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# Global Attractivity in a Higher Order Nonlinear Difference Equation *

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**Abstract**

Our aim in this paper is to investigate the global attractivity of the recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - x_{n-k}}, \quad n = 0, 1, ...,$$

where $\alpha \geq 0$, $\gamma > \beta > 0$ are real numbers and $k \geq 1$ is an integer. We show that one positive equilibrium of the equation is a global attractor with a basin that depends on certain conditions imposed on the coefficients.

## 1 Introduction

The asymptotic stability of the rational recursive sequence

$$x_{n+1} = \frac{\alpha + \beta x_n}{\gamma + \sum_{i=0}^{k} \gamma_i x_{n-i}}, \ n = 0, 1, ..., \tag{1}$$

was investigated when the coefficients $\alpha, \beta, \gamma$ and $\gamma_i$ are nonnegative, see Kocic et al. [7], and Kocic and Ladas [6, 8]. Studying the asymptotic behavior of the rational sequence (1) when some of the coefficients are negative was suggested by Kocic and Ladas in [8]. Recently, Aboutaleb et al. [1] studied the rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-1}}, \ n = 0, 1, ...,$$

where $\alpha, \beta$ and $\gamma$ are nonnegative real numbers and obtained sufficient conditions for the global attractivity of the positive equilibria. Other related results can be found in [2, 3, 4, 5, 9, 10].

Our aim in this paper is to study the global attractivity of the rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - x_{n-k}}, \ n = 0, 1, ..., \tag{2}$$

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where $\alpha \geq 0$, $\gamma > \beta > 0$ are real numbers and $k \geq 1$ is an integer number, and the initial conditions $x_{-k}$ , $x_{-k+1}, \cdots, x_{-1}$ and $x_0$ are arbitrary. We prove that the positive equilibrium $\overline{x}$ of Eq.(2) is a global attractor with a basin that depends on certain conditions of the coefficients. The case where $k = 1$ was investigated in [11].

## 2 Local Stability and Permanence

We start this section with the following known result which will be used in our proofs.

LEMMA 2.1 [**8**]. Assume that $p, q \in R$ and $k \in \{0, 1, \cdots\}$. Then

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \ n = 0, 1, ...$$

Now, let us consider the rational recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - x_{n-k}}, \ n = 0, 1, ..., \tag{3}$$

where

$$\alpha > 0, \ \gamma > \beta > 0, \ k \in \{1, 2, 3, ...\}. \tag{4}$$

If (4) holds, and $\alpha = (\beta+\gamma)^2/4$, then Eq.(3) has a unique positive equilibrium $\overline{x}_0 = (\beta+\gamma)/2$. If (4) holds and $\alpha < (\beta+\gamma)^2/4$, then Eq.(3) has two positive equilibria

$$\overline{x}_{1,2} = \frac{\beta + \gamma \pm \sqrt{(\beta+\gamma)^2 - 4\alpha}}{2}.$$

The linearized equation of Eq.(3) about the equilibrium $\overline{x}_i$ is

$$y_{n+1} + \frac{\beta}{\gamma - \overline{x}_i} y_n - \frac{\overline{x}_i}{\gamma - \overline{x}_i} y_{n-k} = 0, \ i = 0, 1, 2, \ n = 0, 1, ... \ . \tag{5}$$

The characteristic equation associated with Eq.(5) about $\overline{x}_0$ is

$$\lambda^{k+1} + \frac{2\beta}{\gamma - \beta} \lambda^k - \frac{\gamma + \beta}{\gamma - \beta} = 0.$$

Since $(\gamma + \beta)/(\gamma - \beta) > 1$, the equilibrium $\overline{x}_0$ of Eq.(3) is unstable.

The characteristic equation associated with Eq.(5) about $\overline{x}_1$ is

$$\lambda^{k+1} + \frac{2\beta}{\gamma - \beta - \sqrt{(\beta+\gamma)^2 - 4\alpha}} \lambda^k - \frac{\gamma + \beta + \sqrt{(\beta+\gamma)^2 - 4\alpha}}{\gamma - \beta - \sqrt{(\beta+\gamma)^2 - 4\alpha}} = 0.$$

In view of

$$\left| \frac{\left(\gamma + \beta + \sqrt{(\beta+\gamma)^2 - 4\alpha}\right)}{\left(\gamma - \beta - \sqrt{(\beta+\gamma)^2 - 4\alpha}\right)} \right| > 1,$$

the equilibrium $\overline{x}_1$ of Eq.(3) is also unstable.

For the positive equilibrium $\overline{x}_2$, in view of condition (4) and $\alpha < (\beta+\gamma)^2/4$, we have

$$\overline{x}_2 = \frac{\beta+\gamma-\sqrt{(\beta+\gamma)^2-4\alpha}}{2} < \frac{\beta+\gamma}{2} < \gamma.$$

Hence, if

$$0 < \alpha < \beta(\gamma-\beta), \tag{6}$$

then

$$\begin{aligned}\sqrt{(\beta+\gamma)^2-4\alpha} &\geq \sqrt{(\beta+\gamma)^2-4\beta(\gamma-\beta)}\\ &> \sqrt{(\beta+\gamma)^2-(\gamma+3\beta)(\gamma-\beta)}\\ &= \sqrt{(\beta+\gamma)^2-(\beta+\gamma)^2+4\beta^2} = 2\beta,\end{aligned}$$

and so

$$\begin{aligned}\left|\frac{\beta+\overline{x}_2}{\gamma-\overline{x}_2}\right| &= \frac{\beta+\overline{x}_2}{\gamma-\overline{x}_2} = \frac{3\beta+\gamma-\sqrt{(\beta+\gamma)^2-4\alpha}}{\gamma-\beta+\sqrt{(\beta+\gamma)^2-4\alpha}}\\ &< \frac{3\beta+\gamma-2\beta}{\gamma-\beta+2\beta} = \frac{\beta+\gamma}{\gamma+\beta} = 1,\end{aligned}$$

which, by Lemma 2.1, implies that $\overline{x}_2$ ( in the sequel, we will denote $\overline{x}_2$ as $\overline{x}$ ) is locally asymptotically stable.

Before stating our result related to permanence, we list a lemma which is useful in proving our main result.

LEMMA 2.2**.** Let $f(u,v) = (\alpha-\beta u)/(\gamma-v)$ and assume that (4) and (6) hold. Then the following statements are true:

(a) $0 < \overline{x} < \alpha/\beta$, and $\alpha/\beta < \overline{x}_1 < \infty$,

(b) $f(x,x)$ is a strictly decreasing function in $(-\infty, \alpha/\beta]$, and

(c) let $u, v \in (-\infty, \alpha/\beta]$, then $f(u,v)$ is a strictly decreasing function in $u$, and a strictly increasing function in $v$.

PROOF. We only prove (a). The proofs of (b) and (c) are simple and will be omitted. In view of (4) and (6), we have

$$\overline{x} = \frac{\beta+\gamma-\sqrt{(\beta+\gamma)^2-4\alpha}}{2} < \frac{\beta+\gamma}{2} < \gamma.$$

By Eq.(3), we have

$$\overline{x} = \frac{\alpha-\beta\overline{x}}{\gamma-\overline{x}} > 0,$$

and so $\overline{x} < \alpha/\beta$. Also, in view of (4) and (6) we have

$$\begin{aligned}0 &< \frac{\alpha-\beta\overline{x}_1}{\gamma-\overline{x}_1} = \overline{x}_1 = \frac{\beta+\gamma+\sqrt{(\beta+\gamma)^2-4\alpha}}{2}\\ &\geq \frac{\beta+\gamma+\sqrt{(\beta+\gamma)^2-4\beta(\gamma-\beta)}}{2} = \frac{\beta+\gamma+\sqrt{(\gamma-\beta)^2+4\beta^2}}{2}\\ &> \frac{\beta+\gamma+\sqrt{(\gamma-\beta)^2}}{2} = \gamma,\end{aligned}$$

and so $\alpha - \beta\overline{x}_1 < 0$, which implies that $\overline{x}_1 > \alpha/\beta$. The proof is complete.

THEOREM 2.1. Assume that (4) and (6) hold and let $\{x_n\}$ be any solution of Eq.(3). If $x_i \in (-\infty, \alpha/\beta]$ for $i = -k, -(k-1), ..., -1$ and $x_0 \in [0, \alpha/\beta]$, then $0 \le x_n < \alpha/\beta$ for $n = 1, 2, ...$ .

PROOF. By part (c) of Lemma 2.2, we have

$$0 = \frac{\alpha - \beta \cdot \frac{\alpha}{\beta}}{\gamma - x_{-k}} \le x_1 = \frac{\alpha - \beta x_0}{\gamma - x_{-k}} \le \frac{\alpha - \beta \cdot 0}{\gamma - \frac{\alpha}{\beta}} < \frac{\alpha}{\beta},$$

and

$$0 = \frac{\alpha - \beta \cdot \frac{\alpha}{\beta}}{\gamma - x_{-k+1}} \le x_2 = \frac{\alpha - \beta x_1}{\gamma - x_{-k+1}} \le \frac{\alpha - \beta \cdot 0}{\gamma - \frac{\alpha}{\beta}} < \frac{\alpha}{\beta}.$$

The result now follows by induction. The proof is complete.

# 3 Global Attractivity

In this section, we will study the global attractivity of positive solutions of Eq.(3). We show that the positive equilibrium $\overline{x}$ of Eq.(3) is a global attractor with a basin that depends on certain conditions imposed on the coefficients.

LEMMA 3.1 **[3].** Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-k}),\ n = 0, 1, ..., \tag{7}$$

where $k \ge 1$. Let $I = [a, b]$ be some interval of real numbers, and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties:

(a) $f(u, v)$ is a nonincreasing function in $u$, and a nondecreasing function in $v$, and

(b) if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(M, m),\ \text{and}\ M = f(m, M), \tag{8}$$

then $m = M$.

Then Eq.(7) has a unique positive equilibrium point $\overline{x}$ and every solution of Eq.(7) converges to $\overline{x}$.

THEOREM 3.1. Assume that the conditions (4) and (6) hold. Then the positive equilibrium $\overline{x}$ of Eq.(3) is a global attractor with a basin $S = [0, \alpha/\beta]^{k+1}$ .

PROOF. For $u, v \in [0, \alpha/\beta]$, set

$$f(u, v) = \frac{\alpha - \beta u}{\gamma - v}.$$

We claim that $f : [0, \alpha/\beta] \times [0, \alpha/\beta] \to [0, \alpha/\beta]$. In fact, set $a = 0$ and $b = \alpha/\beta$, then

$$f(b, a) = \frac{\alpha - \beta b}{\gamma - a} = \frac{\alpha - \alpha}{\gamma} = 0 = a,$$

and in view of $0 < \alpha < \beta(\gamma - \beta)$, we have

$$f(a,b) = \frac{\alpha - \beta a}{\gamma - b} = \frac{\alpha}{\gamma - \frac{\alpha}{\beta}} < \frac{\alpha}{\beta} = b.$$

Since $f(u,v)$ is decreasing in $u$ and increasing in $v$, it follows that

$$a \leq f(u,v) \leq b, \text{ for } u, v \in [a,b],$$

which implies that our assertion is true. On the other hand, the conditions (a) and (b) of Lemma 3.1 are clearly true. Let $\{x_n\}$ be a solution of Eq.(3) with initial conditions $(x_{-k}, \cdots, x_{-1}, x_0) \in S$. By Lemma 3.1, we have $\lim_{n\to\infty} x_n = \overline{x}$. the proof is complete.

By Theorems 2.1 and 3.1, we have the following more general result.

THEOREM 3.2. Assume that the conditions (4) and (6) hold, then the positive equilibrium $\overline{x}$ of Eq.(3) is a global attractor with a basin $S = (-\infty, \alpha/\beta]^k \times [0, \alpha/\beta]$.

PROOF. Let $\{x_n\}$ be a solution of Eq.(3) with initial conditions $(x_{-k}, \cdots, x_{-1}, x_0) \in S$. Then by Theorem 2.1, we have

$$x_n \in [0, \alpha/\beta], \ n = 1, 2, ..., k, k+1, ... .$$

By Theorem 3.1, we have $\lim_{n\to\infty} x_{n+k} = \overline{x}$, and so $\lim_{n\to\infty} x_n = \overline{x}$. The proof is complete.

In the above discussion, we always assume that $0 < \alpha < \beta(\gamma - \beta)$. In fact, the following example shows that the upper bound $\beta(\gamma - \beta)$ may be the best.

EXAMPLE 3.1. Consider the difference equation

$$x_{n+1} = \frac{1 - x_n}{2 - x_{n-k}}, \ n = 0, 1, ...,$$

where $k \geq 1$. Obviously, $\alpha = \beta(\gamma - \beta)$. When $k$ is odd, however, it is easy to see that the solution of this equation with initial conditions $x_{-k} = 0, x_{-k+1} = 1, ..., x_{-1} = 0$ and $x_0 = 1$ is periodic with period 2.

Motivated by the above example, we shall prove that the following general result is also true if

$$\beta(\gamma - \beta) \leq \alpha < (\gamma - \beta)(\gamma + 3\beta)/4. \tag{9}$$

THEOREM 3.3. Assume that (4) holds. Then Eq.(3) has prime period two non-negative solutions if and only if $k$ is odd and (9) holds.

PROOF. By direct computation, it is easy to see that there exist no period two solutions when $k$ is even and if $k$ is odd the period two solution must be of the form

$$..., \frac{\gamma - \beta + \sqrt{(\gamma + 3\beta)(\gamma - \beta) - 4\alpha}}{2}, \frac{\gamma - \beta - \sqrt{(\gamma + 3\beta)(\gamma - \beta) - 4\alpha}}{2}, ...$$

from which our result follows. The proof is complete.

# 4 The Case $\alpha = 0$

In this section we study the asymptotic stability of the difference equation

$$x_{n+1} = \frac{-\beta x_n}{\gamma - x_{n-k}},\ n = 0, 1, ..., \tag{10}$$

where $\beta, \gamma > 0$, and $k \geq 1$.

By putting $x_n = \beta y_n$, Eq.(10) yields

$$y_{n+1} = \frac{-y_n}{A - y_{n-k}},\ n = 0, 1, ..., \tag{11}$$

where $A = \gamma/\beta$. Eq.(11) has two equilibrium points

$$\overline{y}_1 = 0,\ \overline{y}_2 = 1 + A.$$

The linearized equation associated with Eq.(11) about $\overline{y}_i$ is

$$z_{n+1} + \frac{1}{A - \overline{y}_i} z_n + \overline{y}_i z_{n-k} = 0,\ n = 0, 1, ... . \tag{12}$$

The characteristic equation of (12) about $\overline{y}_2$ is

$$\lambda^{k+1} - \lambda^k + 1 + A = 0.$$

Since $1 + A > 1$, then the equilibrium $\overline{y}_2$ of Eq.(11) is unstable.

The characteristic equation of (12) about $\overline{y}_1$ is

$$\lambda^{k+1} + \frac{1}{A}\lambda^k = 0.$$

This equation has two roots

$$\lambda_1 = 0 \text{ and } \lambda = -\frac{1}{A}.$$

Hence, (1) if $\gamma > \beta$ then $\overline{y}_1$ is asymptotically stable, (2) if $\gamma < \beta$ then $\overline{y}_1$ is a saddle point, and (3) if $\gamma = \beta$ then linearized stability analysis fails.

In the following results we assume that $A \geq 2$.

LEMMA 4.1. Assume that the initial conditions $y_{-i} \in [-1, 1]$ for $i = 1, 2, ..., k$ and $y_0 \in [-1, 0]$. Then $\{\ y_{2n-1}\}$ is nonnegative and monotonically decreasing to zero, while $\{y_{2n}\}$ is non-positive and monotonically increasing to zero.

PROOF. Suppose that $y_{-i} \in [-1, 1]$ for $i = 1, 2, \cdots, k$ and $y_0 \in [-1, 0]$. Clearly, $0 \leq y_1 \leq 1$ and $-1 \leq y_2 \leq 0$. By induction we can see that $0 \leq y_{2n-1} \leq 1$ and $-1 \leq y_{2n} \leq 0$ for $n \geq 1$. Since

$$\frac{y_{2n-1}}{y_{2n+1}} = (A - y_{2n-k})(A - y_{2n-k-1}) > 1,$$

then

$$y_{2n-1} > y_{2n+1},\ n = 1, 2, \ldots .$$

Similarly, we can show that $y_{2n} < y_{2n+2}$, $n = 1, 2, \ldots$ . The proof is complete.

LEMMA 4.2. Assume that the initial conditions $y_{-i} \in [-1, 1]$ for $i = 1, 2, ..., k$, and $y_0 \in [0, 1]$. Then $\{ y_{2n-1}\}$ is non-positive and monotonically increasing to zero, while $\{y_{2n}\}$ is nonnegative and monotonically decreasing to zero.

The proof is similar to that of Lemma 4.1 and will be omitted.

COROLLARY 4.1. The equilibrium $\overline{y}_1 = 0$ of Eq.(11) is a global attractor with a basin $S = [-1, 1]^{k+1}$.

THEOREM 4.1. The equilibrium $\overline{y}_1 = 0$ of Eq.(11) is a global attractor with a basin $S = (-\infty, 1]^k \times [-A + 1, A - 1]$.

PROOF. Assume that $(y_{-k}, \cdots, y_{-1}, y_0) \in S$. We have

$$-1 \leq \frac{A-1}{-(A - y_{-k})} \leq y_1 = \frac{-y_0}{A - y_{-k}} \leq \frac{A-1}{A-1} = 1,$$

and

$$-1 \leq \frac{1}{-(A - y_{-k+1})} \leq y_2 = \frac{-y_1}{A - y_{-k+1}} \leq 1.$$

By induction, it is easy to see that $y_i \in [-1, 1]$ for $i \geq 1$. Our result now follows from Corollary 4.1. The proof is complete.

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