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Mathematical Works Of Jerzy Popenda *

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Abstract

During the past twenty years, there is a renewed interest in the study of difference equations. Jerzy Popenda was one of the active Polish mathematicians who worked in this area. It is most unfortunate that he passed away in 1999. In view of his contributions, we believe that a short survey of his main mathematical works will be beneficial to our mathematics community.

1 Introduction

Jerzy Popenda was born in Kowary, Poland, on November 4, 1948. He received his Ph.D. degree in 1980 at Adam Mickiewicz University in Poznań, Poland, and his post-doctoral degree in 1991. He worked at Poznań University of Technology, Poland as a research fellow from 1971 until his death. He published sixty eight research papers in various fields of mathematics. Besides his professional activity, he shared with us his artistic passion - especially in music and painting. J. Popenda passed away suddenly in Bethume, France on May 31 of 1999, leaving his wife Ewa, his two daughters and his son.

Jerzy Popenda has made many contributions in mathematics (see the list of published papers at the end of this article). We believe that a short survey of his main mathematical works will be beneficial to our mathematics community. In this article, we will concentrate on three of them here:

1. Discrete Gronwall type inequalities and their applications.
2. Qualitative properties of difference equations.
3. Explicit formulas for solutions of difference equations.

For the sake of convenience, we use N to denote the set $\{1, 2, 3, \ldots\}$ of natural numbers, N_0 for $\{n_0, n_0+1, \ldots\}$ where n_0 is fixed nonnegative integer, and R, R_0, R_+ the set of real numbers, nonnegative real numbers and positive real numbers, respectively. The forward difference operator is defined as

$$\Delta x_n = x_{n+1} - x_n,\ \Delta^k x_n = \Delta(\Delta^{k-1} x_n),\ \text{where } \Delta^0 x_n = x_n$$

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and

$$\Delta_a x_n = x_{n+1} - ax_n,\ \Delta_a^k x_n = \Delta_a(\Delta_a^{k-1} x_n),\ \text{where } \Delta_a^0 x_n = x_n,$$

where a is fixed real constant and $k \in N$. A sequence is denoted by $\{x_n\}$ or $\{x(n)\}$.

2 Discrete Gronwall Type Inequalities

One of the well known inequalities in the theory of differential equations is Gronwall's inequality. Jerzy Popenda began his work with D. Bobrowski and J. Werbowski in Gronwall type differential inequalities with delay [1]. Later he got interested in discrete analogs of the Gronwall inequality and wrote nineteen papers, some of which were written jointly with D. Bobrowski, J. Werbowski and R. Agarwal, and contained applications in the stability and asymptotic behaviors of solutions of difference equations (see [4], [8], [10], [13], [16], [19], [20], [24], [27], [32], [33], [36], [37], [40], [41], [43], [49], [53], [62]).

In [4], Popenda and Werbowski considered the functional inequality

$$u(n+1) \leq v(n) + M \sum_{i=n_0}^{n} w(i) f(u(i)). \tag{1}$$

They proved the following theorem: Let $u(n), v(n), w(n), \Delta v(n)$ be nonnegative functions on N_0 such that $v(n_0) > 0, u(n_0) > 0$; f is a nonnegative, continuous and nondecreasing function on R_+ such that $f(v(n_0)) > 0$ and the function F defined by

$$F(x) = \int_{\varepsilon}^{x} \frac{dt}{f(t)},\ \varepsilon > 0, \quad x > 0,$$

tends to ∞ as $x \to \infty$, and the functional inequality (1) holds for $n \geq n_0$ and some constant M. Then

$$u(n+1) \leq F^{-1}\left(F(v(n_0)) + Mw(n_0)f(u(n_0)) + \sum_{i=n_0}^{n-1}\left(\frac{\Delta v(i)}{f(v(i))} + Mw(i+1)\right)\right)$$

for $n \in N_0$.

As special cases, some well-known discrete version of Gronwall's lemmas follow :

If $u : N_0 \to R_+$, $c : N \to R_0$, $C_0, M \in R_+$, $p > 1$ and

$$u(n) \leq C_0 + M \sum_{j=n_0}^{n-1} c(j) u^p(j),$$

then

$$u(n) \leq \left\{C_0^{1-p} + M(p-1) \sum_{j=n_0}^{n-1} c(j)\right\}^{\frac{1}{1-p}}.$$

If $u : N \to R_+$, $c : N \to R_0$, $C \in R_+$ and

$$u(n) \leq C + \sum_{j=n_0}^{n-1} c(j)u(j),\ n \in N,$$

then

$$u(n) \leq C \exp\left\{\sum_{j=n_0}^{n-1} c(j)\right\}.$$

Another functional inequality considered by Jerzy Popenda in [10] is:

$$u(n) \leq c(n) + \sum_{k=1}^{s}\sum_{i=n_0}^{n-1} h_k(i)u(d_k(i)) + b(n)\Psi\left(n, \sum_{i=n_0}^{n-1} B(i, u(d_1(i)), ..., u(d_s(i))\right) \quad (2)$$

for $n \in N_0$, where $b, u : N_0 \to R_+$, $c : N_0 \to R$ and $d_k : N_0 \to N_0$ such that $n_0 \leq d_k(n) \leq n$ and $\lim\limits_{n\to\infty} d_k(n) = \infty$ for $k = 1, ..., s$. Here the terms $d_1, ..., d_k$ are motivated by the concept of delay appearing in differential equations. The estimation of solution $u(n)$ of inequality (2) can be obtained under the following assumptions: the function $B : N_0 \times R^s \to R$ is continuous for each $n \in N_0$ such that $0 \leq B(n, y_1, ..., y_s) \leq B(n, v_1, ..., v_s)$ for $n \in N_0$ and $0 \leq y_i \leq v_i$ for $i = 1, ..., s$, and $B(n, a(t)y_1, ..., a(t)y_s) \leq A(a(t))B(n, y_1, ..., y_s)$ for arbitrary continuous function $a : R_0 \to [1, \infty)$, where $A : [1, \infty) \to R_+$ is a continuous and nondecreasing function.

Some difference inequalities are discrete analogs of continuous inequalities, or vice versa. This duality can be seen in the paper [20], in which the first part deals with integrodifferential inequalities of Gronwall-Bellman type and in the second discrete analogies such as

$$\Delta u(n) \leq u(n_0) + \sum_{j=n_0}^{n-1} a(j)[u(j) + \Delta u(j)] + \sum_{j=n_0}^{n-1} a(j)\sum_{i=n_0}^{j-1} B(i, \Delta u(u(d(i))) \quad (3)$$

are discussed. The technique for dealing with (3) is to consider solutions of difference equations which are possibly easier to solve. Similar procedures were used by Popenda in [27]. In this paper he considered, among others, functional inequalities of the form

$$\Delta u_n \leq c + \sum_{j=1}^{n-1} a_j \left(u_{j+1}^{\alpha} - u_j^{\alpha}\right),$$

$$\Delta u_n \leq c_n + \sum_{j=1}^{n-1} a_j \left((u_j)^{\alpha_j} - b_j \Delta u_j\right),$$

and

$$u_{n+1} \leq c + \sum_{j=1}^{n-1} a_j \left((u_j)^{\alpha_j} + (\Delta u_j)^{\alpha_j}\right).$$

Jerzy Popenda also investigated functional inequalities for functions of several variables. In [16], linear inequalities of the form

$$u(n,m) \leq c_0 + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} a(i,j)u(i,j),$$

$$u(n,m) \leq c(n,m) + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} a(i,j)u(i,j),$$

$$u(n,m) \leq c(n,m) + b(n,m)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} a(i,j)u(i,j),$$

$$u(n,m) \leq c_0 + \sum_{j=0}^{m-1} a(j)u(n,j) + \sum_{i=0}^{n-1} b(i)u(i,m)$$

are considered, and he obtained estimations which are essentially better than those presented in papers [c6], [c8], [c9], [c10].

In [13], there are presented inequalities in distributive lattices of type

$$u(n+1) \leq c(n) \vee \bigvee_{j=1}^{n} [a(j) \wedge u(j)]$$
$$u(n+1) \leq c(n) \wedge \bigwedge_{j=1}^{n} [a(j) \vee u(j)]$$

and in [19] discrete inequalities of Gronwall type in vector lattices. Theorem 6 of this paper, which can be considered as a “transformation” of Pachpatte’s result [c13] for vector lattices, shows that some difference inequalities can be considered without any greater difficulties for more general structures.

An important property of Gronwall’s original inequality is that it is “the best possible”, or roughly, equality in the inequality implies equality in the estimation. In some Gronwall type inequalities, the best possible property is not shown. However, in [16], [20], [33], [49] and [53], Popenda and/or his coauthors showed that if the operators which appear in the right side of the Gronwall inequalities are monotonic on some suitable sets, then sharp estimations can be obtained, so that best estimates of the inequalities are achieved by solutions of corresponding equations.

Jerzy Popenda also tried to show the usefulness of the estimations of difference inequalities in the investigation of the qualitative behaviors of solutions of difference equations such as stability or asymptotic behavior. Such applications can be found in [12], [14] and [29]. In [14] boundedness of solutions is studied. Difference inequalities are used a few times. In [12] and [29], asymptotic properties of solutions of difference equation

$$\Delta^m y(n) + F(n, y(n), ..., y(n+m-1)) = b(n),\ m \geq 1,\ n \in N_0,$$

are examined. The results obtained in [12] are based on the following Gronwall’s lemma which is proved in that paper: Let the function $B : N \times R^m \to R$ be continuous on

R^m for each $n \in N$ and such that for $z_k \geq 0$, we have

$$0 \leq B(n, z_1, ..., z_m) \leq B(n, y_1, ..., y_m) \text{ for } z_k \leq y_k,\ k = 1, ..., m,$$

and

$$B(n, a_n z_1, ..., a_n z_m) \leq A(a_n) B(n, z_1, ..., z_m) \text{ for } a_n \geq \varepsilon > 0,$$

where $A : (\varepsilon, \infty) \to R_+$ is nondecreasing and

$$\int_{\varepsilon}^{\infty} \frac{ds}{A(s)} = \infty,\ \varepsilon > 0,$$

and suppose that the nonnegative sequences $\{u_n\}$ and $\{v_n\}$ satisfy the following conditions

$$u_{n+k} \leq v_{n+k} \left\{ c + \sum_{j=n_0}^{n} B(j, u_j, ..., u_{j+k-1}) \right\},$$

and

$$\sum_{j=n_0}^{\infty} B(j, v_j, ..., v_{j+k-1}) < \infty$$

where k is a given natural number and c is a positive constant. Then there exists a constant $M > 0$ such that $u_n \leq M v_n$ for $n \geq n_0 + k$.

3 Qualitative Properties of Difference Equations

Jerzy Popenda wrote more than forty papers related to qualitative properties of difference equations. He started his research activity in this subject in 1980 by publishing two papers: one with J. Werbowski in Commentationes Mathematicae and one with E. Schmeidel in Fasciculi Mathematici.

Roughly, eight types of solutions of difference equations are considered:

- Asymptotically constant solutions are dealt with in [23], [34], [46], [52], [56], [57], [60] and [66].
- Asymptotically zero solutions are dealt with in [7], [9], [28] and [35].
- Approximate polynomial solutions are dealt with in [6], [18], [25] and [42].
- Periodic and asymptotically periodic solutions are dealt with in [30], [31], [48], [54], [61], [63] and [65].
- Monotonic, nonoscillatory and convex solutions are dealt with in [15], [21], [38], [39] and [45].
- Oscillatory solutions are dealt with in [11], [21], [26], [35], [38], [39], [44], [47], [50], [51], [52] [58], [59], [63] and [64].

In the papers [23], [34], [46], [52] and [56], Jerzy Popenda and his coauthors gave necessary and sufficient conditions for the existence of asymptotically constant solutions. A good example is the following theorem in [23]:

> Let $f : R \to R$ be continuous, $p : N \to R_0$ and c constant such that $f(c) \neq 0$. A necessary and sufficient condition for the existence of a solution of equation $\Delta^2 x_n + p_n f(x_n) = 0$ which satisfies $\lim_{n\to\infty} y_n = c$ is $\sum_{j=1}^{\infty} jp_j < \infty$.

An extension of this result for higher order difference equations can be found in Drozdowicz and Popenda [34]. Furthermore, together with S. McKee, J. Popenda obtained conditions for the existence of asymptotically constant solutions for a system of linear difference equation [56].

Paper [57] written by Magnucka-Blandzi and Popenda deals with Riccati equations with constant coefficients. It is proved that almost all of the trajectories (except for a countable set of finite trajectories) tends to one of the stationary points.

In [60], Magnucka-Blandzi and Popenda considered a system of two rational type difference equations with constant coefficients

$$\begin{cases} x_1(n+1) &= \frac{a}{x_1(n)} + \frac{b}{x_2(n)}, \\ x_2(n+1) &= \frac{a}{x_1(n)}. \end{cases}$$

The asymptotic properties of solutions and the attractor (the almost global attractor of this system) are investigated.

Paper [66] deals with asymptotic properties of solutions of the system of two rational difference equations

$$\begin{cases} x_1(n+1) &= \frac{1}{x_2(n)} + c, \\ x_2(n+1) &= \frac{1}{x_1(n)} + d, \end{cases}$$

where $c \cdot d > 0$. By means of continued fractions, Magnucka-Blandzi and Popenda proved that every solution with initial values of the same sign is convergent. Moreover, an application to a system of two Riccati type equations is given.

Zero convergent solutions are special cases of asymptotically constant solutions. Jerzy Popenda investigated this case separately. In 1980, Popenda and Schmeidel [7] presented assumptions under which nonoscillatory solution of second order difference equation diverges or tends to zero. This result is later generalized in [28].

Nonexistence of solutions which tend to zero is obtained by J. Popenda and E. Schmeidel in [9]. They presented conditions under which a nontrivial solution of equation of the form

$$\Delta^2 x_n + f(n, x_n) = 0$$

which tends to zero does not exist.

J. Popenda studied approximate first order and higher order polynomial solutions of difference equations. In [18] the asymptotic behavior of solutions of m-th order difference equations

$$\Delta^m y_n + f(n, y_n) = 0,\ m \geq 1,$$

is examined. Motivated by the work of Trench [c14] on differential equations, J. Popenda gave sufficient conditions which imply that the above equation has a solution which behaves like a given polynomial of degree less than m as $n \to \infty$.

Paper [30] on periodic and asymptotically periodic solutions was published in 1988 and it contains results obtained together with R. Musielak for equation $\Delta^2 x_n + a_n f(x_n) = 0$. They made the following definitions:

> The function (sequence) y is called asymptotically t-periodic if there exist two sequences u and v such that $y = u + v$, u is t-periodic and $\lim_{n\to\infty} v_n = 0$.
>
> Equation $\Delta^2 x_n + a_n f(x_n) = 0$ possesses a p_t^∞-constant if there exists a constant $p \in R$ such that equation
> $$\Delta^2 x_n + a_n f(x_n) = p$$
> has asymptotically t-periodic solution.

Then they proved the following:

> Let $\lim_{n\to\infty} a_n = 0$. Then equation $\Delta^2 x_n + a_n f(x_n) = 0$ does not have any p_t^∞-constant.
>
> If equation $\Delta^2 x_n + a_n f(x_n) = 0$ possesses a p_t^∞-constant then a is a t-periodic function.

In a joint paper with R. Agarwal [48], various basic statements on the periodicity of the solutions of first order linear difference equations are given together with conditions under which every solution of nonhomogeneous second order linear difference equation is asymptotically ω-periodic.

In 1997, J. Popenda and E. Schmeidel investigated integer valued difference equations and published some results on periodicity of integer valued solutions of difference equations [54].

In [39], J. Popenda and E. Schmeidel gave sufficient conditions for third order linear difference equation to have monotonic nonoscillatory and convex solutions.

M. Migda and J. Popenda in [45] presented sufficient conditions for positivity and monotonicity of solutions of equation

$$x_{n+m} = \sum_{i=0}^{n-1} a_n^i x_{n+i}.$$

Another area of J. Popenda's interests was oscillation theory. J. Popenda and B. Szmanda studied in [11] the oscillation properties of solutions of the difference equations

$$\Delta_a x_n + \delta \sum_{i=1}^{m} q_{i,n} f_i(x_{d_{i,n}}) = 0, \quad \delta = \pm 1,$$

where $\{q_{i,n}\}$ are sequences of real numbers and $\{d_{i,n}\}$ sequences of nonnegative integers. Oscillation criteria are given which can be considered as the discrete analogs of Kitamura-Kusano theorems [c3].

Paper [21] is concerned with the oscillatory behavior of solutions of the second-order difference equation

$$\Delta_a^2 x_n = F(n, x_n, \Delta_b x_n).$$

In [35] J. Korczak and J. Popenda considered, among others, difference equation

$$\Delta[x_n - p x_{n-t}] + \sum_{i=1}^{k} q_{i,n} x_{n-s_i} = 0,\ n > t.$$

They gave conditions under which every solution is oscillatory. The results of this paper are similar in part to those presented in [c1]. However the equations are more general. In [c4] and [c5], Korczak and Migda considered the problem when a given difference equation has a solution (or when all its solutions) approaches a given polynomial. One of the theorems presented in [35] gives a negative answer for the equation

$$\Delta[x_n + p x_{n-s}] + q_n x_{n-s} = 0,\ n \in N_0,\ s \in N,\ s < n_0.$$

In [44], J. Popenda obtained sufficient oscillation conditions for a generalized Emden-Fowler difference equation

$$\Delta^2 y_n + \sum_{i=1}^{t} p^i_{n-s_i} (y_{n-s_i})^{q_n} = 0,\ n = r, r+1, \ldots,$$

were $q : N_r \to R_+$ is a sequence of quotients of odd positive integers, $p^i : N_{r-s_i} \to R_+$, $\sigma = \max_{1 \le i \le t}\{s_i\}$, s_i are fixed nonnegative integers and $t \in N_\sigma$.

Also in [50] oscillatory behavior of a discrete analogue of Emden-Fowler differential equation with retarded arguments

$$\Delta^2 y_n + p_{n-k} y^q_{n-k} = 0, \quad n = k, k+1, \ldots,$$

is investigated, where q is a quotient of odd positive integers, k any fixed nonnegative integer. The above equation with $k = -1$ is considered in [c2].

In [47], J. Popenda gave a slight generalization of one of Patula's result in [c7] which is related to the oscillatory properties of solutions of the second order linear difference equation

$$x_{n+2} = a^1_n x_{n+1} + a^0_n x_n.$$

In [51], J. Popenda and E. Schmeidel investigated fourth order nonlinear difference equation of the form

$$\Delta^4 x_n = f(n, x_{n+2}).$$

The set of its solutions are divided into two disjoint subsets: F_+ and F_--solutions. Conditions under which every F_+-solution is oscillatory are given. The results generalize theorems obtained by Smith and Taylor [c11] and Taylor [c12].

In [52], J. Popenda and E. Schmeidel presented necessary conditions for the existence of a family of oscillatory solutions of the equation

$$c_n^r x_{n+r} + ... + c_n^1 x_{n+1} + c_n^0 x_n = d_n.$$

R. Agarwal and J. Popenda published a joint paper [59] in 1999. The purpose of this paper was to offer several new fundamental concepts in oscillation. The scalar recurrence equation is considered and an oscillation around a constant a, oscillation around a sequence, regular oscillation, periodic oscillation are introduced. The point-wise oscillation property of several orthogonal polynomials (Chebyshev polynomials of the first and second kind, Hermite polynomials and Legendre polynomials) is proved. The global oscillation of sequences of real-valued functions is also defined. Next for a second-order nonlinear continuous-recurrence relation, sufficient conditions to ensure global oscillation of solutions are provided. Moreover oscillations in ordered sets, in linear spaces, in Archimedean spaces, are considered and the concept of $(f, R, \leq)$-oscillation is introduced. For the partial recurrence relations, oscillation across the family $\gamma = \{\gamma_k : k \in N\}$ and oscillation between sets is defined. Finally, the oscillation of continuous-discrete recurrence relations is studied.

In [64], Popenda and Radicheva gave conditions under which every solution of equation

$$\Delta^2 y_n + (p_n|y_n|^\alpha + q_n|y_n|^\beta)\operatorname{sgn}(y_n) = 0,$$

where $\alpha > 0$, $\beta \in (0, 1)$, $\sup\limits_{k\geq n} p_k > 0$ and $\sup\limits_{k\geq n} q_k > 0$ for all $n \in N$, is oscillatory.

4 Explicit Solutions

Analytic formulas of solutions of difference equations are considered by Jerzy Propenda in [22], [58], [63] and [67].

In [22], he gave explicit formulas for the solutions of a linear homogeneous second order equations

$$a_n x_{n+2} + b_n x_{n+1} + c_n x_n = 0,$$

in the form

$$x_n = (-1)^n x_1 \frac{c_1}{a_1} \left[\prod_{j=2}^{n-2} \frac{b_j}{a_j}\right] \sum\nolimits_{(n-4,2)} z_i + (-1)^n x_2 \left[\prod_{j=1}^{n-2} \frac{b_j}{a_j}\right] \sum\nolimits_{(n-3,1)} z_i, \quad n \geq 2$$

where x_1, x_2 are arbitrary constants,

$$z_n = -\frac{c_n a_{n-1}}{b_n b_{n-1}}, \quad \text{for} \quad n > 1,$$

$$\sum\nolimits_{(n,k)} x_i := \sum_{\substack{d_1,...,d_n = 0 \\ d_i d_{i+1} = 0 \\ i=1,...,n-1}}^{1} \prod_{i=1}^{n} x_{i+k}^d \quad \text{for} \quad n > 1, \quad k \geq 0,$$

and

$$\sum\nolimits_{(1,k)} x_i := \sum_{d_1=0}^{1} x_{1+k}^{d_1}, \quad \sum\nolimits_{(0,k)} x_i = 1, \quad \sum\nolimits_{(-1,k)} x_i = 1, \quad \sum\nolimits_{(-2,k)} x_i = 0.$$

Futhermore the general solution of a linear nonhomogeneous equation

$$a_n x_{n+2} + b_n x_{n+1} + c_n x_n = r_n,$$

where $a, b, c : N \to R \setminus \{0\}$ and $r : N \to R$ is presented. J. Popenda also gave an example which shows that a nonlinear equation (the Riccati's equation) can be transformed into a linear homogeneous equation and hence its general solution can be found. The general solution can be used both in theoretical investigations and numerical methods. It should be noted that similar results are not known for differential equations. The methods introduced in this paper can be used, after some modifications, for recurrence equations of higher order.

Jerzy Popenda was also interested in partial difference equations and published with A. Musielak in 1998 a paper [58] which contains explicit formula for the solutions of hyperbolic partial difference equation

$$D^2_{/1,2} y(m,n) = a(m,n) y(m,n),$$

where $D^2_{/1,2} y(m,n) = y(m+1,n+1) - y(m+1,n) - y(m,n+1) + y(m,n)$. They proved the following theorem:

> Let a, ϕ, ψ be functions such that $\phi(0) = \psi(0)$. Then there exists a unique solution of the problem
>
> $$\begin{aligned} D^2_{/1,2} y(m,n) &= a(m,n) y(m,n) \\ y(m,0) &= \phi(m) \\ y(0,n) &= \psi(n). \end{aligned}$$
>
> This solution can be presented in the form
>
> $$\begin{aligned} y(m,n) = \phi(m) + \sum_{r=1}^{m-1} \sum_{\substack{i \in \overline{1,m-1} \\ j \in \overline{0,n-1}}} {}_r\, a(i,j) \circ \phi(i) \\ + \psi(n) + \sum_{r=1}^{n-1} \sum_{\substack{i \in \overline{0,m-1} \\ j \in \overline{1,n-1}}} {}_r\, a(i,j) \circ \psi(j) \\ + (\phi(0)a(0,0) - \phi(0)) \sum_{r=1}^{p} \sum_{\substack{i \in \overline{1,m-1} \\ j \in \overline{1,n-1}}} {}_r\, a(i,j) + (\phi(0)a(0,0) - \phi(0)) \end{aligned}$$

for all $m, n \in N$, where $\rho = \min\{m-1, n-1\}$,

$$\sum_{\substack{i \in \overline{\mu, m} \\ j \in \overline{\nu, n}}}{}_{r}\, x(i,j) \circ \omega(i) := \sum_{\substack{m \geq i_1 > i_2 > \ldots > i_r \geq \mu \\ n \geq j_1 > j_2 > \ldots > j_r \geq \nu}} \left(\prod_{k=1}^{r} x(i_k, j_k) \right) \omega(i_r)$$

for $1 \leq r \leq \min(m-\mu+1, n-\nu+1)$, $m \geq \mu$, $n \geq \nu$ and

$$\sum_{\substack{i \in \overline{\mu, m} \\ j \in \overline{\nu, n}}}{}_{r}\, x(i,j) \circ \omega(i) = 0$$

for $r = 0$, or $r > \min(m-\mu+1, n-\nu+1)$, or $m < \mu$, or $n < \nu$.

Such a formula seems to have escaped the notice of S. S. Cheng in his research monograph [c0].

Finally, Jerzy Popenda and A. Andruch-Sobiło considered difference equations in groups and published a joint paper [67] in which explicit formula of solutions is given.

5 Final Remarks

During the past twenty years, there are renewed interests in the subject of difference equations. Jerzy Propenda was one of the active mathematicians who worked in these areas. The problems he was interested in and studied are still up-to date which can be confirmed by a number of generalizations of his results and citations of his papers seen in recent journals. Some of his ideas are original and offer motivations for further study. Furthermore, a few of his results have already been included in monographs on inequalities and difference equations. However the actual number of his papers is still unknown as some of his papers have not been published yet.

We have tried to give a short survey of the works of Jerzy Popenda based on our best judgment. No doubt some of our assertions related to originality may be unjustified since results in difference equations are too numerous and difficult to arrange in chronological orders. We like to extend our apology to all those who deserve the proper recognition.

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