \mathbf{R} \to \mathbf{R}$ that satisfies the relations:

$$\mathcal{L}(v) = \begin{cases} v & \text{if } v \in \mathcal{B} \setminus \{1\} \\ 0 & \text{if } v = 1. \end{cases}$$

Given $x \in \mathbf{R}$ and $h \geq 0$, we have that

$$\begin{aligned} \Delta_h^2 \mathcal{L}(x) &= \mathcal{L}(x+2h) - 2\mathcal{L}(x+h) + \mathcal{L}(x) \\ &= \mathcal{L}(x) + 2\mathcal{L}(h) - 2\mathcal{L}(x) - 2\mathcal{L}(h) + \mathcal{L}(x) = 0, \end{aligned}$$

since $\mathcal{L}$ is $\mathbf{Q}$-linear. On the other hand, it is clear that $\mathcal{L} \notin \Pi_1$. In fact, $\mathcal{L}$ is nowhere continuous. To prove this it is enough to prove that $\mathcal{L}$ is not continuous at the origin. We can assume without loss of generality that $|v| \geq 1$ for all $v \in \mathcal{B}$. Now, if $v \in \mathcal{B} \setminus \{1\}$ then $v \in \mathbf{R} \setminus \mathbf{Q}$, since $\{v, 1\}$ is $\mathbf{Q}$-linearly independent. This implies that for all $\varepsilon > 0$ there exists integers n, m such that $|mv - n| < \varepsilon$ and $|\mathcal{L}(nv - m)| = |nv| \geq n \geq 1$. Hence $\mathcal{L}$ is nowhere continuous.

On the other hand, any $\mathbf{Q}$-linear map $\mathcal{T} : \mathbf{R} \to \mathbf{R}$ satisfies that for all $\alpha \in \mathbf{R}$, the limit $\lim_{x \to 0; x \in \alpha\mathbf{Q}} \mathcal{T}(x)$ exits and is equal to zero. This motivates us to introduce the following concept:

DEFINITION 1. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. We say that f is $\mathbf{Q}$-continuous at $x_0 \in \mathbf{R}$ if for all $\alpha \in \mathbf{R}$ the relation $\lim_{x \to x_0; x \in \alpha\mathbf{Q}} f(x) = f(x_0)$ holds. We say that f is $\mathbf{Q}$-continuous on a set $\Omega \subset \mathbf{R}$ if f is $\mathbf{Q}$-continuous at x_0 for all $x_0 \in \Omega$.

Of course, the $\mathbf{Q}$-continuity at a point does not imply the continuity at that point, as we have already proved. On the other hand, the map $\mathcal{L}$ defined above is $\mathbf{Q}$-continuous only at $x = 0$.

REMARK 1. Discontinuous $\mathbf{Q}$-linear maps, like the $\mathcal{L}$ above, were introduced in 1905 by G. Hamel [6], as examples of additive functions $f : \mathbf{R} \to \mathbf{R}$ that are discontinuous everywhere, a problem that was open for many years in the XIX century, after the treatise of A. L. Cauchy [2], who proved that the linear maps $f(t) = at$ are the unique solutions of the so called Cauchy functional equation,

$$f(x + y) = f(x) + f(y), \ x, y \in \mathbf{R},$$

that are continuous at least at one point.

We now state the main result of this note.

THEOREM 2. Let us assume that $\Delta_h^{N+1} f(x) = 0$ for all $x \in \mathbf{R}$ and all $h > 0$. If f is $\mathbf{Q}$-continuous on a set $\{x_i\}_{i=0}^N$ of $N + 1$ points, then $f \in \Pi_N$.

PROOF. Let us assume that f is $\mathbf{Q}$-continuous at the points $\{x_0, ..., x_N\} \subset \mathbf{R}$ and denote by $\mathbf{p}(t)$ the unique polynomial from Π_N that interpolates f at these points. We will show that $f = \mathbf{p}$.

Let us fix a value $a \in \mathbf{R}$ and set

$$y_i^{(0)} = i\frac{a}{N}; \ i \in \mathbf{Z}.$$

For all $j \in \mathbf{Z}$, there exists a unique polynomial $\overline{\mathbf{p}}_j(t) \in \Pi_N$ such that $\overline{\mathbf{p}}_j(y_i^{(0)}) = f(y_i^{(0)})$, $i = j, j + 1, ..., j + N$. We denote by $\mathbf{F}$ the set

$$\mathbf{F} = \{z \in \mathbf{R} : f(z) = \overline{\mathbf{p}}_0(z)\}.$$

Clearly, $\{y_i^{(0)}\}_{i=0}^N \subset \mathbf{F}$. On the other hand,

$$\begin{aligned} \left(\Delta_{\frac{a}{N}}^{N+1} f\right)(0) &= \sum_{i=0}^{N+1} \binom{N+1}{i} (-1)^{N+1-i} f\left(i\frac{a}{N}\right) \\ &= \sum_{i=0}^{N} \binom{N+1}{i} (-1)^{N+1-i} f(y_i^{(0)}) + f(y_{N+1}^{(0)}), \end{aligned}$$

so that $\left(\Delta_{\frac{a}{N}}^{N+1} f\right)(0) = 0$ implies that

$$\begin{aligned} f(y_{N+1}^{(0)}) &= -\sum_{i=0}^{N} \binom{N+1}{i} (-1)^{N+1-i} f(y_i^{(0)}) \\ &= -\sum_{i=0}^{N} \binom{N+1}{i} (-1)^{N+1-i} \overline{\mathbf{p}}_0(y_i^{(0)}) \\ &= \overline{\mathbf{p}}_0(y_{N+1}^{(0)}) - \left(\Delta_{\frac{a}{N}}^{N+1} \overline{\mathbf{p}}_0\right)(0) = \overline{\mathbf{p}}_0(y_{N+1}^{(0)}), \end{aligned}$$

since $\left(\Delta_{\frac{a}{N}}^{N+1} \overline{\mathbf{p}}_0\right)(0) = 0$ ($\overline{\mathbf{p}}_0$ is a polynomial of degree less than or equal to N). This means that $\overline{\mathbf{p}}_0 = \overline{\mathbf{p}}_1$. The same argument proves that $\overline{\mathbf{p}}_j = \overline{\mathbf{p}}_{j+1}$ for all $j \in \mathbf{Z}$. Of course, this means that

$$\{y_i^{(0)}\}_{i=-\infty}^{\infty} \subset \mathbf{F}. \tag{1}$$

Let us now set $y_i^{(1)} = i\frac{a}{2N}$; $i \in \mathbf{Z}$ and denote by $\widetilde{\mathbf{p}}_0$ the unique polynomial from Π_N that interpolates f at the nodes $y_i^{(1)} = i\frac{a}{2N}$; $i \in \{0, ..., N\}$. We can follow an argument analogous to the one that we have already used for the proof of (1), to prove that

$$f(y_i^{(1)}) = \widetilde{\mathbf{p}}_0(y_i^{(1)}), \ i \in \mathbf{Z},$$

being the unique difference that now we need to use that $\left(\Delta_{\frac{a}{2N}}^{N+1} f\right)(x) \equiv 0$ (we change the step $h = a/N$ by the step $h = a/(2N)$). Taking into account that $\{y_i^{(0)}\}_{i=-\infty}^{\infty} \subset \{y_i^{(1)}\}_{i=-\infty}^{\infty}$ and $f|_{\{y_i^{(0)}\}_{i=-\infty}^{\infty}} = (\overline{\mathbf{p}}_0)|_{\{y_i^{(0)}\}_{i=-\infty}^{\infty}}$, we conclude that $\widetilde{\mathbf{p}}_0 = \overline{\mathbf{p}}_0$. Hence $\{y_i^{(1)}\}_{i=-\infty}^{\infty} \subset \mathbf{F}$. Once again, the repetition of an analogous argument proves that the set $\{y_i^{(n)}\}_{i=-\infty}^{\infty} := \{i\frac{a}{2^n N}\}_{i=-\infty}^{\infty}$, satisfies $\{y_i^{(n)}\}_{i=-\infty}^{\infty} \subset \mathbf{F}$ for all $n \in \mathbf{N}$. It follows that $\mathbf{F} \cap a\mathbf{Q}$ is a dense subset of $\mathbf{R}$.

From the $\mathbf{Q}$-continuity of f (and $\overline{\mathbf{p}}_0$) at the points $\{x_i\}_{i=0}^{N}$, it follows that for all $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\{y_i^{\delta}\}_{i=0}^{N} \subset a\mathbf{Q} \cap \mathbf{F}$ and $\max_{i=0,\cdots,N} |x_i - y_i^{\delta}| < \delta$ then

$$\max_{i=0,\cdots,N} \left|f(x_i) - f(y_i^{\delta})\right| < \varepsilon/2 \ \text{ and } \max_{i=0,\cdots,N} \left|\overline{\mathbf{p}}_0(x_i) - \overline{\mathbf{p}}_0(y_i^{\delta})\right| < \varepsilon/2.$$

Of course, it follows from the density of $\mathbf{F} \cap a\mathbf{Q}$ in $\mathbf{R}$ that such a set of points $\{y_i^{\delta}\}_{i=0}^{N} \subset \mathbf{F}$ always exists. Hence,

$$\begin{aligned} |f(x_i) - \overline{\mathbf{p}}_0(x_i)| &\leq \left|f(x_i) - f(y_i^{\delta})\right| + \left|f(y_i^{\delta}) - \overline{\mathbf{p}}_0(x_i)\right| \\ &= \left|f(x_i) - f(y_i^{\delta})\right| + \left|\overline{\mathbf{p}}_0(y_i^{\delta}) - \overline{\mathbf{p}}_0(x_i)\right| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

for $i = 0, 1, ..., N$. This implies

$$f(x_i) = \overline{\mathbf{p}}_0(x_i) \text{ for } i = 0, 1, ..., N,$$

since ε was arbitrarily chosen. Thus $\overline{\mathbf{p}}_0(t) = \mathbf{p}(t)$ and

$$f(a) = f(y_N^{(0)}) = \overline{\mathbf{p}}_0(y_N^{(0)}) = \mathbf{p}(y_N^{(0)}) = \mathbf{p}(a).$$

Now, $a \in \mathbf{R}$ was arbitrarily chosen, so that $f = \mathbf{p}$. The proof is complete.

REMARK 2. Theorem 2 is sharp, since for all $N \geq 1$ there are functions $f \notin \Pi_N$ that satisfy $\Delta_h^{N+1} f(x) = 0$ for all $x \in \mathbf{R}$ and all $h > 0$ and are $\mathbf{Q}$-continuous at exactly N distinct points $\{x_1, ..., x_N\}$.

PROOF. We set $\mathcal{F}(t) = R(t)\mathcal{L}(t)$, where $R(t) = (t-1)(t-2)\cdots(t-(N-1))$. It is obvious that $\mathcal{F}(t)$ is $\mathbf{Q}$-continuous exactly at the N distinct points $\{0, 1, ..., N-1\}$, and it is not a polynomial. Thus, we only need to check that $\Delta_h^{N+1}\mathcal{F}(x) = 0$ for all $x \in \mathbf{R}$ and $h > 0$. Now,

$$\begin{aligned} \Delta_h^{N+1}\mathcal{F}(x) &= \mathcal{L}(x)\Delta_h^{N+1}R(x) + \mathcal{L}(h)\sum_{i=0}^{N+1}\begin{pmatrix} N+1 \\ i \end{pmatrix}(-1)^{N+1-i}R(x+ih)i \\ &= \mathcal{L}(h)\Delta_h^{N+1}R^*(x) = 0, \end{aligned}$$

where $R^*(t)$ is defined to be $R(t)\frac{t-x}{h}$ which belongs to Π_N.

In the proof of Theorem 2 we have already shown that if f satisfies $\Delta_h^{N+1} f(x) \equiv 0$, then for all $\alpha \in \mathbf{R}$, its restriction to the set of points $\{\frac{i\alpha}{2^n N}\}_{n=0}^{\infty}$ coincides with a certain polynomial $\mathbf{p}_\alpha \in \Pi_N$. The following proposition is an improvement of this result, and will be useful for the rest of the note.

PROPOSITION 3. Let us assume that $\Delta_h^{N+1} f(t) = 0$ for all $t \in \mathbf{R}$ and all $h > 0$. Then, for every $(x, \alpha) \in \mathbf{R}^2$ there exists a polynomial $p_{x,\alpha}(t) \in \Pi_N$ such that $f(t) = p_{x,\alpha}(t)$ for all $t \in x + \alpha\mathbf{Q}$.

PROOF. Let f be such that it satisfies the hypothesis of the proposition. Let $(x, \alpha) \in \mathbf{R}^2$ be fixed and let us set $g(t) = f(t + x)$, $t \in \mathbf{R}$. Then $\Delta_h^{N+1} g(t) = 0$ for all $t \in \mathbf{R}$ and all $h > 0$. We follow the proof of Theorem 1 to claim that there exists a polynomial $\overline{\mathbf{p}}_0 \in \Pi_N$ such that

$$g\left(\frac{i\alpha}{2^n N}\right) = \overline{\mathbf{p}}_0\left(\frac{i\alpha}{2^n N}\right) \text{ for all } n \in \mathbf{N} \text{ and all } i \in \mathbf{Z}.$$

Let us now fix $\frac{p}{q} \in \mathbf{Q}$. If we change α by $\frac{p}{q}\alpha$ in the above argument, we can also prove that there exists a polynomial $\overline{\mathbf{p}}_1 \in \Pi_N$ such that

$$g\left(\frac{i}{2^n N}\frac{p}{q}\alpha\right) = \overline{\mathbf{p}}_1\left(\frac{i}{2^n N}\frac{p}{q}\alpha\right) \text{ for all } n \in \mathbf{N} \text{ and all } i \in \mathbf{Z}.$$

Now, we set $i = q$ in the above expression to obtain that

$$\overline{\mathbf{p}}_1\left(\frac{q}{2^n N}\frac{p}{q}\alpha\right) = g\left(\frac{q}{2^n N}\frac{p}{q}\alpha\right) = g\left(\frac{p\alpha}{2^n N}\right) = \overline{\mathbf{p}}_0\left(\frac{p\alpha}{2^n N}\right) \text{ for all } n \in \mathbf{N}.$$

Of course, this implies that $\overline{\mathbf{p}}_0 = \overline{\mathbf{p}}_1$. Hence $g(\frac{p}{q}\alpha) = \overline{\mathbf{p}}_0(\frac{p}{q}\alpha)$ for all $p/q \in \mathbf{Q}$. This implies that

$$f\left(x + \frac{p}{q}\alpha\right) = g\left(\frac{p}{q}\alpha\right) = \overline{\mathbf{p}}_0\left(\frac{p}{q}\alpha\right) = \mathbf{p}_{(x,\alpha)}\left(x + \frac{p}{q}\alpha\right), \text{ for all } p/q \in \mathbf{Q},$$

where $\mathbf{p}_{(x,\alpha)}(t) = \overline{\mathbf{p}}_0(t-x)$, $t \in \mathbf{R}$. This ends the proof, since $\mathbf{p}_{(x,\alpha)} \in \Pi_N$. The proof is complete.

Although Theorem 2 is sharp, it is not easy to handle with the concept of **Q**-continuity. Thus, it should be desirable to have another result which depends on the more familiar concept of continuity. Moreover, the hypothesis of continuity of a function at a point is stronger than the corresponding hypothesis of **Q**-continuity, so that if we assume continuity of the function at several points, it should be reasonable to reduce our hypothesis on the number of points where continuity holds in order to guarantee that the function is a polynomial. This is what we make in the next theorem:

THEOREM 4. Let us assume that $\Delta_h^{N+1} f(x) = 0$ for all $x \in \mathbf{R}$ and all $h > 0$. If f is continuous on a set $\{x_i\}_{i=1}^N$ of N points, then $f \in \Pi_N$.

PROOF. Let us assume that f satisfies the hypotheses of the theorem. It follows from Proposition 3 that for each $\alpha \in \mathbf{R}$, there exists a polynomial $\mathbf{p}_\alpha \in \Pi_N$ such that $f(t) = \mathbf{p}_\alpha(t)$ for all $t \in \alpha\mathbf{Q}$. The continuity of f at the points $\{x_i\}_{i=1}^N$ implies that

$$\mathbf{p}_\alpha(x_i) = \lim_{t \to x_i} \mathbf{p}_\alpha(t) = \lim_{t \to x_i; t \in \alpha\mathbf{Q}} \mathbf{p}_\alpha(t) = \lim_{t \to x_i; t \in \alpha\mathbf{Q}} f(t) = f(x_i)$$

for all $\alpha \in \mathbf{R}$. Let $\alpha, \beta \in \mathbf{R} \setminus \{x_i\}_{i=1}^N$, $\alpha \neq \beta$. It follows from Proposition 3 that there exists a polynomial $\mathbf{p}_{\alpha,\beta}(t) \in \Pi_N$ such that

$$f(t) = \mathbf{p}_{\alpha,\beta}(t) \text{ for all } t \in \alpha + (\beta - \alpha)\mathbf{Q}.$$

Clearly, $f(x_i) = \mathbf{p}_{\alpha,\beta}(x_i)$, $i = 1, 2, ..., N$ as a consequence of the continuity of f at the points $\{x_i\}_{i=1}^N$) (at this point, the **Q**-continuity of f at these points is not enough, since we are now taking limits on sets of the form $\alpha + (\beta - \alpha)\mathbf{Q}$ with $\alpha \neq 0$, $\alpha \neq \beta$). Furthermore, $\mathbf{p}_{\alpha,\beta}(\alpha) = \mathbf{p}_\alpha(\alpha)$, $\mathbf{p}_{\alpha,\beta}(\beta) = \mathbf{p}_\beta(\beta)$, since $\{\alpha, \beta\} \subset \alpha + (\beta - \alpha)\mathbf{Q}$. This implies that $\mathbf{p}_\alpha = \mathbf{p}_{\alpha,\beta} = \mathbf{p}_\beta$ since all these polynomials have degree less than or equal to N and $\#\{x_1, ..., x_N, \alpha\} = \#\{x_1, ..., x_N, \beta\} = N + 1$. It follows that there exists a polynomial $\mathbf{p} \in \mathbf{\Pi}_N$ such that

$$f(t) = \mathbf{p}(t) \text{ for all } t \in \bigcup_{\alpha \in \mathbf{R} \setminus \{x_i\}_{i=1}^N} \alpha\mathbf{Q} = \mathbf{R}.$$

This ends the proof.

REMARK 3. The hypotheses of Theorem 4 can be weakened, since we have only used the existence of the limits

$$\lim_{t \to x_i, t \in \alpha + (\beta - \alpha)\mathbf{Q}} f(t) = f(x_i), \; i = 1, ..., N.$$

Here, we would like to remark that the example given at Remark 2 does not enter in contradiction with Theorem 4, formulated in its weakest form, since if we set $t = \alpha + (\beta - \alpha)\frac{p}{q}$, then $\frac{p}{q} = \frac{t-\alpha}{\beta-\alpha}$ and the function $\mathcal{F}$ satisfies:

$$\begin{aligned} \mathcal{F}(t) &= R(t)\mathcal{L}\left(\alpha + (\beta - \alpha)\frac{p}{q}\right) = R(t)\left(\mathcal{L}(\alpha) + (\mathcal{L}(\beta) - \mathcal{L}(\alpha))\frac{p}{q}\right) \\ &= R(t)\left(\mathcal{L}(\alpha) + \frac{(\mathcal{L}(\beta) - \mathcal{L}(\alpha))(t - \alpha)}{\beta - \alpha}\right), \end{aligned}$$

so that,

$$\lim_{t\to x_i, t\in\alpha+(\beta-\alpha)\mathbf{Q}} \mathcal{F}(t) = R(0)\frac{(\mathcal{L}(\alpha)\beta - \mathcal{L}(\beta)\alpha)}{\beta-\alpha} \neq 0 = \mathcal{F}(0).$$

Unfortunately, it seems that the above result is not sharp. At least, in certain cases it is possible, in order to guarantee that $f \in \Pi_N$, to assume continuity at a set of less than N points, once we know that the function $\Delta_h^{N+1} f(x)$ vanishes for all $x \in \mathbf{R}$, $h > 0$. In this sense, we prove the following partial result:

PROPOSITION 5. Let us assume that $\Delta_h^3 f(x) = 0$ for all $x \in \mathbf{R}$, $h > 0$ and f is continuous at a point $x_0 \in \mathbf{R}$. Then $f \in \Pi_2$.

PROOF. We may assume without loss of generality that f is continuous at $x_0 = 0$ and $f(0) = 0$. Let $\alpha, \beta \in \mathbf{R}$ be such that $\alpha \neq \beta$. Proposition 3 implies that there exists polynomials $\mathbf{p}_\alpha, \mathbf{p}_\beta, \mathbf{p}_{\alpha,\beta} \in \Pi_2$ such that

$$(\mathbf{p}_\alpha)|_{\alpha\mathbf{Q}} = f|_{\alpha\mathbf{Q}},\ (\mathbf{p}_\beta)|_{\beta\mathbf{Q}} = f|_{\beta\mathbf{Q}} \text{ and } (\mathbf{p}_{\alpha,\beta})|_{\alpha+(\beta-\alpha)\mathbf{Q}} = f|_{\alpha+(\beta-\alpha)\mathbf{Q}}.$$

Hence

$$\begin{aligned}
\mathbf{p}_{\alpha,\beta}(\alpha) &= \mathbf{p}_\alpha(\alpha) = f(\alpha), \\
\mathbf{p}_{\alpha,\beta}(\beta) &= \mathbf{p}_\beta(\beta) = f(\beta), \\
\mathbf{p}_{\alpha,\beta}(0) &= \mathbf{p}_\alpha(0) = \mathbf{p}_\beta(0) = f(0) = 0.
\end{aligned}$$

If we write $\mathbf{p}_{\alpha,\beta}(t) = At^2 + Bt$ and we use the identities above, we can compute the constants A, B, and then we obtain the identity

$$\mathbf{p}_{\alpha,\beta}(\alpha+\beta) = \frac{(f(\alpha) - f(\beta))(\alpha+\beta)}{\alpha-\beta}.$$

If $f \notin \Pi_2$ then there are points $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 < \beta_0$ and $\mathbf{p}_{\beta_0}(\alpha_0) \neq f(\alpha_0) = \mathbf{p}_{\alpha_0}(\alpha_0)$. Moreover, it follows from the density of $\alpha_0\mathbf{Q}$ in $\mathbf{R}$ and the fact that two distinct polynomials may interpolate each other only at a finite number of points, that there exists a sequence of points $\{x_n\}_{n=1}^\infty \subset \alpha_0\mathbf{Q}$ such that $|x_n - \frac{1}{n}| \leq \frac{1}{2n}$ for all $n \geq 1$ and

$$C_n = |\mathbf{p}_{\alpha_0}(x_n) - \mathbf{p}_{\beta_0}(x_n)| \neq 0,\ n = 1, 2, \cdots$$

Hence fixed $n \geq 1$, we can find a sequence $\{y_{n,m}\}_{m=1}^\infty \subset \beta_0\mathbf{Q}$ such that $\lim_{m\to\infty} y_{n,m} = x_n$ and

$$\lim_{m\to\infty} \mathbf{p}_{\beta_0}(y_{n,m}) = \mathbf{p}_{\beta_0}(x_n),$$

so that we can assume without loss of generality that $|y_{n,m} - x_n| \leq \min\{\frac{1}{2n}, \frac{C_n}{2n}\}$ and

$$|f(x_n) - f(y_{n,m})| = |\mathbf{p}_{\alpha_0}(x_n) - \mathbf{p}_{\beta_0}(y_{n,m})| \geq \frac{C_n}{2}$$

for all $m, n \geq 1$. Thus,

$$\begin{aligned}
\left|\mathbf{p}_{x_n,y_{n,m}}(x_n + y_{n,m})\right| &= \left|\frac{(f(x_n) - f(y_{n,m}))(x_n + y_{n,m})}{x_n - y_{n,m}}\right| \\
&\geq \frac{C_n\,|x_n + y_{n,m}|}{2\,|x_n - y_{n,m}|} \geq \frac{2nC_n}{2nC_n} = 1.
\end{aligned}$$

Finally, it follows from the density of $x_n + (y_{n,m} - x_n)\mathbf{Q}$ in $\mathbf{R}$ and the continuity of $\mathbf{p}_{x_n,y_{n,m}}$ that there exists $\xi_{n,m} \in x_n + (y_{n,m} - x_n)\mathbf{Q}$, $|\xi_{n,m} - (x_n + y_{n,m})| \leq \frac{1}{4n}$ such that

$$\left|\mathbf{p}_{x_n,y_{n,m}}(x_n + y_{n,m}) - f(\xi_{n,m})\right| = \left|\mathbf{p}_{x_n,y_{n,m}}(x_n + y_{n,m}) - \mathbf{p}_{x_n,y_{n,m}}(\xi_{n,m})\right| < \frac{1}{2},$$

so that $\lim_{n,m\to\infty} \xi_{n,m} = 0$ and $|f(\xi_{n,m})| \geq 1/2$ for all n, m. This is in contradiction with the continuity of f at $x_0 = 0$. The proof is complete.

Notice that Theorem 4 implies that if $(\Delta_h^2 f)(x) \equiv 0$ and f continuous at one point then $f \in \Pi_1$. On the other hand, Proposition 5 claims that continuity at a point is also enough to guarantee that the functions f such that $(\Delta_h^3 f)(x) \equiv 0$ are parabolas. This motivate us to end this note with the conjecture that a Cauchy-type theorem holds for the functional equation we are considering:

Conjecture: If f is continuous at one point and $(\Delta_h^{N+1} f)(x) = 0$ for all $x \in \mathbf{R}$ and $h > 0$, then $f \in \Pi_N$.

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