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A Survey On The Oscillation Of Solutions Of First Order Linear Differential Equations With Deviating Arguments *

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Abstract

In this paper, a survey of the most basic results on the oscillation of solutions of first order linear differential equations with deviating arguments is presented.

1 Introduction

The qualitative properties of the solutions to the linear delay differential equations

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

and

$$x'(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0, \tag{2}$$

where and in the sequel $p(t) \in C([t_0, \infty), \mathbf{R})$, $\tau \in (0, \infty), \tau(t) \in C([t_0, \infty), \mathbf{R}^+)$, and $\lim_{t\to\infty} \tau(t) = \infty$, have been the subject of many investigations. Since 1950 when Myshkis [33] obtained the first oscillation criterion for (1), the oscillatory behavior of (1) and (2) has been discussed extensively in the literature. We refer to the papers [1-53] and the references cited therein.

By a solution of (1) (or (2)), we mean a function $x(t) \in C([\bar{t}_{-1}, \infty), \mathbf{R})$ for some $\bar{t} \geq t_0$, where $\bar{t}_{-1} = \inf\{\tau(t) : t \geq \bar{t}\}$, which satisfies equation (1) (or (2)) for all $t \geq \bar{t}$. As is customary, a solution $x(t)$ of (1) (or (2)) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, $x(t)$ is said to be nonoscillatory.

In this paper, our main purpose is to present the state of the art on the oscillation of solutions of (1) and (2).

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2 Oscillation Criteria with Nonnegative Coefficients

2.1 Oscillation Criteria of Inferior Limit

In this section, we always assume that $p(t) \geq 0$ and $\tau(t) < t$.

In 1950, Myshkis [33] first studied the oscillation of (1) and obtained the following theorem.

THEOREM 2.1 [33]. Assume that

$$\limsup_{t\to\infty}[t-\tau(t)] < \infty \text{ and } \liminf_{t\to\infty}[t-\tau(t)]\liminf_{t\to\infty}p(t) > \frac{1}{e}. \tag{3}$$

Then all solutions of (1) oscillate.

In 1979, Ladas [22] established integral conditions for the oscillation of (2). Tomaras [45, 46, 47] extended this result to (1) with variable delay. For related results see Ladde [27, 28, 29] and Koplatadze and Canturija [17]. The following most general result is due to Koplatadze and Canturija.

THEOREM 2.2 [17]. If

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{4}$$

then all solutions of (1) oscillate; if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds < \frac{1}{e}, \tag{5}$$

then (1) has a nonoscillatory solution.

In 1998, Tang [44] proved the following result which further improves (4).

THEOREM 2.3 [44]. Assume that there exists a $T \geq t_0$ such that

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \text{ for t} \geq \text{T}, \tag{6}$$

and

$$\int_{T}^{\infty} p(t)\left[\exp\left(\int_{\tau(t)}^{t} p(s)ds - \frac{1}{e}\right) - 1\right] dt = \infty. \tag{7}$$

Then all solutions of (1) oscillate.

When $\int_{\tau(t)}^{t} p(s)ds - 1/e$ oscillates, the aforementioned oscillation criteria fail to fit (1) or (2). For this case, in 1986 Domshlak [2] established a sufficient condition for oscillation of all solutions of (2).

THEOREM 2.4 [2]. Assume that

$$\liminf_{t\to\infty}\left(\int_{t}^{t+\tau} p(s)ds \exp\int_{t}^{t+\tau} \frac{p(s)ds}{\int_{s}^{s+\tau} p(\xi)d\xi}\right) > 1. \tag{8}$$

Then all solutions of (2) oscillate.

In 1996, Li [31] presented an infinite integral condition for oscillation of (2) which is very effective in the case when $\int_t^{t+\tau} p(s)ds - 1/e$ is oscillatory.

THEOREM 2.5 [31]. Assume that $\int_t^{t+\tau} p(s)ds > 0$ for $t \geq T_0$ for some $T_0 \geq t_0$ and

$$\int_{T_0}^{\infty} p(t) \ln \left(e \int_t^{t+\tau} p(s)ds \right) dt = \infty. \tag{9}$$

Then all solutions of (2) oscillate.

In 1998, Tang and Shen [38] obtained a sufficient condition related to but independent of (9).

THEOREM 2.6 [38]. Assume that there exist a $T_0 \geq t_0 + n\tau$ and a positive integer n such that

$$p_n(t) \geq \frac{1}{e^n} \text{ and } \overline{p}_n(t) \geq \frac{1}{e^n},\ t \geq t_0 \tag{10}$$

and

$$\int_{T_0}^{\infty} p(t) \left[\exp\left(e^{n-1} p_n(t) - \frac{1}{e} \right) - 1 \right] dt = \infty, \tag{11}$$

where

$$p_1(t) = \int_{t-\tau}^{t} p(s)ds,\ p_{k+1}(t) = \int_{t-\tau}^{t} p(s)p_k(s)ds,\ t \geq t_0 + (k+1)\tau,$$

and

$$\overline{p}_1(t) = \int_t^{t+\tau} p(s)ds,\ \overline{p}_{k+1}(t) = \int_t^{t+\tau} p(s)\overline{p}_k(s)ds,\ t \geq t_0,\ k = 1, 2, \dots .$$

Then all solutions of (2) oscillate.

For (1) with variable delay, in 1998 Li [32] and Shen and Tang [37], and in 2000 Tang and Yu [41] established the following theorems respectively.

THEOREM 2.7 [32]. Assume that $\tau(t)$ is nondecreasing and there exists a positive integer k such that

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s_1) \int_{\tau(s_1)}^{s_1} p(s_2) \dots \int_{\tau(s_{k-1})}^{s_{k-1}} p(s_k) ds_k \dots ds_1 > \frac{1}{e^k}. \tag{12}$$

Then all solutions of (1) oscillate.

THEOREM 2.8 [37]. Assume that $\tau(t)$ is strictly increasing on $[t_0, \infty)$ and its inverse is $\tau^{-1}(t)$. Let $\tau^{-k}(t)$ be defined on $[t_0, \infty)$ by

$$\tau^{-(k+1)}(t) = \tau^{-1}(\tau^{-k}(t)),\ k = 1, 2, \dots . \tag{13}$$

Suppose that there exist a positive integer n and $T_0 \geq \tau^{-n}(t_0)$ such that

$$p_n(t) \geq \frac{1}{e^n} \text{ and } \overline{p}_n(t) \geq \frac{1}{e^n},\ t \geq T_0, \tag{14}$$

and

$$\int_{T_0}^{\infty} p(t)\left[\exp\left(e^{n-1}p_n(t)-\frac{1}{e}\right)-1\right]dt=\infty, \tag{15}$$

where

$$p_1(t)=\int_{\tau(t)}^{t} p(s)ds,\ p_{k+1}(t)=\int_{\tau(t)}^{t} p(s)p_k(s)ds,\ t\geq \tau^{-k-1}(t_0)$$

and

$$\overline{p}_1(t)=\int_{t}^{\tau^{-1}(t)} p(s)ds,\ \overline{p}_{k+1}(t)=\int_{t}^{\tau^{-1}(t)} p(s)\overline{p}_k(s)ds,\ t\geq t_0,\ k=1,2,\dots .$$

Then all solutions of (1) oscillate.

THEOREM 2.9 [41]. Assume that $\tau(t)$ is nondecreasing and

$$\int_{t_0}^{\infty} p(t)\ln\left[e\int_{t}^{\tau^{-1}(t)} p(s)ds+1-\text{sign}\ \left(\int_{t}^{\tau^{-1}(s)} p(s)ds\right)\right]dt=\infty, \tag{16}$$

where $\tau^{-1}(t)=\min\{s\geq t_0:\tau(s)=t\}$. Then all solutions of (1) oscillate.

Note that Theorem 2.9 substantially improves Theorem 2.5 by removing the condition $\int_t^{t+\tau} p(s)ds>0$ in the case when $\tau(t)\equiv t-\tau$.

EXAMPLE 2.1. Consider the delay differential equation

$$x'(t)+p(t)x(t-\pi/3)=0,\ t\geq 0, \tag{17}$$

where $p(t)=\max\{0,a\sin t\}, 1>a>2/(2+\sqrt{3})^{\sqrt{3}/2}$. Clearly,

$$\int_{t}^{t+\pi/3} p(s)ds=0 \text{ for } t\in\bigcup_{n=0}^{\infty}\left[2n\pi+\pi,2n\pi+\frac{5}{3}\pi\right]$$

and

$$\limsup_{n\to\infty}\int_{t-\pi/3}^{t} p(s)ds=a<1.$$

So conditions (4), (6), (9), (10) and (12) are not satisfied. By direct calculation, we have

$$\begin{aligned}&\int_{0}^{2\pi} p(t)\ln\left[e\int_{t}^{t+\pi/3} p(s)ds+1-\text{sign}\ \left(\int_{t}^{t+\pi/3} p(s)ds\right)\right]\\ =\ &\frac{a}{2}\ln\frac{(2+\sqrt{3})^{2\sqrt{3}}a^4}{16}>0.\end{aligned}$$

It follows that

$$\int_{0}^{\infty} p(t)\ln\left[e\int_{t}^{t+\pi/3} p(s)ds+1-\text{sign}\ \left(\int_{t}^{t+\pi/3} p(s)ds\right)\right]dt=\infty.$$

Therefore, by Theorem 2.9, all solutions of (17) oscillate.

2.2 Oscillations in Critical State

In this section, we discuss the oscillation of solutions of (1) or (2) in the critical case when
$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}. \tag{18}$$

In 1986, Domshlak [2] first observed the following special critical situation:

Among the equations of the form (2) with
$$\lim_{t\to\infty} p(t) = \frac{1}{\tau e} \tag{19}$$
there exist equations such that their solutions are oscillatory in spite of the fact that the corresponding "limiting" equation
$$x'(t) + \frac{1}{\tau e}x(t-\tau) = 0,\ t \geq t_0 \tag{20}$$
admits a non-oscillatory solution, namely $x(t) = e^{-t/\tau}$.

Later, Domshlak [3], Elbert and Stavroulakis [7], Kozakiewicz [19], Li [30,31], Tang and Yu [39], Yu and Tang [48], Tang *et al.* [40] further investigated the oscillation of (1) or (2) in the critical case.

In 1996, Domshlak and Stavroulakis [5] obtained the following results in the special critical case $\liminf_{t\to\infty} p(t) = 1/\tau e$.

THEOREM 2.10 [5]. (i) Assume that
$$\liminf_{t\to\infty} p(t) = \frac{1}{\tau e},\ \liminf_{t\to\infty}\left[\left(p(t) - \frac{1}{\tau e}\right)t^2\right] = \frac{\tau}{8e}, \tag{21}$$
and
$$\liminf_{t\to\infty}\left\{\left[\left(p(t) - \frac{1}{\tau e}\right)t^2 - \frac{\tau}{8e}\right]\ln^2 t\right\} > \frac{\tau}{8e}. \tag{22}$$
Then all solutions of (2) oscillate.

(ii) Assume that for sufficiently t
$$p(t) \leq \frac{1}{\tau e} + \frac{\tau}{8et^2}\left(1 + \frac{1}{\ln^2 t}\right). \tag{23}$$
Then (2) has an eventually positive solution.

In 1998, Diblik [1] generalized this theorem as follows: Set $\ln_1 t = \ln t$, $\ln_{k+1} t = \ln(\ln_k t)$ for $k = 1, 2, \ldots$.

THEOREM 2.11 [1]. (i) Assume that for an integer $k \geq 2$ and a constant $\theta > 1$
$$\begin{aligned} p(t) \quad \geq \quad & \frac{1}{\tau e} + \frac{\tau}{8et^2}\Big[1 + (\ln_1 t)^{-2} + (\ln_1 t \ln_2 t)^{-2} + \ldots \\ & + (\ln_1 t \ln_2 t \ldots \ln_{m-1} t)^{-2} + \theta(\ln_1 t \ln_2 t \ldots \ln_m t)^{-2}\Big] \end{aligned} \tag{24}$$

as $t \to \infty$. Then all solutions of (2) oscillate.

(ii) Assume that for a positive integer k

$$p(t) \leq \frac{1}{\tau e} + \frac{1}{8et^2}\left[1 + (\ln_1 t)^{-2} + (\ln_1 t \ln_2 t)^{-2} + ... + (\ln_1 t \ln_2 t ... \ln_m t)^{-2}\right] \quad (25)$$

as $t \to \infty$. Then there exists a positive solution $x = x(t)$ of (2). Moreover as $t \to \infty$,

$$x(t) < e^{\frac{-t}{\tau}} \sqrt{t \ln t \ln_2 t ... \ln_k t}.$$

In 1999, Tang *et al.* [40] established the following comparison theorem.

THEOREM 2.12 [40]. Assume that for sufficiently large t

$$p(t) \geq \frac{1}{\tau e}. \quad (26)$$

Then all solutions of (2) oscillate if and only if all solutions of the following second order ordinary differential equation

$$y''(t) + \frac{2e}{\tau}\left(p(t) - \frac{1}{\tau e}\right) y(t) = 0, \ t \geq t_0 \quad (27)$$

oscillate.

Employing this comparison theorem and a wealth of results on oscillation of (27), many interesting oscillation and nonoscillation criteria can be obtained. One of them is the above Theorem 2.11.

In 2000 Tang and Yu [42] established the following more general comparison theorem in the case when $p(t) - 1/(\tau e)$ is oscillatory and $\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds = 1/e$.

THEOREM 2.13 [42]. Assume that $r(t) \in C([t_0, \infty), [0, \infty))$ is a τ-periodic function and satisfies the following hypothesis

$$\int_{t-\tau}^{t} r(s)ds \equiv \frac{1}{e}. \quad (28)$$

Suppose that

$$p(t) - r(t) \geq 0 \text{ for sufficiently large } t. \quad (29)$$

Then all solutions of (2) oscillate if and only if the Riccati inequality

$$\omega'(t) + r(t)\omega^2(t) + 2e^2[p(t) - r(t)] \leq 0, \ t \geq t_0 \quad (30)$$

has no eventually positive solution.

As a application of Theorem 2.13, the following theorem is also given in [42].

THEOREM 2.14 [42]. Assume that there is a τ-periodic function $r(t) \in C([t_0, \infty), [0, \infty))$ such that (28) and (29) hold. Then the following statements are valid.

(i) If

$$\liminf_{t\to\infty}\left[\int_{t_0}^{t} r(s)ds \int_{t}^{\infty} (p(s) - r(s))ds\right] > \frac{1}{8e^2}, \quad (31)$$

then all solutions of (2) oscillate;

(ii) If there exist a $T \geq t_0$ such that for $t \geq T$

$$\int_T^t r(s)ds \int_t^\infty (p(s) - r(s))ds \leq \frac{1}{8e^2}, \tag{32}$$

then (2) has an eventually positive solution.

EXAMPLE 2.2. Applying Theorem 2.14 to the following delay equation

$$x'(t) + \left[\frac{1}{\pi e}(1 + \sin 2t) + Ct^{-\beta}\right] x(t - \pi) = 0,\ t \geq \frac{\pi}{4}, \tag{33}$$

where $C > 0$ and $\beta \in \mathbf{R}$. We see that all solutions of (33) oscillate if and only if $\beta < 2$ or $\beta = 2$ and $C > \pi/8e$.

In 1998, Tang [44] proved Theorem 2.3 in the critical case where

$$\liminf_{t\to\infty} \int_{\tau(t)}^t p(s)ds = 1/e,$$

which extends a special case obtained in 1995 by Li [30]. In the sequel, we always assume that $t_0 < t_1 < t_2 < ...$ and $t_{k-1} = \tau(t_k)$ for $k = 1, 2, ...$.

THEOREM 2.15 [30]. Assume that there exists a $T_0 \geq t_0 + \tau$ such that

$$\int_{t-\tau}^t p(s)ds \geq \frac{1}{e} \text{ for } t \geq T_0 \tag{34}$$

and

$$\int_{T_0}^\infty p(t)\left[\exp\left(\int_{t-\tau}^t p(s)ds - \frac{1}{e}\right) - 1\right] dt = \infty. \tag{35}$$

Then all solutions of (2) oscillate.

DEFINITION 2.1 [7]. The piecewise continuous function $p : [t_0, \infty) \to [0, \infty)$ belongs to $\mathcal{A}_\lambda$ if

$$\int_{\tau(t)}^t p(s)ds \geq \frac{1}{e} \text{ for sufficiently large } t \tag{36}$$

and

$$\int_{\tau(t)}^t p(s)ds - \frac{1}{e} \geq \lambda_k \left(\int_{t_k}^{t_{k+1}} p(s)ds - \frac{1}{e}\right),\ t_k < t \leq t_{k+1},\ k = 1, 2, ..., \tag{37}$$

for some $\lambda_k \geq 0$, and $\liminf_{k\to\infty} \lambda_k = \lambda > 0$.

In 1995, Elbert and Stavroulakis [7] proved the following theorem.

THEOREM 2.16 [7]. Assume that $\tau(t)$ is strictly increasing on $[t_0, \infty)$ and that $p(t) \in \mathcal{A}_\lambda$ for some $\lambda \in (0, 1]$ and either

$$\lambda \limsup_{k\to\infty} k \sum_{i=k}^\infty \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e}\right) > \frac{2}{e} \tag{38}$$

or

$$\lambda \liminf_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e} \right) > \frac{1}{2e}. \tag{39}$$

Then all solutions of (1) oscillate.

In [7], Elbert and Stavroulakis put forth the following open problem.

OPEN PROBLEM 2.1 Can the bounds in conditions (38) and (39) of Theorem 2.16 be replaced by smaller ones?

In 2000 Tang and Yu [39] proved the following theorem.

THEOREM 2.17 [39]. Assume that $\tau(t)$ is strictly increasing on $[t_0, \infty)$, (36) holds, and that

$$\limsup_{k\to\infty} k \int_{t_k}^{\infty} p(t) \left(\int_{\tau(t)}^{t} p(s)ds - \frac{1}{e} \right) dt > \frac{1}{e^2}. \tag{40}$$

Then all solutions of (1) oscillate.

REMARK 2.1. If $p(t) \in \mathcal{A}_\lambda$ for some $\lambda \in (0, 1]$, then (40) reduces to

$$\lambda \limsup_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e} \right) > \frac{1}{e}, \tag{41}$$

which shows that the right-hand side of (38) can be replaced by $1/e$ which is less than the original $2/e$.

In 2002 Yu and Tang [48] proved the following theorems.

THEOREM 2.18 [48]. (i) Assume that

$$\liminf_{t\to\infty} \left[\left(\int_{\tau(t)}^{t} p(s)ds - \frac{1}{e} \right) \left(\int_{t_0}^{t} p(s)ds \right)^2 \right] > \frac{1}{8e^3}. \tag{42}$$

Then all solutions of (1) oscillate.

(ii) Assume that (36) holds and

$$\limsup_{t\to\infty} \left[\left(\int_{\tau(t)}^{t} p(s)ds - \frac{1}{e} \right) \left(\int_{t_0}^{t} p(s)ds \right)^2 \right] < \frac{1}{8e^3}. \tag{43}$$

Then (1) has an eventually positive solution.

THEOREM 2.19 [48]. Assume that (34) holds and $p(t) \not\equiv 0$ on any subinterval of $[t_0, \infty)$ and

$$\liminf_{t\to\infty} \left[\left(\int_{t_0}^{t} p(s)ds \right)^2 \int_{t}^{\infty} p(s) \left(\int_{\tau(s)}^{s} p(\xi)d\xi - \frac{1}{e} \right) ds \right] > \frac{1}{8e^3}. \tag{44}$$

Then all solutions of (1) oscillate.

THEOREM 2.20 [48]. Assume that (44) holds and

$$\int_{\tau(t)}^{t} p(s)ds > \frac{1}{e} \text{ for sufficiently large } t. \tag{45}$$

Then all solutions of (1) oscillate.

COROLLARY 2.1 [48]. Assume that (2.43) holds and $\tau(t)$ is strictly increasing on $[t_0, \infty)$. If

$$\liminf_{k\to\infty} k \int_{t_k}^{\infty} p(s) \left(\int_{\tau(s)}^{s} p(\xi)d\xi - \frac{1}{e} \right) ds > \frac{1}{8e^2}, \tag{46}$$

then all solutions of (1) oscillate.

COROLLARY 2.2 [48]. Assume that $\tau(t)$ is strictly increasing on $[t_0, \infty)$ and $p(t) \in \mathcal{A}_\lambda$ for some $\lambda \in (0, 1]$, and that

$$\lambda \liminf_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e} \right) > \frac{1}{8e}. \tag{47}$$

Then all solutions of (1) oscillate.

REMARK 2.2. The following example shows that 1/8e in (47) is the best possible. Thus, Theorems 2.17 and 2.20 or Corollary 2.1 or 2.2 answer the Open Problem 2.1.

EXAMPLE 2.3. Consider the delay differential equation

$$x'(t) + \frac{1}{e\ln 2} \left(\frac{1}{t} + \frac{C}{t(\ln t)^{1+\alpha}} \right) x\left(\frac{t}{2}\right) = 0,\ t \geq e, \tag{48}$$

where C, $\alpha > 0$. Here $\tau(t) = t/2$,

$$p(t) = \frac{1}{e\ln 2} \left(\frac{1}{t} + \frac{C}{t(\ln t)^{1+\alpha}} \right),$$

and

$$\int_{\tau(t)}^{t} p(s)ds = \frac{1}{e} + \frac{C}{\alpha e \ln 2} \left[\frac{1}{(\ln t - \ln 2)^\alpha} - \frac{1}{(\ln t)^\alpha} \right]. \tag{49}$$

Note that

$$\begin{aligned} & \left(\int_{\tau(t)}^{t} p(s)ds - \frac{1}{e} \right) \left(\int_{e}^{t} p(s)ds \right)^2 \\ = \ & \frac{C}{\alpha(e\ln 2)^3} \left[\frac{1}{(\ln t - \ln 2)^\alpha} - \frac{1}{(\ln t)^\alpha} \right] \left[\ln t - 1 + \frac{C}{\alpha}\left(1 - \frac{1}{(\ln t)^\alpha}\right) \right]^2, \end{aligned}$$

and

$$\lim_{t\to\infty} \left(\int_{\tau(t)}^{t} p(s)ds - \frac{1}{e} \right) \left(\int_{e}^{t} p(s)ds \right)^2 = \begin{cases} C/\left[e^3(\ln 2)^2\right], & \alpha = 1, \\ 0, & \alpha > 1, \\ \infty, & \alpha < 1. \end{cases}$$

By Theorem 2.18, every solution of (48) oscillates when $\alpha < 1$ or $\alpha = 1$ and $C > (\ln 2)^2/8$, but (48) has an eventually positive solution when $\alpha > 1$ or $\alpha = 1$ and $C < (\ln 2)^2/8$.

REMARK 2.3. When $\alpha = 1$, condition (49) implies that $p(t) \in \mathcal{A}_\lambda$ for $\lambda = 1$. Let $t_1 = e$, $t_n = 2t_{n-1} = 2^n e$, $n = 1, 2, \ldots$. Then

$$\begin{aligned}\lambda \lim_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e} \right) &= \frac{c}{e} \lim_{k\to\infty} k \sum_{i=k}^{\infty} \frac{1}{(i \ln 2 + 1)(i \ln 2 + 1 - \ln 2)} \\ &= \frac{c}{e(\ln 2)^2} \begin{cases} > 1/8e, & c > (\ln 2)^2/8, \\ < 1/8e, & c < (\ln 2)^2/8. \end{cases}\end{aligned}$$

This shows that $1/(8e)$ in condition (47) is the best possible.

2.3 Oscillation Criteria of Superior Limit

In this section, we always assume that $\tau(t) < t$ is nondecreasing on $[t_0, \infty)$ and $p(t) \geq 0$ for $t \geq t_0$ and define

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \text{ and } A := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds.$$

In 1972, Ladas *et al.* proved the following theorem which is a special case of the results in [25].

THEOREM 2.21 [25]. If

$$A := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag{50}$$

then all solutions of (1) oscillate.

Clearly, when the limit $\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ does not exist, there is a gap between conditions (5) and (50). How to fill this gap is an interesting open problem which has been investigated by several authors.

In 1988, Erbe and Zhang [9] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of (1). Their result, when formulated in terms of the numbers α and A says that all the solutions of (1) are oscillatory, if $0 < \alpha \leq \frac{1}{e}$ and

$$A > 1 - \frac{\alpha^2}{4}. \tag{51}$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$.

In 1991 Jian [15] derived the condition

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)}, \tag{52}$$

while in 1992 Yu and Wang [50] and Yu *et al.* [51] obtained the condition

$$A > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{53}$$

In 1990 Elbert and Stavroulakis [6] and in 1991 Kwong [21], using different techniques, improved (51), in the case where $0 < \alpha \le 1/e$, to the conditions

$$A > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \tag{54}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{55}$$

respectively, where λ_1 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$.

In 1994 Koplatadze and Kvinikadze [18] improved (53), while in 1998 Philos and Sficas [35], in 1999 Zhou and Yu [53] and Jaroš and Stavroulakis [14] derived the conditions

$$A > 1 - \frac{\alpha^2}{2(1-k)} - \frac{\alpha^2}{2}\lambda, \tag{56}$$

$$A > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} - (1 - \frac{1}{\sqrt{\lambda_1}})^2, \tag{57}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \tag{58}$$

respectively, and in 2000 Tang and Yu [41] the conditions

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s) \exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(\xi)d\xi\right) ds > 1 - \frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha-\alpha^2}\right),$$

$$\limsup_{t\to\infty} \left[\int_{\tau^2(t)}^{\tau(t)} p(s)ds + \frac{\int_{\tau(t)}^{t} p(s) \int_{\tau(s)}^{\tau(t)} p(\xi)d\xi}{1 - \int_{\tau(t)}^{t} p(s)ds}\right] > \frac{1+\ln \lambda_1}{\lambda_1},$$

where λ_1 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$.

Consider (1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that

$$p(\tau(t))\tau'(t) \ge \theta p(t)$$

eventually for all t. Under this additional condition, in 2000 Kon *et al.* [16] and in 2001 Sficas and Stavroulakis [36] established the conditions

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1-\alpha-\sqrt{(1-\alpha)^2 - 4\Theta}}{2} \tag{59}$$

and

$$A > \frac{\ln \lambda_1}{\lambda_1} - \frac{1+\sqrt{1+2\theta-2\theta\lambda_1 M}}{\theta\lambda_1} \tag{60}$$

respectively, where λ_1 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$, Θ is given by

$$\Theta = \frac{e^{\lambda_1\theta\alpha} - \lambda_1\theta\alpha - 1}{(\lambda_1\theta)^2},$$

and

$$M = \frac{1-\alpha-\sqrt{(1-\alpha)^2 - 4\Theta}}{2}.$$

REMARK 2.4. Observe that when $\theta = 1$, then

$$\Theta = \frac{\lambda_1 - \lambda_1\alpha - 1}{{\lambda_1}^2},$$

and (59) reduces to

$$A > 2\alpha + \frac{2}{\lambda_1} - 1, \tag{61}$$

while in this case it follows that

$$M = 1 - \alpha - \frac{1}{\lambda_1}.$$

and (60) reduces to

$$A > \frac{\ln\lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2k\lambda_1}}{\lambda_1}, \tag{62}$$

In the case where $\alpha = 1/e$, then $\lambda_1 = e$ and (62) leads to

$$A > \frac{\sqrt{7-2e}}{e} \approx 0.459987065.$$

It is to be noted that as $\alpha \to 0$, then all the previous conditions (51)-(59) and (61) reduce to the condition (50), i.e.,

$$A > 1.$$

However the condition (62) leads to

$$A > \sqrt{3} - 1 \approx 0.732$$

which is an essential improvement. Moreover (62) improves all the above conditions when $0 < \alpha \leq 1/e$ as well. Note that the value of the lower bound on A can not be less than $1/e \approx 0.367879441$. Thus the aim is to establish a condition which leads to a value *as close as possible to* $1/e$. For illustrative purpose, we give the values of the lower bound on A under these conditions when $\alpha = 1/e$:

0.966166179	(51)
0.892951367	(52)
0.863457014	(53)
0.845181878	(54)
0.735758882	(55)
0.709011646	(56)
0.708638892	(57)
0.599215896	(58)
0.471517764	(61)
0.459987065	(62)

We see that the condition (62) essentially improves all the known results in the literature.

EXAMPLE 2.4 [36]. Consider the delay differential equation

$$x'(t) + px\left(t - q\sin^2\sqrt{t} - \frac{1}{pe}\right) = 0,$$

where $p > 0$, $q > 0$ and $pq = 0.46 - \frac{1}{e}$. Then

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} pds = \liminf_{t\to\infty} p\left(q\sin^2\sqrt{t} + \frac{1}{pe}\right) = \frac{1}{e}$$

and

$$A = \limsup_{t\to\infty} \int_{\tau(t)}^{t} pds = \limsup_{t\to\infty} p\left(q\sin^2\sqrt{t} + \frac{1}{pe}\right) = pq + \frac{1}{e} = 0.46.$$

Thus, according to Remark 2.4, all solutions of this equation oscillate. Observe that none of the conditions (51)-(59) and (61) apply to this equation.

3 Oscillation of (1) and (2) with oscillating coefficients

In this section, the coefficient $p(t)$ and the deviating argument $\tau(t)$ are allowed to be oscillatory. Throughout this section, we will use the following notations:

$$\tau^0(t) = t,\ \tau^i(t) = \tau(\tau^{i-1}(t)),\ i = 1, 2, ..., \tag{63}$$

where $\tau^i(t)$ is defined on the set

$$E^i = \{t : \tau^{i-1}(t) \geq t_0\},\ i = 1, 2, ..., \tag{64}$$

and

$$\tau^{-1}(t) = \min\{s \geq t_0 : \tau(s) = t\}. \tag{65}$$

Clearly, $\lim_{t\to\infty} \tau^i(t) = \infty$ for $i = -1, 0, 1, 2, ...$, and $\tau(\tau^{-1}(t)) = t$, $\tau^{-1}(\tau(t)) \leq t$.

In 1982 Ladas *et al.* [23] first established the following theorems.

THEOREM 3.1 [23]. Assume that $p(t) > 0$ (at least) on a sequence of disjoint intervals $\{(\xi_n, t_n)\}_{n=1}^{\infty}$ with $t_n - \xi_n = 2\tau$. If

$$\limsup_{t\to\infty} \int_{t_n-\tau}^{t_n} p(s)ds \geq 1$$

then all solutions of (2) oscillate.

THEOREM 3.2 [23]. Assume that $p(t) > 0$ (at least) on a sequence of disjoint intervals $\{(\xi_n, t_n)\}_{n=1}^{\infty}$ with $t_n - \xi_n = 2\tau$ and $\lim_{n\to\infty}(t_n - \xi_n) = \infty$. If

$$\liminf_{t\to\infty} \int_{t-\frac{\tau}{2}}^{t} p(s)ds > 0 \text{ for } t \in \bigcup_{n=1}^{\infty} (\xi_n + \frac{\tau}{2},\ t_n)$$

and

$$\liminf_{t\to\infty}\int_{t-\tau}^{t} p(s)ds > \frac{1}{e} \text{ for } t \in \bigcup_{n=1}^{\infty}(\xi_n+\tau,\ t_n)$$

then all solutions of (2) oscillate.

EXAMPLE 3.1 [23]. Consider the differential equations

$$x'(t) + (\sin t)x(t-\frac{\pi}{2}) = 0,$$

$$x'(t) + p(t)x(t-1) = 0,$$

$$x'(t) + (\sin t)x(t-2\pi) = 0,\ \text{(see [12, p.197])}$$

and

$$x'(t) + \frac{\sin t}{2+\sin t}x(t-\frac{\pi}{2}) = 0,$$

where

$$p(t) = \begin{cases} p > 1/e & t \in [2^n\pi, 2^{n+1}\pi],\ n \text{ odd} \\ \cos t & t \in (2^n\pi, 2^{n+1}\pi),\ n \text{ even} \end{cases}.$$

From Theorems 3.1 and 3.2 it follows that all solutions of the first two equations oscillate. However the last two equations admit the nonoscillatory solutions $x_1(t) = e^{\cos t}$ and $x_2(t) = 2+\cos t$ respectively. As expected, the conditions of Theorems 3.1 and 3.2 are violated for the last two equations.

In 1984, Kulenovic and Grammatikopoulos [20] and Fukagai and Kusano [10] obtained the following theorems respectively.

THEOREM 3.3 [20]. Let $T > T_0 \geq t_0$ and $\mu > 0$ such that

$$\tau(t) \geq t-\mu \quad for \quad t \in A(T,\tau), \tag{66}$$

where

$$A(T,\tau) = [T,\infty)\bigcap\{t : \tau(t) < t, t \geq T_0\}. \tag{67}$$

Suppose that there exists a sequence of intervals $\{(a_n,b_n)\}_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty}(a_n,b_n) \subseteq A(T,\tau) \text{ and } \lim_{n\to\infty}(b_n-a_n) = \infty.$$

If

$$p(t) \geq 0 \text{ for } t \in \bigcup_{n=n_0}^{\infty}(a_n,b_n),\ n_0 \geq 1, \tag{68}$$

and

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{1}{e},\ t \in \bigcup_{n=n_0}^{\infty}(a_n+\mu,b_n), \tag{69}$$

then all solutions of (1) oscillate.

THEOREM 3.4 [10]. Assume that

(i) $\tau(t) < t$ and $\tau'(t) \le 0$ for $t \ge t_0$;
(ii) there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$ such that

$$p(t) \ge 0 \text{ for } t \in \bigcup_{n=1}^{\infty} [\tau^n(t_n), t_n] \tag{70}$$

and

$$\int_{\tau(t)}^{t} p(s)ds \ge c > \frac{1}{e} \text{ for } t \in \bigcup_{n=1}^{\infty} [\tau^{n-1}(t_n), t_n]. \tag{71}$$

Then all solutions of (1) oscillate.

Ladas *et al.* in [23] also presented the following open problem.

OPEN PROBLEM 3.1. Extend Theorem 2.2 to (1) with oscillating coefficients.

In 1986 Domshlak [2] and in 1988 Erbe and Zhang [9] proved the following theorems respectively, which answer this open problem.

THEOREM 3.5 [2]. Assume $c > 1/e$ and $\nu > 0$ is the root of the equation

$$\frac{\nu}{\sin \nu} \exp\left(-\frac{\nu}{\tan \nu}\right) = c.$$

Let (a_n, b_n), $n = 1, 2, ...$, be intervals such that $a_n \to \infty$,

$$p(t) \ge 0 \text{ for all } t \in G := \bigcup_{n=1}^{\infty} (\tau(a_n), b_n),$$

and

$$\int_{a_n}^{b_n} p(t)dt \ge \frac{\pi c}{\nu} \text{ for all } n \text{ and } \lim_{\substack{t \in G \\ t \to \infty}} \int_{\tau(t)}^{t} p(s)ds = c.$$

Then any solution of (1) has at least one root on each interval $(\tau^2(a_n), \tau(b_n))$.

THEOREM 3.6 [9]. Assume that
(i) $\tau(t) < t, \tau'(t) \ge 0$ for $t \ge t_0$.
(ii) there exists a sequence $t_n \to \infty$ such that

$$p(t) \ge 0 \text{ for } t \in \bigcup_{n=1}^{\infty} [\tau^{N+1}(t_n), t_n] \tag{72}$$

and

$$\int_{\tau(t)}^{t} p(s)ds \ge c > \frac{1}{e} \text{ for } t \in \bigcup_{n=1}^{\infty} [\tau^{N}(t_n), t_n], \tag{73}$$

where

$$N = \left[\frac{2(\ln 2 - \ln c)}{1 + \ln c}\right] + 1, \tag{74}$$

and $[\cdot]$ denotes greatest integer. Then all solutions of (1) oscillate.

In 1988 Domshlak and Aliev [4] derived the following

THEOREM 3.7 [4]. Suppose that $\tau(t) \le t$ is increasing, $\lim_{t\to\infty} \tau(t) = \infty$ and for $p(t) = p_+(t) - p_-(t)$, there exist $\tilde{p}_+(t)$ and $\tilde{p}_-(t)$ such that $p_+(t) \ge \tilde{p}_+(t) \ge 0$, $\tilde{p}_-(t) \ge p_-(t) \ge 0$, and

$$A_+ := \lim_{t\to\infty} \int_{\tau(t)}^{t} \tilde{p}_+(s)ds,\ A_- := \lim_{t\to\infty} \int_{\tau(t)}^{t} \tilde{p}_-(s)ds,$$

with

$$A_+ - A_- > \frac{1}{e}$$

Then all solutions of (1) are oscillatory.

EXAMPLE 3.2 [4]. Consider the delay equation

$$x'(t) + p(t)x(t-1) = 0,$$

where

$$p(t) := 2a\sin^2 n\pi t - bt^{-\alpha}\sin^2 \omega\pi t = p_+(t) - p_-(t),\ \omega, a, b, \alpha \in \mathbf{R}^+,\ n \in \mathbf{N}.$$

We have

$$\tilde{p}_+ = p_+,\ \tilde{p}_- = p_-,\ A_+ = a,\ A_- = 0$$

and from the above theorem it follows that if

$$a > \frac{1}{e}$$

then all solutions of this equation are oscillatory. Note that $p(t)$ will be necessarily oscillating in case ω is irrational. Since n and ω may be arbitrary large, oscillation rapidity of $p(t)$ may be arbitrary high.

In 1992 Yu *et al.* [51] established a completely different sufficient condition for oscillation.

THEOREM 3.8 [51]. Assume that $\tau(t)$ is nondecreasing, and that

(i) there exist a sequence $\{b_n\}$ and a positive integer $k \ge 3$ such that

$$b_n \to \infty \text{ as } n \to \infty,\ \tau^k(b_n) < b_n \text{ for } n = 0, 1, 2, ... \tag{75}$$

and

$$p(t) \ge 0,\ \tau(t) < t \text{ for } t \in \bigcup_{n=0}^{\infty} [\tau^k(b_n), b_n]. \tag{76}$$

(ii) there exists $\alpha \in [0, 1)$ such that

$$\int_{\tau(t)}^{t} p(s)ds \ge \alpha,\ t \in \bigcup_{n=0}^{\infty} [\tau^{k-2}(b_n), b_n]; \tag{77}$$

(iii) for some $i \in \{0, 1, \cdots, k-3\}$

$$\int_{\tau(t)}^{t} p(s)ds > 1 - A_i \text{ for } t \in \bigcup_{n=0}^{\infty} [\tau^{k-2}(b_n), \tau^{i+1}(b_n)], \tag{78}$$

where

$$A_0 = \alpha^2/2(1-\alpha),\ A_j = A_{j-1}^2 + \alpha A_{j-1} + \alpha^2/2,\ j = 1, 2, \dots . \tag{79}$$

Then all solutions of (1) oscillate.

In 1994 Domshlak [3] and in 2000 Tang and Yu [43] established the following results for (2) respectively.

THEOREM 3.9 [3]. Let $\{b_n\}_{n=1}^{\infty}$ where $\lim_{n\to\infty} b_n = \infty$, be an arbitrary sequence, $\mu \in (0, \pi/2)$ be a positive number, $G := \cup_{n=1}^{\infty} (b_n - \tau, b_n \exp(\pi/\mu))$ and

$$\liminf_{t\in G, t\to\infty} p(t) = \frac{1}{e\tau}, \tag{80}$$

and

$$\liminf_{t\in G, t\to\infty} \left[\left(p(t) - \frac{1}{e\tau} \right) t^2 \right] = D > \frac{\tau}{8e} \tag{81}$$

Then all solutions of (2) oscillate.

THEOREM 3.10 [43]. Assume that

(i) there exists a sequence of intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ such that $b_n \leq a_{n+1}$ and $b_n - a_n \geq 2\tau$ for $n = 1, 2, \cdots$, and that

$$p(t) \geq 0 \text{ for } t \in \bigcup_{n=1}^{\infty} [a_n, b_n]; \tag{82}$$

(ii)

$$\int_{t_0}^{\infty} \overline{p}(t) \ln \left[e \int_t^{t+\tau} \overline{p}(s) ds + 1 - \text{sign} \left(\int_t^{t+\tau} \overline{p}(s) ds \right) \right] dt = \infty, \tag{83}$$

where

$$\overline{p}(t) = \begin{cases} p(t), & t \in \cup_{n=1}^{\infty} [a_n + \tau, b_n); \\ 0, & t \in [t_0, a_1 + \tau) \cup \cup_{n=1}^{\infty} [b_n, a_{n+1} + \tau). \end{cases} \tag{84}$$

Then all solutions of (2) oscillate.

COROLLARY 3.1 [43]. Assume that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$p(t) \geq 0 \text{ for } t \in \bigcup_{n=1}^{\infty} [t_n - (N+1)\tau, t_n] \tag{85}$$

and

$$\int_{t-\tau}^{t} p(s) ds \geq c > \frac{1}{e} \text{ for } t \in \bigcup_{n=1}^{\infty} [t_n - N\tau, t_n], \tag{86}$$

where N is defined by $N = \left[\frac{1}{1 + \ln c} \right] + 1$ and $[x]$ denotes the greatest integer of x. Then all solutions of (2) oscillate.

REMARK 3.1. The result in Theorem 3.5 is essentially stronger than the results in Theorem 3.6 and Corollary 3.1. Indeed, in case

$$\tau(t) := t - 1,\ p(t) := \frac{1}{e}(1 + \alpha),$$

where $\alpha << 1$ on $\bigcup_{n=1}^{\infty}(a_n - 1, b_n)$, we obtain $(b_n - a_n) \sim 2(\ln 2 + 1)/\alpha$ in Theorem 3.6 and $(b_n - a_n) \sim 1/\alpha$ in Corollary 3.1, while from Theorem 3.5 we obtain $\nu \sim \sqrt{2\alpha}$ which implies $(b_n - a_n) \sim \pi/\sqrt{2\alpha}$ which is an essential improvement.

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