Applied Mathematics E-Notes, 2(2002), 66-71 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Solutions for a Class of Integro-differential Equations with Time Periodic Coefficients *

José Márcio Machado and Masayoshi Tsuchida †

Received 28 November 2001

Abstract

In this work, a series solution is found for the integro-differential equation $y''(t) = -(\omega_c^2 + \omega_f^2 \sin^2 \omega_p t)y(t) + \omega_f (\sin \omega_p t)\, z'(0) + \omega_f^2 \omega_p \sin \omega_p t \int_0^t (\cos \omega_p s)\, y(s) ds$, which describes the charged particle motion for certain configurations of oscillating magnetic fields. As an interesting feature, the terms of the solution are related to distinct sequences of prime numbers.

The integro-differential equation

$$\frac{d^2y}{dt^2} = -a(t)y + b(t) \int_0^t (\cos \omega_p s)\, y(s) ds + g(t), \tag{1}$$

where $a(t)$, $b(t)$ and $g(t)$ are given periodic functions of time may be easily found in the charged particle dynamics for some field configurations. Taking for instance the three mutually orthogonal magnetic field components $B_x = B_1 \sin \omega_p t$, $B_y = 0$, $B_z = B_0$, the nonrelativistic equations of motion for a particle of mass m and charge q in this field configuration are

$$m\frac{d^2x}{dt^2} = q\left(B_0 \frac{dy}{dt}\right), \tag{2}$$

$$m\frac{d^2y}{dt^2} = q\left(B_1 \sin \omega_p t \frac{dz}{dt} - B_0 \frac{dx}{dt}\right), \tag{3}$$

$$m\frac{d^2z}{dt^2} = q\left(-B_1 \sin \omega_p t \frac{dy}{dt}\right). \tag{4}$$

By integration of (4) and (2) and replacement of the time first derivatives of z and x in (3) one has (1) with

$$a(t) = \omega_c^2 + \omega_f^2 \sin^2 \omega_p t,\ b(t) = \omega_f^2 \omega_p \sin \omega_p t, \tag{5}$$

$$g(t) = \omega_f (\sin \omega_p t)\, z'(0) + \omega_c^2 y(0) + \omega_c x'(0), \tag{6}$$

*Mathematics Subject Classifications: 34A34, 34C15.

†UNESP-State University of São Paulo at São José do Rio Preto, Department of Computing 15054-000 - São José do Rio Preto - Brazil.

where $\omega_c = qB_0/m$ and $\omega_f = qB_1/m$. Making the additional simplification that $x'(0) = 0$ and $y(0) = 0$, (1) is finally written as

$$\begin{aligned}\frac{d^2y}{dt^2} &= -(\omega_c^2 + \omega_f^2 \sin^2 \omega_p t)y + \omega_f (\sin \omega_p t)\, z'(0) \\ &\quad + \omega_f^2 \omega_p (\sin \omega_p t) \int_0^t (\cos \omega_p s)\, y(s) ds. \end{aligned} \tag{7}$$

In this work, a series solution $y(t) = \sum_{k=0}^{\infty} y_k(t)$ will be constructed for (7) by the Adomian's method [1, 2]. According to this method, the highest-ordered differential operator in (7) is $L = d^2/dt^2$, and therefore this equation may be written as

$$Ly(t) = h(t) - Ry(t), \tag{8}$$

where R is the "remainder operator"

$$Ry(t) = a(t)y(t) - b(t) \int_0^t (\cos \omega_p s)\, y(s) ds \tag{9}$$

and

$$h(t) = \omega_f (\sin \omega_p t)\, z'(0). \tag{10}$$

The application of the inverse operator L^{-1} in both sides of (8) results in

$$LL^{-1}y = y - y(0) - y'(0)t = L^{-1}h - L^{-1}Ry, \tag{11}$$

and therefore

$$y(t) = y(0) + y'(0)t + L^{-1}h - L^{-1}Ry(t). \tag{12}$$

By explicit calculation of L^{-1} in the right side of (12), one has

$$\begin{aligned} y(t) &= y(0) + y'(0)t + \int_0^t \left(\int_0^s \omega_f (\sin \omega_p u)\, z'(0) du \right) ds \\ &\quad - \int_0^t \left(\int_0^s (\omega_c^2 + \omega_f^2 \sin^2 \omega_p u) y(u) du \right) ds \\ &\quad + \int_0^t \int_0^s \left(\omega_f^2 \omega_p \sin \omega_p u \left(\int_0^u (\cos \omega_p v)\, y(v) dv \right) du \right) ds \end{aligned} \tag{13}$$

The series approximation $y(t) = \sum_{k=0}^{\infty} y_k(t)$ may be replaced in both sides of (13), and the first term y_0 of this expansion is clearly identified with

$$\begin{aligned} y_0 &= y(0) + y'(0)t + \int_0^t \left(\int_0^s \omega_f (\sin \omega_p u)\, z'(0) du \right) ds = \\ &\quad y(0) + y'(0)t + \frac{\omega_f}{\omega_p^2} (\omega_p t - \sin \omega_p t) z'(0), \end{aligned} \tag{14}$$

otherwise the remaining terms in (13) will lead to the equation

$$\begin{aligned} & y_1(t) + y_2(t) + y_3(t)... \\ = & -\int_0^t \left(\int_0^s \left[\omega_c^2 + \omega_f^2 \sin^2 \omega_p u\right] (y_0(u) + y_1(u) + y_2(u) + ...)\, du \right) ds \\ & + \int_0^t \int_0^s \omega_f^2 \omega_p \sin \omega_p u \int_0^u \cos \omega_p v (y_0(v) + y_1(v) + y_2(v) + ...) dv du ds. \quad (15) \end{aligned}$$

Therefore, since y_0 is given in (14), a sequence of functions may be derived from (15):

$$\begin{aligned} y_1(t) &= -\int_0^t \left(\int_0^s (\omega_c^2 + \omega_f^2 \sin^2 \omega_p u) y_0(u) du \right) ds \\ &\quad + \int_0^t \int_0^s \left(\omega_f^2 \omega_p \sin \omega_p u \left(\int_0^u (\cos \omega_p v)\, y_0(v) dv \right) du \right) ds, \\ y_2(t) &= -\int_0^t \left(\int_0^s (\omega_c^2 + \omega_f^2 \sin^2 \omega_p u) y_1(u) du \right) ds \\ &\quad + \int_0^t \int_0^s \left(\omega_f^2 \omega_p \sin \omega_p u \left(\int_0^u (\cos \omega_p v)\, y_1(v) dv \right) du \right) ds, \\ ... &= ... \\ y_n(t) &= -\int_0^t \left(\int_0^s (\omega_c^2 + \omega_f^2 \sin^2 \omega_p u) y_{n-1}(u) du \right) ds \\ &\quad + \int_0^t \int_0^s \left(\omega_f^2 \omega_p \sin \omega_p u \left(\int_0^u (\cos \omega_p v)\, y_{n-1}(v) dv \right) du \right) ds. \quad (16) \end{aligned}$$

Recalling the action of the remainder operator in (9), the same result would have been obtained by expanding Ry in a power series

$$Ry = \sum_{k=0}^{\infty} A_k \left(y_0, y_1, ..., y_k\right) \quad (17)$$

where $A_0 = a(t)y_0 - b(t) \int_0^t (\cos \omega_p s)\, y_0(s) ds,\ A_1 = a(t)y_1 - b(t) \int_0^t (\cos \omega_p s)\, y_1(s) ds$, and so on. Tracking studies of the equations (2), (3) and (4) performed by the Runge-Kutta method have demonstrated that a solution for (7) seems to be decomposed into two components, an oscillation with frequency ω_c and a perturbation of very complex structure, which depends on t, ω_c, ω_p and ω_f. These tracking simulations we have executed also showed resonance phenomena for some particular choices of the cyclotron frequencies ω_c, ω_f and ω_p.

In order to clarify this question and calculate the terms of $\sum_{k=0}^{\infty} y_k(t)$ according to (16), an algebraic computation routine was prepared to run in the Mathematica package environment [3]. The general solution $y(t)$ obtained for (7) by this routine under the initial conditions $x'(0) = 0$ and $y(0) = 0$ is

$$y(t) = y'(0) \Bigg\{ \left(\sin \omega_c t\right) / \omega_c + F_1(\omega_c, \omega_f, \omega_p, t) + \sum_{n=1}^{\infty} A_{2n}(\omega_c, \omega_f, \omega_p, t) \cos[\omega_p (2n) t]$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty} B_{2n+1}(\omega_c,\omega_f,\omega_p)\sin[\omega_p(2n+1)t] + \sum_{n=1}^{\infty} C_{2n}(\omega_c,\omega_f,\omega_p,t)\sin[\omega_p(2n)t] \\
&\left.+\sum_{n=1}^{\infty} D_{2n}(\omega_c,\omega_f,\omega_p)\sin[\omega_p(2n)t]\right\} \\
&+z'(0)\left\{F_2(\omega_c,\omega_f,\omega_p,t) + \sum_{n=1}^{\infty} P_{2n}(\omega_c,\omega_f,\omega_p,t)\cos[\omega_p(2n)t]\right. \\
&+\sum_{n=0}^{\infty} Q_{2n+1}(\omega_c,\omega_f,\omega_p)\sin[\omega_p(2n+1)t] + \sum_{n=1}^{\infty} R_{2n}(\omega_c,\omega_f,\omega_p,t)\sin[\omega_p(2n)t] \\
&\left.+\sum_{n=1}^{\infty} S_{2n}(\omega_c,\omega_f,\omega_p)\sin[\omega_p(2n)t]\right\}, \qquad (18)
\end{aligned}
$$

where the functions $F_1, F_2, A_{2n}, B_{2n+1}$, etc., are also given by power series with a very characteristic structure. The aforementioned harmonic component with frequency ω_c is clearly present in (18). Retaining for instance only the terms y_0, y_1, y_2 and y_3 in the series for $y(t)$, one has

$$
\begin{aligned}
F_1 \;=\;& t\left\{-\frac{3\omega_f^2}{2^2\omega_p^2} + \frac{5\cdot 7\omega_f^4}{2^6\cdot 3\omega_p^4} - \frac{3\omega_c^2\omega_f^2}{2\omega_p^4} - \frac{11\cdot 7\omega_f^6}{2^8\cdot 3\cdot 5\omega_p^6}\right. \\
&\left.+\frac{1163\cdot 7\omega_c^2\omega_f^4}{2^8\cdot 3^3\omega_p^6} - \frac{3^2\omega_c^4\omega_f^2}{2^2\omega_p^6} + \ldots\right\} \\
&+t^3\left\{\frac{5\omega_c^2\omega_f^2}{2^3\cdot 3\omega_p^2} - \frac{5^3\omega_c^2\omega_f^4}{2^7\cdot 3^2\omega_p^4} + \frac{\omega_c^4\omega_f^2}{3\omega_p^4} + \ldots\right\} \\
&+t^5\left\{-\frac{7\omega_c^4\omega_f^2}{2^5\cdot 3\cdot 5\omega_p^2} + \ldots\right\} + \ldots \qquad (19)
\end{aligned}
$$

it is clear that the coefficient of the odd powers in t are constructed as follows: one term for the power 2 of ω_p in denominators, two terms for power 4, 3 for power 6, and so on. All integer coefficients in (19) are shown decomposed in prime factors, whose magnitude increases as one adds more terms in the expansion. There are similar structures for

$$
\begin{aligned}
F_2 \;=\;& t\left\{\frac{\omega_f}{\omega_p} - \frac{5\omega_f^3}{2^2\cdot 3\omega_p^3} + \frac{\omega_c^2\omega_f}{\omega_p^3} + \frac{3\cdot 7\omega_f^5}{2^6\cdot 5\omega_p^5} - \frac{167\omega_c^2\omega_f^3}{2^2\cdot 3^3\omega_p^5} + \frac{\omega_c^4\omega_f}{\omega_p^5}\right. \\
&\left.-\frac{11\cdot 13\omega_f^7}{2^8\cdot 3\cdot 5\cdot 7\omega_p^7} + \frac{577\cdot 1109\omega_c^2\omega_f^5}{2^8\cdot 3^3\cdot 5^3\omega_p^7} - \frac{1663\omega_c^4\omega_f^3}{2\cdot 3^5\omega_p^7} + \frac{\omega_c^6\omega_f}{\omega_p^7} + \ldots\right\} \\
&+t^3\left\{-\frac{\omega_c^2\omega_f}{3\cdot 2\omega_p} + \frac{11\omega_c^2\omega_f^3}{2^3\cdot 3^2\omega_p^3} - \frac{\omega_c^4\omega_f}{2\cdot 3\omega_p^3} - \frac{353\omega_c^2\omega_f^5}{2^7\cdot 3^2\cdot 5\omega_p^5}\right. \\
&\left.+\frac{5^2\cdot 11\omega_c^4\omega_f^3}{2^3\cdot 3^4\omega_p^5} - \frac{\omega_c^6\omega_f}{2\cdot 3\omega_p^5} + \ldots\right\}
\end{aligned}
$$

$$+t^5\left\{\frac{\omega_c^4\omega_f}{2^3\cdot 3\cdot 5\omega_p}-\frac{17\omega_c^4\omega_f^3}{2^5\cdot 3^2\cdot 5\omega_p^3}+\frac{\omega_c^6\omega_f}{2^3\cdot 3\cdot 5\omega_p^3}+...\right\}+..., \tag{20}$$

$$B_1=\frac{\omega_f^2}{\omega_p^3}-\frac{7\omega_f^4}{2^3\cdot 3\omega_p^5}+\frac{2\omega_c^2\omega_f^2}{\omega_p^5}+\frac{11\omega_f^6}{2^6\cdot 5\omega_p^7}-\frac{397\omega_c^2\omega_f^4}{2^3\cdot 3^3\omega_p^7}+\frac{3\omega_c^4\omega_f^2}{\omega_p^7}+..., \tag{21}$$

$$\begin{aligned} Q_1 \quad = \quad & -\frac{\omega_f}{\omega_p^2}+\frac{5\omega_f^3}{2^3\omega_p^4}-\frac{\omega_c^2\omega_f}{\omega_p^4}-\frac{7\omega_f^5}{2^6\omega_p^6}+\frac{3^2\omega_c^2\omega_f^3}{2^2\omega_p^6}-\frac{\omega_c^4\omega_f}{\omega_p^6} \\ & +\frac{13\cdot 11\omega_f^7}{2^{10}\cdot 3\cdot 5\omega_p^8}-\frac{7\cdot 11\omega_c^2\omega_f^5}{2^6\omega_p^8}+\frac{13\cdot 3\omega_c^4\omega_f^3}{2^3\omega_p^8}-\frac{\omega_c^6\omega_f}{\omega_p^8}+... \end{aligned} \tag{22}$$

$$A_2=t\left\{\frac{\omega_c^2\omega_f^2}{2^2\omega_p^4}-\frac{5\cdot 7\omega_c^2\omega_f^4}{2^4\cdot 3^2\omega_p^6}+\frac{\omega_c^4\omega_f^2}{2\omega_p^6}+...\right\}+t^3\left\{-\frac{\omega_c^4\omega_f^2}{2^3\cdot 3\omega_p^4}+...\right\}+..., \tag{23}$$

$$P_2=t\left\{\frac{\omega_c^2\omega_f^3}{2^2\omega_p^5}-\frac{23\omega_c^2\omega_f^5}{2^4\cdot 3^2\omega_p^7}+\frac{3\omega_c^4\omega_f^3}{2^2\omega_p^7}+...\right\}+t^3\left\{-\frac{\omega_c^4\omega_f^3}{2^3\cdot 3\omega_p^5}+...\right\}+..., \tag{24}$$

$$D_2=-\frac{\omega_f^2}{2^3\omega_p^3}+\frac{7\omega_f^4}{2^5\cdot 3\omega_p^5}-\frac{3\omega_c^2\omega_f^2}{2^3\omega_p^5}-\frac{11\omega_f^6}{2^{10}\omega_p^7}+\frac{421\omega_c^2\omega_f^4}{2^5\cdot 3^3\omega_p^7}-\frac{5\omega_c^4\omega_f^2}{2^3\omega_p^7}+..., \tag{25}$$

$$S_2=-\frac{\omega_f^3}{2^3\omega_p^4}+\frac{\omega_f^5}{2^5\omega_p^6}-\frac{\omega_c^2\omega_f^3}{2\omega_p^6}-\frac{13\cdot 11\omega_f^7}{2^{10}\cdot 3^2\cdot 5\omega_p^8}+\frac{5^2\omega_c^2\omega_f^5}{2^3\cdot 3^2\omega_p^8}-\frac{3^2\omega_c^4\omega_f^3}{2^3\omega_p^8}+... \tag{26}$$

$$C_2=t^2\left\{\frac{\omega_c^2\omega_f^2}{2^4\omega_p^3}-\frac{13\omega_c^2\omega_f^4}{2^6\cdot 3\omega_p^5}+\frac{3\omega_c^4\omega_f^2}{2^4\omega_p^5}+...\right\}+t^4\left\{-\frac{\omega_c^4\omega_f^2}{2^6\cdot 3\omega_p^3}+...\right\}+... \tag{27}$$

$$R_2=t^2\left\{\frac{\omega_c^2\omega_f^3}{2^4\omega_p^4}-\frac{3\omega_c^2\omega_f^5}{2^6\omega_p^6}+\frac{\omega_c^4\omega_f^3}{2^2\omega_p^6}+...\right\}+t^4\left\{-\frac{\omega_c^4\omega_f^3}{2^6\cdot 3\omega_p^4}+...\right\}+..., \tag{28}$$

etc. A more detailed structure for these coefficient functions, as well as for those related to higher harmonics of ω_p will require more terms of the sequence (16). However, it is also clear that the relative magnitude of the frequencies ω_c, ω_f and ω_p (ω_p much greater, for example) may allow that higher order terms in ω_p and higher harmonics be neglected. The role played by the prime numbers to form all functions $F_1, F_2, A_{2n}, B_{2n+1}$, etc., becomes evident by adding more terms in the expansion. As an example, by retaining the first seven terms of $y(t)=\sum_{k=0}^{\infty} y_k(t)$, (21) becomes

$$\begin{aligned} B_1 \quad = \quad & \frac{\omega_f^2}{\omega_p^3}-\frac{7\omega_f^4}{2^3\cdot 3\omega_p^5}+\frac{2\omega_c^2\omega_f^2}{\omega_p^5}+\frac{11\omega_f^6}{2^6\cdot 5\omega_p^7}-\frac{397\omega_c^2\omega_f^4}{2^3\cdot 3^3\omega_p^7}+\frac{3\omega_c^4\omega_f^2}{\omega_p^7} \\ & -\frac{13\cdot 11\omega_f^8}{2^{10}\cdot 3^2\cdot 7\omega_p^9}+\frac{29\cdot 1237\omega_c^2\omega_f^6}{2^4\cdot 3^3\cdot 5^3\omega_p^9}-\frac{5477\omega_c^4\omega_f^4}{2^2\cdot 3^5\omega_p^9}+\frac{2^2\omega_c^6\omega_f^2}{\omega_p^9} \\ & +\frac{13\cdot 17\cdot 19\omega_f^{10}}{2^{14}\cdot 3^4\cdot 5\cdot 7\omega_p^{11}}-\frac{13\cdot 197\cdot 188999\omega_c^2\omega_f^8}{2^{10}\cdot 3^4\cdot 5^3\cdot 7^3\omega_p^{11}}+\frac{3607\cdot 26627\omega_c^4\omega_f^6}{2^5\cdot 3^5\cdot 5^5\omega_p^{11}} \\ & -\frac{110939\omega_c^6\omega_f^4}{2^2\cdot 3^7\omega_p^{11}}+\frac{5\omega_c^8\omega_f^2}{\omega_p^{11}}+\dots . \end{aligned}$$

Further considerations about the solution (18), as well as application results, will be reported soon.

References

[1] G. Adomian, A new approach to nonlinear partial differential equations, J. Math. Anal. Appl., 102(1984), 420-434.

[2] E. Deeba, S. A. Khuri, S. S. Xie, An algorithm for solving a nonlinear integro-differential equation, Appl. Math. Comput., 115(2000), 123-131.

[3] R. Gass, Mathematica for Scientists and Engineers, Prentice Hall, 1998.