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Exponential Stability of Modified Stochastic Approximation Procedure *†

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Abstract

The modified stochastic approximation procedure

$$x_{i+1} = x_i + \alpha_i g(x_i + \xi_{i+1}), x_0 = \zeta \tag{1}$$

is considered. Here $\{\alpha_j\}$ is the sequence of positive numbers, $\{\xi_n\}$ is a sequence of martingale-differences, function g is twice differentiable, $ug(u) \le 0$ for $u \ne 0$, ζ is the initial value. Results on the almost-sure boundedness and the exponential stability of procedure (1) are obtained. The theorem of the convergence of nonnegative semimartingale has been applied.

1 Introduction

Stochastic approximation, originally proposed by Robbins and Monro in 1951 [7], is concerned with the problem of finding the root of the function $y = R(x)$ which is neither known nor directly observed. Let the result of the measurement at the point x_k at moment k be equal to $Y_k = R(x_k) + \xi_{k+1}$, where $\xi_1, \ldots, \xi_k, \ldots$ are independent random values with zero mean. For an arbitrary initial point $X(0) = x$ and an arbitrary sequence $\{\gamma_k\}$ of positive numbers, Robbins and Monro suggested the following procedure

$$X_{k+1} = X_k - \gamma_k Y_k,\ Y_k = R(x_k) + \xi_{k+1}. \tag{2}$$

The generalizations of Robbins and Monro method were investigated in numerous publications. We mention here just two outstanding books: by Nevelson and Khasminskii (cf. [5]) and Kushner (cf. [2]). Over the years, stochastic approximation has been proven to be a powerful and useful tool. Here we discuss the application of the stochastic approximation to credibility.

Let x denote the claim size, with distribution P_θ that depends on some random parameter θ (with the prior distribution $\pi(\theta)$). It can be proven (cf. [1] and [3]) that

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for some distributions P and π (the Normal/Normal, the Poisson/Gamma, *etc.*) the tradition credibility formula

$$\hat{\mu}_n = (1 - \alpha_n)m + \alpha_n \bar{x}_n, \tag{3}$$

takes place. Here $\hat{\mu}_n$ is an estimation of the fair premium, m is the collective fair premium, $\bar{x}_n$ is the mean of n years of individual experience $x_1, x_2, \ldots, x_n$. It is not difficult to show that (3) can be rewritten as a stochastic recursion of the type (2). The latter is particularly suited for the sequential evaluation of the fair premium. However in some situations (see [3])) tradition credibility formula fails and the stochastic approximation gives rise to some kind of quasi-credibility. In this case, instead of the stochastic approximation procedure (2), we need to consider the following modified procedure

$$x_{i+1} = x_i + \alpha_i g(x_i + \xi_{i+1}),\ x_0 = \zeta. \tag{4}$$

In this paper we investigate the exponential stability of procedure (4), where the errors of the observation ξ_i are martingale-differences, the function g is twice differentiable and $ug(u) \leq 0$ for $u \neq 0$. The theorem of the convergence of nonnegative semimartingale will be needed (cf. [6]).

2 Definitions and Auxiliary Lemmas

Let the probability space (Ω, F, P) with filtration $F = \{\mathcal{F}_n\}_{n=1,2,\ldots}$ be given. Let the stochastic sequence $\{m_n\}$ be an $\mathcal{F}_n$-martingale with $m_0 = 0$. We put $\xi_n = m_n - m_{n-1}$ for $n \geq 1$ where $\xi_0 = 0$. Then the stochastic sequence $\{\xi_n\}$ is an $\mathcal{F}_n$-martingale-difference. For detailed definitions and facts of random processes, the reader can see e.g. [4]. We present below two necessary lemmas which will be used in this paper.

LEMMA 1. Let $\{\xi_n\}$ be an $\mathcal{F}_n$-martingale-difference. Then there exists an $\mathcal{F}_n$-martingale-difference $\{\mu_n\}$ and a positive $\mathcal{F}_{n-1}$-measurable stochastic sequence $\{\eta_n\}$ such that for every $n = 1, 2, \ldots,$

$$\xi_n^2 = \mu_n + \eta_n \quad \text{a.s.} \tag{5}$$

LEMMA 2. Let $Z_n = Z_0 + A_n^1 - A_n^2 + M_n$ be a non-negative semimartingale, where M_n is a martingale, A_n^1, A_n^2 are a.s. non-decreasing processes. Then $\left\{\omega :: A_\infty^1 < \infty\right\} \subseteq \{Z \rightarrow\} \cap \left\{A_\infty^2 < \infty\right\}$ a.s.

Here $\{Z \rightarrow\}$ denotes the set of all $\omega \in \Omega$ for which $Z_\infty = \lim\limits_{t\to\infty} Z_t$ exists and is finite. The notation a.s. means almost-surely.

3 Boundedness of Solution

Let the function g be twice differentiable and for any $u \in \Re$,

$$ug(u) \leq 0, \quad u \neq 0, \tag{6}$$

$$g^2(u) \leq K_1 u^2, \tag{7}$$

$$|g''(u)| \leq K, \tag{8}$$

where $K_1, K > 0$ are nonrandom numbers. Let $\{\xi_n\}$ be a sequence of $\mathcal{F}_n$-martingale-differences with $\xi_0 = 0$ and decomposition (5) takes place. Let $\{\alpha_j\}$ be a sequence of positive numbers such that a.s.

$$\sum_{j=0}^{\infty} \alpha_j = \infty, \tag{9}$$

$$\sum_{j=0}^{\infty} \alpha_j^2 < \infty, \tag{10}$$

$$\sum_{j=0}^{\infty} \alpha_j \eta_{j+1} < \infty. \tag{11}$$

THEOREM 1. Let conditions (6)-(11) be fulfilled, then the solution x_i to equation (4) has the following properties:

$$\sup_{0<i<\infty} x_i^2 < \infty \quad \text{and} \quad \liminf_{i\to\infty} x_i = 0 \quad \text{a.s.}$$

PROOF. Applying Taylor's expansion to $g(x_i + \xi_{i+1})$ we have

$$\begin{aligned} x_{i+1}^2 - x_i^2 &= (x_i + \alpha_i g(x_i + \xi_{i+1}))^2 - x_i^2 \\ &\leq 2\alpha_i x_i \left[g(x_i) + g'(x_i)\xi_{i+1} + g''(u)\frac{\xi_{i+1}^2}{2} \right] + \alpha_i^2 K_1 (x_i + \xi_{i+1})^2, \end{aligned}$$

where u lies between x_i and ξ_{i+1}. From above and using the decomposition of ξ_i^2 (see Lemma 1 and (5)) we have

$$\begin{aligned} x_{i+1}^2 - x_i^2 \leq\; & 2\alpha_i x_i g(x_i) + (K\alpha_i|x_i| + 2K_1\alpha_i^2)\eta_{i+1} + 2K_1\alpha_i^2 x_i^2 + 2\alpha_i x_i g'(x_i)\xi_{i+1} \\ & + (K\alpha_i|x_i| + 2K_1\alpha_i^2)\mu_{i+1}. \end{aligned}$$

Let $\Delta m_i = 2\alpha_i x_i g'(x_i)\xi_{i+1} + (K\alpha_i|x_i| + 2K_1\alpha_i^2)\mu_{i+1}$, which is a martingale-difference. Applying the estimation $|x_i| \leq 1 + x_i^2$, we get

$$x_{i+1}^2 - x_i^2 \leq 2\alpha_i x_i g(x_i) + (K\alpha_i + 2K_1\alpha_i^2)\eta_{i+1} + (K\alpha_i\eta_{i+1} + 2K_1\alpha_i^2)x_i^2 + \Delta m_i, \tag{12}$$

and then

$$x_{i+1}^2 \leq 2\alpha_i x_i g(x_i) + (1+\beta_i)x_i^2 + (K\alpha_i + 2K_1\alpha_i^2)\eta_{i+1} + \Delta m_i, \tag{13}$$

where $\beta_i = K\alpha_i\eta_{i+1} + 2K_1\alpha_i^2$. Note that (10)-(11) imply

$$\prod_{j=1}^{i}(1+\beta_j) < M \tag{14}$$

for some nonrandom $M > 0$ and every $i = 1, 2, \ldots$ Letting

$$x_{i+1} = \prod_{j=1}^{i}(1+\beta_j)^{1/2} y_{i+1}$$

and substituting it in (13) we get

$$\begin{aligned}\prod_{j=1}^{i}(1+\beta_j)(y_{i+1}^2-y_i^2) \quad \le \quad & 2\alpha_i\prod_{j=1}^{i-1}(1+\beta_j)^{1/2}y_i g\Big(\prod_{j=1}^{i-1}(1+\beta_j)^{1/2}y_i\Big)\\ & +(K\alpha_i+2K_1\alpha_i^2)\eta_{i+1}+\Delta m_i.\end{aligned}$$

Let

$$\prod_{j=1}^{i}(1+\beta_j)^{-1}\Delta m_i=\Delta m_i^1,$$

which is a martingale-difference, therefore

$$\begin{aligned}y_{i+1}^2-y_i^2 \quad \le \quad & 2\prod_{j=1}^{i}(1+\beta_j)^{-1}\alpha_i\prod_{j=1}^{i-1}(1+\beta_j)^{1/2}y_i g\Big(\prod_{j=1}^{i-1}(1+\beta_j)^{1/2}y_i\Big)\\ & +\prod_{j=1}^{i}(1+\beta_j)^{-1}(K\alpha_i+2K_1\alpha_i^2)\eta_{i+1}+\Delta m_i^1. \qquad (15)\end{aligned}$$

Taking the sum of (15) from $i=0$ to $i=n-1$, we have

$$\begin{aligned}\sum_{i=0}^{n-1}y_{i+1}^2-\sum_{i=0}^{n-1}y_i^2 \quad \le \quad & 2\sum_{i=0}^{n-1}\prod_{j=1}^{i}(1+\beta_j)^{-1}\alpha_i\prod_{j=1}^{i-1}(1+\beta_j)^{1/2}y_i g\Big(\prod_{j=1}^{i-1}(1+\beta_j)^{1/2}y_i\Big)\\ & +\sum_{i=0}^{n-1}\prod_{j=1}^{i}(1+\beta_j)^{-1}(K\alpha_i+2K_1\alpha_i^2)\eta_{i+1}+\sum_{i=0}^{n-1}\Delta m_i^1.\end{aligned}$$

Therefore

$$y_n^2\le U_n=y_0^2+A_n^1-A_n^2+m_n^1, \qquad (16)$$

where

$$A_n^1=\sum_{i=0}^{n-1}\prod_{j=1}^{i}(1+\beta_j)^{-1}(K\alpha_i+2K_1\alpha_i^2)\eta_{i+1}$$

and

$$A_n^2=-2\sum_{i=0}^{n-1}\prod_{j=1}^{i}(1+\beta_j)^{-1}\alpha_i\prod_{j=1}^{i-1}(1+\beta_j)^{1/2}y_i g\Big(\prod_{j=1}^{i-1}(1+\beta_j)^{1/2}y_i\Big).$$

It should be noted that A_n^1 and A_n^2 are increasing processes a.s., m_n^1 is a martingale, U_n is nonnegative semimartingale and $P\{A_\infty^1<\infty\}=1$ due to the convergence of series in (10) and (11). Applying Lemma 2, that is

$$\{A_\infty^1<\infty\}\subset\{U_i\to\}\cap\{A_\infty^2<\infty\},$$

we have

$$P\{U_i\to\}=1.$$

This implies that there exists some a.s. finite random value $H = H(\omega)$ such that $P\{\sup\limits_{0<i<\infty} U_i \le H\} = 1$. Then $P\{\sup\limits_{0<i<\infty} y_i^2 \le H\} = 1$ and the first part of the theorem is proved.

Suppose now that $P\{\liminf\limits_{i\to\infty} y_i^2 > 0\} = p_0 > 0$. Then there exist random variables $\zeta_0 = \zeta_0(\omega) > 0$ and $N_0 = N_0(\omega) > 0$ such that $P(\Omega_0) = p_0$, where $\Omega_0 = \{\omega : y_i^2 > \zeta_0(\omega)/2 \text{ for } i > N_0\}$. Since $\prod\limits_{j=1}^{i}(1+\beta_j)^{1/2} > 1$, we have $y_i^2 \prod\limits_{j=1}^{i}(1+\beta_j)^{1/2} > \zeta_0(\omega)/2$ for $i > N_0$ and $\omega \in \Omega_0$. Due to the continuity and negativity (for $u \neq 0$) of the function $\phi(u) = ug(u)$ we can find $k_0 = k_0(\omega)$ and $N_1 = N_1(\omega) \ge N_0(\omega)$ such that

$$-\phi\left(\prod_{j=1}^{i}(1+\beta_j)^{1/2} y_i\right) > k_0(\omega)$$

for $\omega \in \Omega_0$ and $i > N_1$. Then

$$\begin{aligned}
& -2\sum_{i=0}^{n-1}\alpha_i \prod_{j=1}^{i}(1+\beta_j)^{1/2} y_i(\omega) g\left(\prod_{j=1}^{i}(1+\beta_j)^{1/2} y_i(\omega)\right) \\
= \;& -2\sum_{i=0}^{N_1-1} -2\sum_{i=N_1}^{n-1} \\
\ge \;& -2\sum_{i=N_1}^{n-1}\alpha_i \phi\left(\prod_{j=1}^{i}(1+\beta_j)^{1/2} y_i(\omega)\right) \\
\ge \;& 2k_0(\omega)\sum_{i=N_1}^{n-1}\alpha_i \to \infty
\end{aligned}$$

as $n \to \infty$. Hence $P\{A_\infty^2 = \infty\} \ge p_0 > 0$ which contradicts (16) and conditions (10)-(11). Then $P\{\liminf\limits_{i\to\infty} y_i^2 > 0\} = 0$ and from (14) we have: $P\{\liminf\limits_{i\to\infty} x_i^2 \le M \liminf\limits_{i\to\infty} y_i^2 = 0\} = 1$. The theorem is completely proved.

4 Exponential Stability

In addition to the conditions from the previous section let two more conditions be fulfilled

$$H_1\,|x| \le |g(x)|\,, \tag{17}$$

$$\sum_{i=0}^{\infty}\alpha_i \eta_{i+1}\prod_{j=1}^{i}(1-2\alpha_j H_1)^{-1} < \infty \text{ a.s.}, \tag{18}$$

where $H_1 > 0$ is some constant and $\eta_{i+1} \to 0$ when $i \to \infty$.

REMARK 1. Let $H_1|x| \le |g(x)|$ and $xg(x) < 0$ for all $x \in \Re$ and $x \neq 0$. The following is correct: if $x > 0$, then

$$-xg(x) = x|g(x)| \ge |x|H_1|x| = H_1x^2;$$

and if $x < 0$, then

$$-xg(x) = |x||g(x)| \geq |x|H_1|x| = H_1 x^2.$$

Therefore

$$xg(x) \leq -H_1 x^2. \tag{19}$$

THEOREM 2. Let conditions (6)-(11) and (17)-(18) be fulfilled. Then for any $\kappa > 0$,

$$\exp\left[(1-\kappa)2H_1 \sum_{i=0}^{n} \alpha_i\right] x_n^2 \to 0$$

a.s. where x_n is a solution of equation (4).

PROOF. Substituting (19) in (12) we get

$$x_{i+1}^2 - x_i^2 \leq -(2\alpha_i H_1 - K\alpha_i \eta_{i+1} - 2K_1\alpha_i^2)x_i^2 + (K\alpha_i + 2K_1\alpha_i^2)\eta_{i+1} + \Delta m_i. \tag{20}$$

Let

$$\tau_i = 2\alpha_i H_1 - K\alpha_i\eta_{i+1} - 2K_1\alpha_i^2.$$

Due to the positivity of K_1, K and η_{i+1} from condition (18) we have

$$\begin{aligned}
&\sum_{i=0}^{\infty}(K\alpha_i - 2K_1\alpha_i^2)\eta_{i+1} \prod_{j=0}^{i}(1 - 2\alpha_j H_1 + 2K_1\alpha_j^2 + K\alpha_j\eta_{j+1})^{-1} \\
\leq \quad & K\sum_{i=0}^{\infty}\alpha_i\eta_{i+1}\prod_{j=0}^{i}(1-2\alpha_j H_1)^{-1} < \infty. \qquad (21)
\end{aligned}$$

From (20) we get

$$x_{i+1}^2 \leq (1-\tau_i)x_i^2 + (K\alpha_i + 2K_1\alpha_i^2)\eta_{i+1} + \Delta m_i. \tag{22}$$

Let

$$z_i^2 = \prod_{j=1}^{i-1}(1-\tau_j)^{-1}x_i^2$$

which implies that

$$x_i^2 = \prod_{j=1}^{i-1}(1-\tau_j)z_i^2.$$

Substituting it in (22) we get,

$$\prod_{j=1}^{i}(1-\tau_j)z_{i+1}^2 \leq (1-\tau_i)\prod_{j=1}^{i-1}(1-\tau_j)z_i^2 + (K\alpha_j + 2K_1\alpha_j^2)\eta_{j+1} + \Delta m_i$$

and

$$z_{i+1}^2 - z_i^2 \leq \prod_{j=1}^{i}(1-\tau_j)^{-1}(K\alpha_j + 2K_1\alpha_i j^2)\eta_{j+1} + \Delta m_i^1, \tag{23}$$

where $\Delta m_i^1 = \prod_{j=1}^{i}(1-\tau_j)^{-1}\Delta m_i$. Taking the sum of (23) from $i=0$ to $i=n-1$, we obtain

$$z_n^2 \leq z_0^2 + \sum_{i=0}^{n-1}\prod_{j=1}^{i}(1-\tau_j)^{-1}(K\alpha_j + 2K_1\alpha_j^2)\eta_{j+1} + m_n^1.$$

Applying (21) and using the same arguments as in Theorem 1 we obtain that there exists some a.s. finite random value $H = H(\omega)$ such that $P\{\sup_{0<i<\infty} z_i^2 \leq H\} = 1$. As $\alpha_i, \eta_{i+1} \to 0$ when $i \to \infty$ for any $\varepsilon > 0$ there exists the random integer $N = N(\omega) > 0$ such that for any $j \geq N$

$$2K_1\alpha_j + K\eta_{j+1} < 2H_1\varepsilon.$$

Then for some $H_2 > 0$

$$\begin{aligned} x_i^2 &\leq HH_2\prod_{j=N}^{i}(1-2H_1(1-\varepsilon)\alpha_j) = HH_2\exp\{\sum_{j=N}^{i}\ln\bigl[1-2H_1(1-\varepsilon)\alpha_j\bigr]\} \\ &\leq HH_2\exp\{-2H_1\sum_{j=N}^{i}(1-\varepsilon)\alpha_j\}. \end{aligned} \tag{24}$$

If we take some $\kappa > 0$ and $\varepsilon < \kappa/2$, then from (24) we obtain that $P\{\lim_{i\to\infty}\exp\{(1-\kappa)2H_1\sum_{j=N}^{i}\alpha_j\}x_i^2 = 0\} = 1$. The proof is complete.

In the following example we investigate the fulfillment of the condition (18). We consider two different cases for α_i, and η_i.

EXAMPLE. a) Let $\alpha_i = \frac{1}{i}$ and $\eta_i \leq C(\omega)/i^{\varepsilon+2H_1}$ for some $\varepsilon > 0$ and a.s. finite random variable $C(\omega) > 0$. Then

$$\prod_{j=N}^{i}(1-2H_1\alpha_j)^{-1} = \prod_{j=N}^{i}e^{-\ln(1-2H_1\alpha_j)} = e^{-\sum_{j=N}^{i}\ln(1-2H_1\alpha_j)} \leq K_1' i^{2H_1},$$

and a.s.

$$\sum_{i=1}^{\infty}\alpha_i\eta_{i+1}\prod_{j=1}^{i}(1-2\alpha_j H_1)^{-1} \leq K_2'\sum_{i=1}^{\infty}\frac{1}{i}i^{2H_1}\frac{C(\omega)}{i^{\varepsilon+2H_1}} < \infty,$$

where K_1' and K_2' are some constants.

b) Let $\alpha_i = (i\ln i)^{-1}$ and $\eta_i \leq C(\omega)/i^{\varepsilon+2H_1}$ for some $\varepsilon > 0$ and a.s. finite random variable $C(\omega) > 0$. Then

$$\prod_{j=N}^{i}(1-2H_1\alpha_j)^{-1} \leq K_1'\ln^{2H_1} i,$$

and a.s.

$$\sum_{i=0}^{\infty}\alpha_i\eta_{i+1}\prod_{i=1}^{i}(1-2\alpha_i H_1)^{-1} \leq K_2'\sum_{i=1}^{\infty}\frac{1}{i\ln i}\ln^{2H_1} i\frac{C(\omega)}{i^{\varepsilon+2H_1}} < \infty.$$

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