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Optimization Analysis Involving Set Functions *

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Abstract

We review optimization problems involving set functions, which are defined on a family $S \subset \Gamma$ of measurable subsets in a nonatomic finite measure space (X, Γ, μ), and their related convexity, differentiability and subdifferentiability. In particular, sufficient optimality theorems and dual models for fractional programming involving set functions are presented in the framework of generalized convexity.

1 Introduction

Various types of functions appear in optimization problems. The common function $f : A \to B$ maps an element or a point of A to an element or a point in B. But there are also functions that map a point to a set, a set to a point, and a set to a set as well.

Since 1970 there are at least two directions of concern in optimization theory. One is to generalize the classical nonlinear programming problem. The involved functions take their values in ordered topological vector spaces, and are defined in some topological vector spaces (see for example Zowe [32]). The other one is to study scalar valued set functions defined on a family of measurable subsets of a measurable space (see, e.g. [1-12], [15-17], [19-24]). Clearly, it is of importance to study optimization problems involving set functions taking values in ordered topological vector space and defined on measurable subsets of a measure space (see Lai *et al.* [13, 14]).

In this review we will confine ourselves to set functions defined on a σ-algebra of measurable subsets to points of real numbers or to points in an ordered topological vector spaces. Optimization problems with set functions arise in many situation dealing with optimal selection for measurable subsets. These problems have been encountered in fluid flow, electrical insulator design, optimal plasma confinement and regional design problems (see e.g. [1,5,22,23]).

The analysis of optimization problems involving set functions has been developed by many researchers, for example, Chou *et al.* [2-4], Corley *et al.* [5,7], Jo *et al.* [8], Lai *et al.* [9-16], Lin [20-21], Morris [22], Preda [23-24], Rosenmuller *et al.* [25-30], Tanaka *et al.* [27], Zalmai [28-27], etc.

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Morris [22] was the first to develop the general theory for optimizing set functions. Corley [7] and Lin [20-24] developed an optimization theory for programming problems with n-set functions. Recently, Lai *et al.* [15] considered set functions with generalized convexity, and developed fractional programming with generalized convex set functions (cf. [16,17] and [23,24]). Although a σ-algebra is not a linear space, the convexity and differentiability can be defined in similar ways as those in linear spaces, and optimization theory with set functions are developed in various situations.

The convexity, differentiability and basic theory for set functions are discussed in Section 2. Section 3 is concerning for optimality conditions in ordered topological vector spaces. The optimality criteria for programming problem involving n-set functions and n-fold set functions are presented in Section 4 under certain generalized convexity conditions. In general, the main tasks in optimization problem are to find the necessary and/or sufficient conditions for a feasible solution to be an optimal solution. Usually, the sufficient conditions are more difficult to establish. Indeed, extra assumptions have to be imposed. Thus many researchers try to replace the usual convexity conditions with generalized convexity conditions, see for example [20,23,30,31]. Recently, Lai and Liu [15] defined the $(\mathcal{F}, \rho, \theta)$-convexity and established several sufficient optimality conditions. The authors also investigated fractional programming involving set functions. Under the framework of generalized convexity, three kinds of dual models can be constructed, and duality theorems can be established in weak, strong and strictly converse forms. These results are reviewed in Sections 5 and 6.

2 Convexity, Differentiability and Optimality

Throughout the paper, let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable. We write $L_1(X, \Gamma, \mu) = L_1$ and $L_\infty(X, \Gamma, \mu) = L_\infty$ for brevity. Since $\mu(X) < +\infty$, for each $\Omega \in \Gamma$, the characteristic function $\chi_\Omega \in L_\infty \subset L_1$. If $f \in L_1$, $\Omega \in \Gamma$ and $\chi_\Omega \in L_\infty$, then the integral $\int_\Omega f d\mu$ will be denoted by the dual pair $\langle f, \chi_\Omega \rangle$. By separability of L_1, for any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0,1]$, and any sequences $\{\Omega_n\}$ in Ω and $\{\Lambda_n\}$ in Λ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda\chi_{\Omega-\Lambda} \text{ and } \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda-\Omega} \Rightarrow \chi_{\Omega_n\cup\Lambda_n\cup(\Omega\cap\Lambda)} \xrightarrow{w^*} \lambda\chi_\Omega + (1-\lambda)\,\chi_\Lambda \quad (1)$$

as $n \to \infty$, where w^* stands for weak*-topology in L_∞.

We remark that in [22], Morris considers all sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ in Γ such that (1) holds. Actually the proof of Proposition 3.2 in [22] leads us to consider $\Omega_n \subset \Omega\backslash\Lambda$ and $\Lambda_n \subset \Lambda\backslash\Omega$ for every n such that (1) holds. This practice will be followed throughout the rest of our discussions.

We call the sequence $\{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\} \equiv \{M_n\}$ a Morris' sequence. Although a σ-algebra Γ is not a linear space, the convexity of a subfamily S for measurable subsets in Γ can be defined as follows: S is convex if for any $(\Omega, \Lambda, \lambda) \in S \times S \times [0,1]$ associated with a Morris sequence $\{M_n\}$ in Γ such that $M_n = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda) \in S$. The convexity of a set function can then be defined as follows.

DEFINITION 1. Let S be a convex family of measurable subsets in X. A set function $F : S \subset \Gamma \to \mathcal{R}$ is convex if for any $(\Omega, \Lambda, \lambda) \in S \times S \times [0,1]$, there exists a Morris sequence $\{M_n\}$ such that $\overline{\lim} F(M_n) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda)$.

We remark that in Rosenmuller *et al.* [25,26], set functions defined on a family of subsets of a finite set appear in a different fashion in probability theory as well as in discrete game theory, and their convexity are defined in different manners.

DEFINITION 2. We say that a set function $F:\Gamma\to\mathcal{R}$ is differentiable at $\Omega_0\in\Gamma$ if there exists $f_{\Omega_0}\in L_1(X,\Gamma,\mu)$, namely the derivative of F at Ω_0, such that

$$F(\Omega)=F(\Omega_0)+\langle f_{\Omega_0},\chi_\Omega-\chi_{\Omega_0}\rangle+o(\rho(\Omega,\Omega_0))$$

where ρ is a pseudometric on Γ which is defined by $\rho(\Omega_1,\Omega_2)=\mu(\Omega_1\triangle\Omega_2)$, $\Omega_1,\Omega_2\in\Gamma$ and Δ denotes the symmetric difference of sets.

We remark that each $\Omega\in\Gamma$ is regarded as $\chi_\Omega\in L_\infty\subset L_1$ throughout the paper, where all topologies induced in Γ is topology induced by w^*-topology on $\{\chi_\Omega,\Omega\in\Gamma\}\subset L_\infty$. In particular if a set function F is countably additive and is absolutely continuous with respect to μ, then the Radon-Nikodym derivative $dF/d\mu$ is simply identified by f_Ω. Furthermore, if $\widetilde{F}:L_1\to\mathcal{R}$ is Fréchet differentiable with $F:\Gamma\to\mathcal{R}$ defined by $F(\Omega)=\widetilde{F}(\chi_\Omega)$ for $\Omega\in\Gamma$, then F is a differentiable set function in the sense of Definition 2 and the differential $DF(\Omega)$ coincides with the Fréchet differential $\widetilde{F}'(\chi_\Omega)$ defined on L_1. That is (cf. [9] and [10]),

$$\widetilde{F}'(\chi_\Omega)=DF(\Omega)=\frac{dF}{d\mu}(\Omega)=f_\Omega\in L_1^*=L_\infty\subset L_1.$$

In the same manner, a subgradient of a set function F can be defined as follows.

DEFINITION 3. An element $f\in L_1$ is a subgradient of a convex set function F at $\Omega_0\in\Gamma$ if it satisfies the inequality

$$F(\Omega)\geq F(\Omega_0)+\langle f,\chi_\Omega-\chi_{\Omega_0}\rangle,\ \Omega\in\Gamma.$$

The set of all subgradients of a set function F at Ω_0 is called the subdifferential of F at Ω_0, and is denoted by

$$\partial F(\Omega_0)=\{f\in L_1\mid F(\Omega)\geq F(\Omega_0)+\langle f,\chi_\Omega-\chi_{\Omega_0}\rangle,\ \Omega\in\Gamma\}.$$

If F is convex and differentiable at Ω_0, then $\partial F(\Omega_0)=\{f_{\Omega_0}\}=\{DF(\Omega_0)\}$ is a singleton. It can be shown (cf. [11, Theorem 3.5]) that if a set function F is properly convex and w^*-lower semicontinuous on its convex domain $S\subset\Gamma$, then $\partial F(\Omega)\neq\emptyset$ and $\operatorname{Dom}F^*\neq\emptyset$. Here F^* stands for the conjugate function of F, and is defined by

$$F^*(f)=\sup_{\Omega\in\Gamma}\left[\langle f,\chi_\Omega\rangle-F(\Omega)\right],\ f\in L_1.$$

The conjugate set Γ^* is defined as a subset of L_1 by

$$\Gamma^*=\left\{f\in L_1\mid\sup_{\Omega\in\Gamma}\left[\langle f,\chi_\Omega\rangle-F(\Omega)\right]<\infty\right\}.$$

THEOREM 1 (Fenchel-Moreau, [11, Theorem 3.6]). Let F be a proper convex set function which is w^*-lower semicontinuous on a convex domain $S\subset\Gamma$. Then $F(\Omega)=F^{**}(\Omega)$ for all $\Omega\in S$. Here F^{**} is the conjugate function of F^*, and is defined by

$$F^{**}(\Omega)=\begin{cases}\sup_{f\in L_1}\left[\langle f,\chi_\Omega\rangle-F^*(f)\right] & \text{if }\Omega\in\mathrm{Dom}F\\ +\infty & \text{if }\Omega\notin\mathrm{Dom}F\end{cases}.$$

Accordingly, we have the following consequence.

COROLLARY 1. Let $f_0 \in \operatorname{Dom} F^* = \Gamma^* \subset L_1$ and $\Omega_0 \in S$. Then,

$$f_0 \in \partial F(\Omega_0) \Longleftrightarrow \Omega_0 \in \partial F^*(f_0).$$

PROOF. Indeed, for $f_0 \in \partial F(\Omega_0)$, we have $F(\Omega) \geq F(\Omega_0) + \langle f_0, \chi_\Omega - \chi_{\Omega_0} \rangle$ and $\langle f_0, \chi_{\Omega_0} \rangle - F(\Omega_0) \geq \langle f_0, \chi_\Omega \rangle - F(\Omega)$ for all $\Omega \in S$. Then

$$F(\Omega_0) + F^*(f_0) = \langle f_0, \chi_{\Omega_0} \rangle . \tag{2}$$

By Theorem 1, we have $F^{**}(\Omega_0) + F^*(f_0) = \langle f_0, \chi_{\Omega_0} \rangle$. Hence, $\Omega_0 \in \partial F^*(f_0)$ if, and only if, $f_0 \in \partial F(\Omega_0)$. The proof is complete.

Note that if the set function F is convex on Γ, and $\Omega_0 \in \Gamma$ minimizes $F(\Omega)$, then by (2),

$$F(\Omega_0) = \min_{\Omega \in \Gamma} F(\Omega) = \langle f_0, \chi_{\Omega_0} \rangle - F^*(f_0) = \sup_{f \in \Gamma^*} \left[\langle f, \chi_{\Omega_0} \rangle - F^*(f) \right].$$

If we define a Lagrangian functional $L : \Gamma \times \Gamma^* \to \mathcal{R}$ by $L(\Omega, f) = \langle f, \chi_\Omega \rangle - F^*(f)$, then it can be shown that $L(\Omega, f)$ has a saddle point $(\Omega_0, f_0) \in \Gamma \times \Gamma^*$ (cf. [9]). That is, $L(\Omega_0, f) \leq L(\Omega_0, f_0) \leq L(\Omega, f_0)$ for all $f \in \Gamma^*$ and $\Omega \in \Gamma$. Consequently,

$$\sup_{f \in \Gamma^*} \left[\langle f, \chi_{\Omega_0} \rangle - F^*(f) \right] = L(\Omega_0, f_0) = F(\Omega_0) = \min_{\Omega \in \Gamma} F(\Omega).$$

Let $F : S \subset \Gamma \to \mathcal{R}$ be a convex set function, and let $G : \Im \subset \Gamma \to \mathcal{R}$ be a concave set function on $\Im$ with $S \cap \Im$ having nonempty interior. Consider the minimization problem:

$$(P_r) : \text{Minimize } [F(\Omega) - G(\Omega)] \text{ subject to } \Omega \in S \cap \Im.$$

The Fenchel duality theorem (cf. [10]) is valid.

THEOREM 2 (Fenchel duality theorem). If $\mu = \inf_\Omega [F(\Omega) - G(\Omega)]$ is finite such that the epigraphs $[F, S]$ or $[G, \Im)$ have nonempty interiors, then

$$\inf_{\Omega \in S \cap \Im} [F(\Omega) - G(\Omega)] = \max_{f \in S^* \cap \Im^*} [G^*(f) - F^*(f)] = \mu. \tag{3}$$

An example explaining the result (3) of problem (P_r) can be found in [10]. We give another example as follows.

EXAMPLE. Let $X = \left\{ (x, y) \mid x^2 + y^2 \leq 5^2 \right\} \subset \mathcal{R}^2$, $\Gamma = \{ \Omega \mid \Omega \subset X \}$ a σ-algebra and $\mu = m$ is the Lebesgue measure. Then (X, Γ, m) is a finite atomless measure space. Define

$$F(\Omega) = \int \int_\Omega f_1(x, y)\, dm, \ f_1(x, y) \in L_1.$$

Then, for any $f \in L_1$, the conjugate functional of F is given by

$$\begin{aligned} F^*(f) &= \sup_{\Omega \in \Gamma} [\langle f, \chi_\Omega \rangle - F(\Omega)] = \sup_{\Omega \in \Gamma} \int \int_\Omega [f(x, y) - f_1(x, y)]\, dm \\ &= \int \int_{X_1} [f(x, y) - f_1(x, y)]\, dm, \end{aligned}$$

where $X_1 = \{(x,y) \in X \mid f(x,y) > f_1(x,y)\} \subset X$. The conjugate set is $\Gamma^* = L_1(X, \Gamma, m)$. Now if we take $f_1(x,y) = x^2 - y^2 + 6$, and define

$$F(\Omega) = \int\int_\Omega (x^2 - y^2 + 6)dm,$$

and

$$G(\Omega) = 6m(\Omega) = 6\int\int_\Omega dm,\ \Omega \in \Gamma,$$

then $F(\Omega)$ is convex (linear), and $G(\Omega)$ is concave (linear) and

$$F^*(f) = \int\int_{X_1} \left[f(x,y) - x^2 + y^2 - 6\right] dm,$$

$$G^*(f) = \inf_{\Omega\in\Gamma} \left[\langle f, \chi_\Omega\rangle - G(\Omega)\right] = \int\int_{X_2} \left[f(x,y) - 6\right] dm,$$

where $X_2 = \{(x,y) \in X \mid f(x,y) \le 6\}$. Consequently,

$$\begin{aligned} & \max_{f\in L_1} \left[G^*(f) - F^*(f)\right] \\ = & \max_{f\in L_1} \int\int_{X_2} \left[f(x,y) - 6\right] dm - \int\int_{X_1} \left[f(x,y) - x^2 + y^2 - 6\right] dm \\ = & \max_{f\in L_1} v(f) \end{aligned}$$

where $v(f)$ is a functional on L_1. The maximum of $v(f)$ can be solved from $v'(f) = 0$, where $v'(f)$ stands for the Fréchet derivative of $v(f)$, that is,

$$v'(f) = \lim_{h\to 0} \frac{|v(f+h) - v(f)|}{\|h\|} = \lim_{h\to 0} \frac{1}{\|h\|} \left|\int\int_{X_2} h(x,y)dm - \int\int_{X_1} h(x,y)dm\right| = 0.$$

This implies that

$$X_1 \cap X_2 = \left\{(x,y) \in X \mid x^2 - y^2 + 6 \le f(x,y) \le 6\right\} = \Omega_0.$$

Consequently, we have

$$\max_{f\in L_1} v(f) = \int\int_{\Omega_0} \left(x^2 - y^2\right) dm = -\frac{625}{2} \ \text{ for } x^2 - y^2 \le 0.$$

On the other hand

$$\begin{aligned} \min_{\Omega\in\Gamma} \left[F(\Omega) - G(\Omega)\right] &= \min_\Omega \left\{\int\int_\Omega \left[x^2 - y^2 + 6 - 6\right] dm\right\} \\ &= \int\int_{\Omega_0} \left(x^2 - y^2\right) dm = -\frac{625}{2}. \end{aligned}$$

with $\Omega_0 = \left\{(x,y) \in X \mid x^2 - y^2 \le 0\right\}$. This shows that

$$\min_{\Omega\in\Gamma} \left[F(\Omega) - G(\Omega)\right] = \max_{f\in L_1} \left[G^*(f) - F^*(f)\right] = -\frac{625}{2}.$$

3 Optimization in Ordered Vector Spaces

The alternative theorems for saddle-point results of convex programming problems for set functions with values in ordered vector spaces was recently investigated by Lai and Szilagyi in [14]. Here we define the order relations in (Y, C) by

$$\begin{aligned} y_1 <_C y_2 &\Longleftrightarrow y_2 - y_1 \in C - \{0\}, \\ y_1 \leq_C y_2 &\Longleftrightarrow y_2 - y_1 \in C, \\ y_1 \ll_C y_2 &\Longleftrightarrow y_2 - y_1 \in \operatorname{int} C. \end{aligned}$$

Consider a programming problem in ordered vector space as follows

(P_0) Minimize $F(\Omega)$ subject to $\Omega \in \mathcal{S} \subset \Gamma$ and $G(\Omega) \leq_D \theta$,

where θ stands for the zero vector, $F : \mathcal{S} \to Y$ and $G : \mathcal{S} \to Z$ are C-convex and D-convex set mappings, respectively. Here C and D are normal cones in the locally convex Hausdorff vector spaces Y and Z over real field $\mathcal{R}$ respectively, so that (Y, C) and (Z, D) are ordered vector lattices with $\text{int}C \neq \phi$ and $\text{int}D \neq \phi$.

A Farkas type theorem follows from Gordan and Fan in finite dimensional space was generalized to convex set function as follows (see [14]).

THEOREM 3. Let $F : \mathcal{S} \to Y$ be a C-convex set function. Then the inequality system

$$F(\Omega) \ll_C \theta \tag{4}$$

has no solution in $\mathcal{S}$ if, and only if, there exists $y^* \neq \theta$ in C^*, the conjugate cone of C in Y^* (the topological dual space of Y) such that

$$\langle y^*, F(\Omega) \rangle \geq 0 \text{ for all } \Omega \in S. \tag{5}$$

PROOF. The sufficiency part is trivial. Indeed, if there is a nonzero vector $y^* \in C^*$ such that $\langle y^*, F(\Omega) \rangle \geq 0$ for all $\Omega \in \mathcal{S}$, then $F(\Omega) \notin \operatorname{int}(-C)$. That is (4) has no solution. As for the necessity part, if (4) has no solution, then the set $A = \{y \in Y \mid F(\Omega) \ll_C y \text{ for } \Omega \in \mathcal{S}\}$ does not contain the origin θ. It is not hard to prove that A is convex in Y when F is a convex set function. Further, $\operatorname{int} A \neq \emptyset$ since $\operatorname{int} C \neq \emptyset$. In view of $\theta \notin A$, the separation theorem is applicable so that there exists a nonzero $y^* \in Y^*$ such that $\langle y^*, y \rangle \geq 0$ for all $y \in A$. Putting $y = F(\Omega) + c$ for any $c \in \operatorname{int} C$, we then have

$$\langle y^*, F(\Omega) \rangle + \langle y^*, c \rangle \geq 0. \tag{6}$$

We assert that $y^* \in C^*$. Indeed, if the contrary holds, there would be a $\bar{c} \in C$ such that $\langle y^*, \bar{c} \rangle < 0$. Since C is a cone, if we choose $c_0 \in \operatorname{int} C$ then $nc_0 \in \operatorname{int} C$ for any positive integer n. Thus (6) implies $0 \leq \langle y^*, F(\Omega) \rangle + n \langle y^*, c_0 \rangle < 0$ for sufficiently large n. This is a contradiction. Hence $y^* \in C^*$. Again from (6), as y^* is a continuous linear functional, letting $c \to \theta$ we obtain (5).

COROLLARY 3. Let $F : \mathcal{S} \to Y$ and $G : \mathcal{S} \to Z$ be C-convex and D-convex set functions, respectively. If the inequality system

$$\begin{cases} F(\Omega) \ll_C \theta \\ G(\Omega) <_D \theta \end{cases} \tag{7}$$

has no solution in $\mathcal{S}$, then there exists a nonzero vector $(y^*, z^*) \in C^* \times D^*$ such that

$$\langle y^*, F(\Omega)\rangle + \langle z^*, G(\Omega)\rangle \geq 0,\ \Omega \in S.$$

In general, the convex version of the Farkas theorem for convex set functions can be stated as follows.

THEOREM 4 [14]. Let F and G be the convex set functions in Corollary 3. Moreover suppose the constraint qualification (or Slater's type condition) of (P_0) holds, that is, there exists $\tilde{\Omega} \in \mathcal{S}$ satisfying $G(\tilde{\Omega}) \ll_D \theta$. Then the system of inequalities

$$\begin{cases} F(\Omega) \ll_C \theta \\ G(\Omega) \leq_D \theta \end{cases} \tag{8}$$

has no solution in $\mathcal{S}$ if, and only if, there exists

$$W_0 \in B^+(Z, Y) \equiv \{W \in B(Z, Y) \mid W(D) \subset C\},$$

the positive continuous linear operators from Z into Y, such that there is no $\Omega \in S$ satisfying

$$F(\Omega) + W_0(G(\Omega)) \ll_C \theta.$$

PROOF. The sufficiency part is trivial. As for the necessity part, if (8) has no solution, then the system (7) has no solution. By Corollary 3, there exists $(y^*, z^*) \notin (\theta, \theta)$ in $C^* \times D^*$ such that

$$\langle y^*, F(\Omega)\rangle + \langle z^*, G(\Omega)\rangle \geq 0 \text{ for all } \Omega \in \mathcal{S}.$$

It is easy to show that $y^* \neq \theta$, and so $\langle y^*, y\rangle > 0$ for any $y \in \operatorname{int} C (\neq \emptyset)$. Thus we can choose $y_0 \in \operatorname{int} C$ such that $\langle y^*, y\rangle = 1$. Define $W_0 : Z \to Y$ by $W_0(z) = \langle z^*, z\rangle\, y_0$. Then $W_0 \in B^+(Z, Y)$ and, for any $\Omega \in \mathcal{S}$,

$$\langle y^*, F(\Omega) + W_0(G(\Omega))\rangle = \langle y^*, F(\Omega)\rangle + \langle z^*, G(\Omega)\rangle \geq 0.$$

It follows that $F(\Omega) + W_0(G(\Omega)) \notin \operatorname{int}(-C)$ since $\langle y^*, y\rangle > 0$ for all $y \in \operatorname{int} C$. This shows that there does not exist $\Omega \in \mathcal{S}$ satisfying $F(\Omega) + W_0(G(\Omega)) \ll_C \theta$. The proof is complete.

Applying the above theorems, we obtain the saddle point optimality conditions for problem (P_0).

THEOREM 5 [14]. Let F and G be C-convex and D-convex set functions, respectively. Assume that (P_0) satisfies the constraint qualification, and Ω_0 is a weak minimal point of (P_0). Then there exists $W_0 \in B^+(Z, Y)$ such that (Ω_0, W_0) is a weak saddle point of the Lagrangian

$$L(\Omega, W) \equiv F(\Omega) + W(G(\Omega)). \tag{9}$$

COROLLARY 5. Under the assumptions of Theorem 5 and $Y = R$, Ω_0 is a minimal point of (P_0) if, and only if, there exists $z_0^* \in D^*$ such that (Ω_0, z_0^*) is a (usual) saddle point of the above Lagrangian $L(\Omega, z^*) \equiv F(\Omega) + \langle z^*, G(\Omega)\rangle$.

A generalized Lagrangian (or Fritz John Lagrangian) is defined as

$$\widetilde{L}(\Omega, y^*, z^*) = \langle y^*, G(\Omega)\rangle + \langle z^*, G(\Omega)\rangle \,, \text{ for all } \Omega \in \mathcal{S},\ y^* \in Y^*,\ z^* \in Z^*.$$

We say that a point $(\Omega_0, y_0^*, z_0^*) \in \mathcal{S} \times C^* \times D^*$ is a weak saddle point of $\widetilde{L}(\Omega, y^*, z^*)$ if it satisfies the inequalities $\widetilde{L}(\Omega_0, y_0^*, z^*) \leq \widetilde{L}(\Omega_0, y_0^*, z_0^*) \leq \widetilde{L}(\Omega, y_0^*, z_0^*)$ for all $\Omega \in \mathcal{S}$ and $z^* \in D^*$.

We have the following theorem.

THEOREM 6 [14]. Let F and G be C-convex and D-convex on a convex family $\mathcal{S} \subset \Gamma$ respectively. Suppose that Ω_0 is a weak minimal point of (P_0). Then there exists a nonzero vector $(y_0^*, z_0^*) \in C^* \times D^*$ such that $\langle z_0^*, G(\Omega_0)\rangle = 0$ and (Ω_0, y_0^*, z_0^*) is a weak saddle point of the generalized Lagrangian $L(\Omega, y^*, z^*)$.

Some special cases for $Y = Z = R$ can be deduced from Theorems 5 and 6.

4 Multiobjective Set Function and n-Fold Set Function

Let $F = (F_1, ..., F_n) : \mathcal{S} \to \mathcal{R}^n$ and $G = (G_1, ..., G_m) : S \to \mathcal{R}^m$ be multi convex set functions defined on a convex family $\mathcal{S}$ of a σ-filed Γ in a nonatomic finite measure space (X, Γ, μ). Consider a multiobjective programming problem:

(P_v) Minimize $F(\Omega) = (F_1(\Omega), ..., F_n(\Omega))$ subject to $\Omega \in \mathcal{S}$ and $G_j(\Omega) \leq 0$ for $j = 1, ..., m$.

Then the Moreau-Rockafellar type theorem for point functions can be extended to set functions as follows.

THEOREM 7 (cf. [12]). Let $F_1, F_2, ..., F_n : \Gamma \to \mathcal{R} \cup \{\infty\}$ be proper convex set functions on $S = \operatorname{Dom} F_i$, $i = 1, 2, \cdots, n$. Then

$$\partial F_1(\Omega) + ... + \partial F_n(\Omega) \subset \partial (F_1 + ... + F_n)(\Omega) \text{ for all } \Omega \in \Gamma. \tag{10}$$

If $\overline{\mathcal{S}}$ contains a relative interior point and all functions F_i, $i = 1, ..., n$, except possibly one, are w^*-continuous on $\mathcal{S}$, then (10) becomes

$$\partial F_1(\Omega) + ... + \partial F_n(\Omega) = \partial (F_1 + ... + F_n)(\Omega) \text{ for all } \Omega \in \Gamma. \tag{11}$$

By this theorem, a necessary optimality theorem (Fritz John type theorem) for (P_v) can be stated as follows.

THEOREM 8 [12, Theorem 12]. Suppose Ω_0 is a Pareto optimal solution of problem (P_v). Suppose that for each $i \in \{1, 2, ..., n\}$, there is a $\Omega_i \in \mathcal{S}$ such that

$$\begin{cases} G_k(\Omega_i) < 0,\ k = 1, ..., m \\ F_j(\Omega_i) < F_j(\Omega_0) \text{ for } j = 1, ..., , n,\ j \neq i \end{cases} \tag{12}$$

and that all $F_1, ..., F_n$ and $G_1, ..., G_m$, except possibly one, are w^*-continuous on $\mathcal{S}$. If $\overline{\mathcal{S}}$ contains a relative interior point, then there exist $\alpha \in \mathcal{R}^n$ with $\alpha_i \geq 1$, $i = 1, ..., n$ and $\lambda \in \mathcal{R}^m_+$ such that

$$\begin{cases} 0 \in \langle \alpha, \partial F(\Omega_0) \rangle_n + \langle \lambda, \partial G(\Omega_0) \rangle_m + N_{\mathcal{S}}(\Omega_0) \\ \langle \lambda, G(\Omega_0) \rangle_m = 0 \end{cases} \tag{13}$$

where $N_{\mathcal{S}}(\Omega_0) = \{f \in L_1 \mid \langle f, \chi_\Omega - \chi_{\Omega_0} \rangle \leq 0 \text{ for all } \Omega \in \mathcal{S}\}$ is the normal cone for F at $\Omega_0 \in S$ and $\langle \cdot, \cdot \rangle_n$, $\langle \cdot, \cdot \rangle_m$ denote the inner products of $\mathcal{R}^n$ and $\mathcal{R}^m$ respectively.

We remark that Theorem 8 furnishes a Kuhn-Tucker type necessary optimality condition for an optimal solution of (P_v).

COROLLARY 8. Let the assumptions of Theorem 8 hold, where (12) is replaced by

$$G_k(\Omega_i) < 0, \ k = 1, ..., m, \text{ for some } \Omega_i \in \mathcal{S}$$

that is, the Slater's condition holds for problem (P_v). Then (13) is reduced to

$$\begin{cases} 0 \in \partial F(\Omega_0) + \langle \lambda, G(\Omega_0) \rangle_m + N_{\mathcal{S}}(\Omega_0), \\ \langle \lambda, G(\Omega_0) \rangle_m = 0. \end{cases} \tag{14}$$

Next we consider the n-fold product Γ^n of a σ-algebra Γ of subsets in the set X. A pseudometric d on Γ^n is defined by

$$d(\Omega, \Lambda) = \left\{ \sum_{k=1}^{n} [\mu(\Omega_k \Delta \Lambda_k)]^2 \right\}^{\frac{1}{2}}, \ \Omega = (\Omega_1, ..., \Omega_n), \ \Lambda = (\Lambda_1, ..., \Lambda_n) \in \Gamma^n.$$

One can consider the nonlinear programming problem for n-set functions as the following

(Q_v) Minimize $F(\Omega_1, ..., \Omega_n)$ subject to $\Omega = (\Omega_1, ..., \Omega_n) \in \Gamma^n$ and $G_j(\Omega_1, ..., \Omega_n) \leq 0$ for $j = 1, ..., m$.

Then the differentiability and convexity of n-set functions can be developed (cf. Corley [7], Lin [20], Preda [24], Zalmai [30 - 31]). An n-set function $F : \Gamma^n \to \mathcal{R}$ is said to have a partial derivative at $\Omega = (\Omega_1, ...\Omega_n) \in \Gamma^n$ with respect to its k-th argument if the set function :

$$\varphi(\Omega_k) = F(\Omega_1, ..., \Omega_{k-1}, \Omega_k, \Omega_{k+1}, ..., \Omega_n)$$

has derivative $D\varphi(\Omega_k)$, and the k-th partial derivative of F at Ω is given by

$$D_k F(\Omega) = D\varphi(\Omega_k), 1 \leq k \leq n. \tag{15}$$

Denote $DF(\Omega) = (D_1 F, ..., D_p F)(\Omega)$.

We say that the n-set function $F : \Gamma^n \to \mathcal{R}$ is differentiable at $\Omega^0 \in \Gamma^n$ if there exist $DF(\Omega^0)$ and a functional $\psi : \Gamma^n \times \Gamma^n \to \mathcal{R}$ which satisfy

$$F(\Omega) = F(\Omega^0) + \sum_{k=1}^{n} \left\langle D_k F(\Omega^0), \chi_{\Omega_k} - \chi_{\Omega_k^0} \right\rangle + \psi(\Omega, \Omega^0) \tag{16}$$

with $\psi\left(\Omega,\Omega^0\right) = o\left(d\left(\Omega,\Omega^0\right)\right)$.

In order to define generalized convexity, we need to let functional $\mathcal{F} : \Gamma \times \Gamma \times L_1(X,\Gamma,\mu) \to \mathcal{R}$ be sublinear in the L_1 space, that is, for any $\Omega, \Omega_0 \in \Gamma$,

$$\mathcal{F}\left(\Omega,\Omega_0 : f_1 + f_2\right) \le \mathcal{F}\left(\Omega,\Omega_0 : f_1\right) + F\left(\Omega,\Omega_0 : f_2\right)$$

and $\mathcal{F}\left(\Omega,\Omega_0 : \alpha f\right) = \alpha\mathcal{F}\left(\Omega,\Omega_0 : f\right)$ for $f, f_1, f_2 \in L_1$ and $\alpha \ge 0$.

Let $\rho \in \mathcal{R}$ and $\theta : \Gamma \times \Gamma \to R_+ = [0,\infty)$ with the property that $\theta(\Omega_1,\Omega_2) = 0$ if and only if $\mu(\Omega_1 \Delta \Omega_2) = 0$. If the set function $F : \Gamma \to \mathcal{R}$ is subdifferentiable, we define the $(\mathcal{F},\rho,\theta)$-convexity as follows.

DEFINITION 4 (cf. [15]). Given $\Omega_0 \in \Gamma$ and $f \in \partial F(\Omega_0)$, F is called $(\mathcal{F},\rho,\theta)$-convex at Ω_0 if for all $\Omega \in \Gamma$,

$$F(\Omega) - F(\Omega_0) \ge \mathcal{F}(\Omega,\Omega_0; f) + \rho\theta\left(\Omega,\Omega_0\right);$$

F is called $(\mathcal{F},\rho,\theta)$-quasiconvex [strictly quasiconvex] at Ω_0 if for each $\Omega \in \Gamma$,

$$F(\Omega) \le F(\Omega_0) \ \text{[resp. } F(\Omega) < F(\Omega_0)] \Rightarrow \mathcal{F}(\Omega,\Omega_0; f) \le -\rho\theta\left(\Omega,\Omega_0\right);$$

F is called $(\mathcal{F},\rho,\theta)$-pseudoconvex [strictly pseudoconvex] at Ω_0 if for each $\Omega \in \Gamma$,

$$\mathcal{F}(\Omega,\Omega_0; f) \ge -\rho\theta\left(\Omega,\Omega_0\right) \Rightarrow F(\Omega) \ge F(\Omega_0) \ \text{[resp. } F(\Omega) > F(\Omega_0)].$$

If $\mathcal{F}(\Omega,\Omega_0; f) = \langle \chi_\Omega - \chi_{\Omega_0}, f\rangle$ in Definition 4, then the $(\mathcal{F},\rho,\theta)$-convexity reduces to ordinary convexity (see [15. Theorem 3.2]). The existence for optimal solutions of problem (P_v) follows from the converse of Theorem 8 by adding $(\mathcal{F},\rho,\theta)$-convexity which we state as Theorem 9. Throughout this paper, we use inner products in $\mathcal{R}^n$ and $\mathcal{R}^m$ by

$$\alpha^\top F\left(\Omega\right) = \left\langle \alpha, F\left(\Omega\right)\right\rangle_n = \sum_{i=1}^{n} \alpha_i F_i\left(\Omega\right)$$

and

$$\lambda^\top G(\Omega) = \left\langle \lambda, G\left(\Omega\right)\right\rangle_m = \sum_{j=1}^{m} \lambda_j G_j\left(\Omega\right)$$

respectively.

THEOREM 9 (Sufficient optimality condition). Let Ω_0 be a feasible solution of (P_v) and suppose there exist $\alpha \in \mathcal{R}^n$ with each component $\alpha_i \ge 1$ and $\lambda \in \mathcal{R}^m_+$ satisfying (13), and $\mathcal{F}(\Omega,\Omega_0; -h) \ge 0$ for $h \in N_{\mathcal{S}}(\Omega_0)$ and all feasible solution Ω of (P_v). Then Ω_0 is a Pareto optimal solution of (P_v) provided any one of the following conditions holds:

a. F_i is $(\mathcal{F},\rho_{1i},\theta)$-convex at Ω_0, $i = 1, ..., n$, G_j is $(\mathcal{F},\rho_{2j},\theta)$-convex at Ω_0, $j = 1, ..., m$ and $\left\langle \alpha, \rho_1\right\rangle_n + \left\langle \alpha, \rho_2\right\rangle_m \ge 0$, where $\rho_1 \in \mathcal{R}^n$ and $\rho_2 \in \mathcal{R}^m$.

b. $\alpha^\top F + \lambda^\top G$ is $(\mathcal{F},\rho,\theta)$-convex at Ω_0 and $\rho \ge 0$.

c. $\alpha^\top F + \lambda^\top G$ is strictly $(\mathcal{F},\rho,\theta)$-quasiconvex at Ω_0 and $\rho > 0$.

d. $\alpha^\top F$ is $(\mathcal{F}, \rho_1, \theta)$-pseudoconvex, $\lambda^\top G$ is $(\mathcal{F}, \rho_2, \theta)$-quasiconvex and $\rho_1 + \rho_2 \geq 0$.

e. $\alpha^\top F$ is $(\mathcal{F}, \rho_1, \theta)$-pseudoconvex, $\lambda^\top G$ is strictly $(\mathcal{F}, \rho_2, \theta)$-quasiconvex and $\rho_1 + \rho_2 \geq 0$.

f. $\alpha^\top F$ is strictly $(\mathcal{F}, \rho_1, \theta)$-quasiconvex, $\lambda^\top G$ is strictly $(\mathcal{F}, \rho_2, \theta)$-quasiconvex and $\rho_1 + \rho_2 > 0$.

PROOF. We sketch the proof as follows. By (13), there exist $f_i \in \partial F_i(\Omega_0)$, $i = 1, ..., n$, $g_j \in \partial G_j(\Omega_0)$, $j = 1, ..., m$ and $h \in N_{\mathcal{S}}(\Omega_0)$ which satisfy $\langle \alpha, f \rangle_n + \langle \lambda, g \rangle_m + h = \theta$ (in L_1), and so

$$\mathcal{F}(\Omega, \Omega_0; \langle \alpha, f \rangle_n + \langle \lambda, g \rangle_m + h) = 0. \tag{17}$$

If Ω_0 is not a Pareto minimum of (P_v), then there exists $\Omega_1 \in \mathcal{F}_{P_v}$ such that $F_i(\Omega_1) \leq F_i(\Omega_0)$ for $i = 1, ..., n$, and $F_k(\Omega_1) < F_k(\Omega_0)$ for $k \neq i$, and so $\langle \alpha, F \rangle (\Omega_1) < \langle \alpha, F \rangle (\Omega_0)$. On the other hand, $\langle \lambda, G(\Omega_1) \rangle_m \leq 0 = \langle \lambda, G(\Omega_0) \rangle_m$. From the above inequalities, one can get

$$\langle \alpha, F \rangle_n (\Omega_1) + \langle \lambda, G \rangle_m (\Omega_1) < \langle \alpha, F \rangle_n (\Omega_0) + \langle \lambda, G \rangle_m (\Omega_0). \tag{18}$$

If condition (a) holds, then by $(\mathcal{F}, \rho, \theta)$-convexity, we have

$$F_i(\Omega_1) - F_i(\Omega_0) \geq \mathcal{F}(\Omega_1, \Omega_0; f_i) + \rho_{1i}\theta(\Omega_1, \Omega_0), \ i = 1, ..., n, \tag{19}$$

$$G_j(\Omega_1) - G_j(\Omega_0) \geq \mathcal{F}(\Omega_1, \Omega_0; g_j) + \rho_{2i}\theta(\Omega_1, \Omega_0), \ j = 1, ..., m. \tag{20}$$

Multiplying (19) by α_i, (20) by λ_j, and summing up the resulting inequalities, we have

$$\begin{aligned} & \langle \alpha, F \rangle_n (\Omega_1) + \langle \lambda, G \rangle_m (\Omega_1) \\ \geq \ & \langle \alpha, F \rangle_n (\Omega_0) + \langle \lambda, G \rangle_m (\Omega_0) + \mathcal{F}(\Omega_1, \Omega_0; \langle \alpha, f \rangle_n + \langle \lambda, g \rangle_m) \\ & + (\langle \alpha, \rho_1 \rangle_n + \langle \alpha, \rho_2 \rangle_m)\,\theta(\Omega_1, \Omega_0). \end{aligned}$$

From (17), (18) and $\mathcal{F}(\Omega_1, \Omega_0; -h) \geq 0$, we obtain $0 > (\langle \alpha, \rho_1 \rangle_n + \langle \lambda, \rho_2 \rangle_m)\theta(\Omega_1, \Omega_0)$. Since $\theta(\Omega_1, \Omega_0) > 0$, $\langle \alpha, \rho_1 \rangle_n + \langle \lambda, \rho_2 \rangle_m < 0$. This contradicts the assumption in (a).

If condition (b) holds, the inequality (18) yields

$$\mathcal{F}(\Omega_1, \Omega_0; \langle \alpha, f \rangle_n + \langle \lambda, g \rangle_m) < -\rho\theta(\Omega_1, \Omega_0). \tag{21}$$

By (18), (21) and $\mathcal{F}(\Omega_1, \Omega_0; -h) \geq 0$, we have $\rho\theta(\Omega_1, \Omega_0) < 0 \Rightarrow \rho < 0$ which is a contradiction since $\rho \geq 0$.

If condition (c) holds, the same conclusion can be proved in a way similar to the previous case.

If condition (d) holds, then $\lambda^\top G$ is $(\mathcal{F}, \rho_2, \theta)$-quasiconvex, and so

$$\langle \lambda, G(\Omega_1) \rangle_m \leq 0 = \langle \lambda, G(\Omega_0) \rangle_m \Rightarrow \mathcal{F}(\Omega_1, \Omega_0; \langle \lambda, g \rangle_m) \leq -\rho_2\theta(\Omega_1, \Omega_0).$$

By (17) and the sublinearity of $\mathcal{F}$, we have

$$\mathcal{F}(\Omega_1, \Omega_0; \langle \alpha, f \rangle_n) + \mathcal{F}(\Omega_1, \Omega_0; \langle \lambda, g \rangle_m) + \mathcal{F}(\Omega_1, \Omega_0; h) \geq 0.$$

It follows that

$$\mathcal{F}(\Omega_1,\Omega_0;\langle\alpha,f\rangle_n)+(-\rho_2\theta(\Omega_1,\Omega_0))+\mathcal{F}(\Omega_1,\Omega_0;h)\geq 0.$$

Since $\langle\alpha,F\rangle_n$ is $(\mathcal{F},\rho_1,\theta)$-pseudoconvex, the above inequality and $\mathcal{F}(\Omega_1,\Omega_0;\langle\alpha,f\rangle_n)<-\rho_1\theta(\Omega_1,\Omega_0)$ then imply

$$-\rho_1\theta(\Omega_1,\Omega_0)-\rho_2\theta(\Omega_1,\Omega_0)+\mathcal{F}(\Omega_1,\Omega_0;h)>0.$$

Since $\mathcal{F}(\Omega_1,\Omega_0;-h)\geq 0$, it follows that $(\rho_1+\rho_2)\,\theta(\Omega_1,\Omega_0)<0$. This contradicts the fact that $\rho_1+\rho_2\geq 0$.

The other two cases are similarly proved. The proof is complete.

We remark that necessary optimality conditions for problem (Q_v) can be obtained as in Corley [7] (cf. also Zalmai [30]), while sufficient optimality conditions as in Theorem 9.

5 Fractional Programming Involving Set Functions

Suppose that $F=(F_1,...,F_n)$, $-G=(-G_1,...,-G_n):\mathcal{S}\to\mathcal{R}^n$ and $H=(H_1,\cdots,H_m):\mathcal{S}\to\mathcal{R}^m$ are convex set functions, and that all set functions are subdifferentiable. Then consider a minimax fractional programming problem as follows

(P) Minimize

$$\varphi(\Omega)\equiv\max_{1\leq i\leq n}\frac{F_i(\Omega)}{G_i(\Omega)}$$

subject to $\Omega\in\mathcal{S}$ and $H_j(\Omega)\leq 0$ for $1\leq j\leq m$,

where $\mathcal{S}$ is a convex family of Γ with $\overline{\mathcal{S}}$ containing a relative interior point; $G_i(\Omega)>0$ and $F_i(\Omega)\geq 0$ for $1\leq i\leq n$. We denote the set of all feasible solutions of (P) by $\mathcal{F}_P$. Let $\lambda=F_i(\Omega)/G_i(\Omega)$, $1\leq i\leq m$. The fractional programming problem (P) can be turned into a nonfractional parametric problem

(EP) Minimize λ subject to $F_i(\Omega)-\lambda G_i(\Omega)\leq 0$ for $1\leq i\leq n$ and $H_j(\Omega)\leq 0$ for $1\leq j\leq m$ with $\Omega\in\mathcal{S}$.

The problems (P) and (EP) are equivalent (cf. Zalmai [30]).

LEMMA 10. Let Ω^* be an optimal solution of (P). Then

$$\lambda^*=\max_{1\leq i\leq n}F_i(\Omega^*)/G_i(\Omega^*)$$

as well as Ω^* together constitute an optimal solution of (EP). Conversely, if (Ω^0,λ^0) is an optimal solution of (EP), then Ω^0 is an optimal solution of (P).

It can be shown that for each feasible solution $\Omega\in\mathcal{S}$ of (P),

$$\varphi(\Omega)\equiv\max_{1\leq i\leq n}\frac{F_i(\Omega)}{G_i(\Omega)}=\max_{u\in I}\frac{\langle u,F(\Omega)\rangle}{\langle u,G(\Omega)\rangle}\tag{22}$$

where

$$I=\left\{u\in\mathcal{R}_{+}^{n}\mid\sum_{i=1}^{n}u_i=1\right\}.$$

Similar to Theorem 8, we can state the necessary optimality theorem for (P) (cf. [17]) as follows.

THEOREM 11 [17, Theorem 3.1]. Suppose that F, $-G$ and H are proper convex vector set functions in problem (P), and that Ω^* is an optimal solution of (P) with optimal value λ^*. If the Slater's condition holds for (EP), then there exist $u^*\in I\subset\mathcal{R}_{+}^{n}$ and $z^*\in\mathcal{R}_{+}^{m}$ such that $(\Omega^*,\lambda^*,u^*,z^*)$ satisfies the Euler-Lagrange type conditions:

$$0\in\partial\left({u^*}^{\top}F\right)(\Omega^*)+\lambda^*\partial\left(-{u^*}^{\top}G\right)(\Omega^*)+\partial\left({z^*}^{\top}H\right)(\Omega^*)+N_{\mathcal{S}}(\Omega^*),\tag{23}$$

$${u^*}^{\top}\left[F(\Omega^*)-\lambda^*\partial G(\Omega^*)\right]=0,\tag{24}$$

$${z^*}^{\top}H(\Omega^*)=0.\tag{25}$$

We say that a solution Ω^* of (P) is regular if there does not exist any $h\in\partial\left({z^*}^{\top}H\right)(\Omega)$ for $z\in\mathcal{R}_{+}^{m}$ or any $\eta\in N_S(\Omega^*)$ such that $h+\eta=0$.

In order to construct parameter-free duality theorems for problem (P), we replace the optimal value λ^* of (P) by

$$\max_{1\le i\le n}\frac{F_i(\Omega^*)}{G_i(\Omega^*)}=\max_{u\in I}\frac{\langle u,F(\Omega^*)\rangle}{\langle u,G(\Omega*)\rangle}\equiv\varphi(\Omega^*)=\lambda^*\tag{26}$$

(see (22)). Then (23) yields the following corollary.

COROLLARY 11. The results of Theorem 11 can be restated as

$$\begin{aligned}0\in\;&{u^*}^{\top}G(\Omega^*)\left[\partial\left({u^*}^{\top}F\right)(\Omega^*)+\partial\left({u^*}^{\top}H\right)(\Omega^*)\right]\\&+{u^*}^{\top}F(\Omega^*)\,\partial\left(-{u^*}^{\top}G\right)(\Omega^*)+N_S(\Omega^*)\end{aligned}\tag{27}$$

where

$$u^*\in I=\left\{u\in\mathcal{R}_{+}^{m}\mid\sum_{i=1}^{m}u_i=1\right\}.$$

The existence of optimal solution for (P) can be obtained from the results of necessary optimality conditions by adding extra assumptions. We use $(\mathcal{F},\rho,\theta)$-convexity defined in Definition 4 to establish the following sufficient optimality conditions (cf. [17]).

In order to simplify the notations, for a given $\Omega^*\in\mathcal{F}_P$, let

$$A(\Omega)={u^*}^{\top}G(\Omega^*)\,{u^*}^{\top}F(\Omega)-{u^*}^{\top}F(\Omega^*)\,{u^*}^{\top}G(\Omega),$$

$$B(\Omega)={z^*}^{\top}H(\Omega),$$

$$C(\Omega)=A(\Omega)+{u^*}^{\top}G(\Omega^*)\,B(\Omega)$$

for $\Omega \in \mathcal{F}_P$. Then the sufficient optimality theorem can be stated as Theorem 12.

THEOREM 12 (Sufficient optimality conditions). Let $\Omega^* \in \mathcal{F}_P$ be a feasible solution and suppose there exist $u^* \in I \subset \mathcal{R}_+^n$ and $z^* \in \mathcal{R}_+^m$ satisfying (25), (26), (27) and $\mathcal{F}(\Omega, \Omega^*, -\eta) \geq 0$ for each $\eta \in N_S(\Omega^*)$ and $\Omega \in \mathcal{F}_P$. Then Ω^* is an optimal solution of (P) provided any one of the following conditions holds:

a. at Ω^*, ${u^*}^\top F$ is $(\mathcal{F}, \rho_1, \theta)$-convex, $-{u^*}^\top G$ is $(\mathcal{F}, \rho_2, \theta)$-convex, ${z^*}^\top H$ is $(\mathcal{F}, \rho_3, \theta)$-convex and

$$ {u^*}^\top G(\Omega^*)\,\rho_1 + {u^*}^\top F(\Omega^*)\,\rho_2 + {u^*}^\top G(\Omega^*)\,\rho_3 \geq 0; $$

b. A is $(\mathcal{F}, \rho_1, \theta)$-pseudoconvex, B is $(\mathcal{F}, \rho_2, \theta)$-quasiconvex, and

$$ \rho_1 + {u^*}^\top G(\Omega^*)\,\rho_2 \geq 0; $$

c. A is $(\mathcal{F}, \rho_1, \theta)$-quasiconvex, B is strictly $(\mathcal{F}, \rho_2, \theta)$-pseudoconvex, and

$$ \rho_1 + {u^*}^\top G(\Omega^*)\,\rho_2 > 0; $$

d. A is strictly $(\mathcal{F}, \rho_1, \theta)$-quasiconvex, B is $(\mathcal{F}, \rho_2, \theta)$-quasiconvex and

$$ \rho_1 + {u^*}^\top G(\Omega^*)\,\rho_2 > 0; $$

e. C is $(\mathcal{F}, \rho, \theta)$-pseudoconvex at Ω^* and $\rho \geq 0$;

f. C is strictly $(\mathcal{F}, \rho, \theta)$-quasiconvex and $\rho > 0$.

The proof can be carried out by arguments similar to those in the proof of Theorem 9.

6 Duality Theorems

Applying Theorems 11 and 12 for (P), we can construct three dual models including two parameter-free dual problems and a parametric dual problem with respect to the primary problem (P).

There are at least three main results that need to be shown for the duality problem (D). The first theorem is to show that the minimum value of (P) is greater than or equal to the maximum value of (D), that is $\max(D) \leq \min(P)$. The second theorem is to show that the optimal solution of (P) yields the optimal solution of (D) under appropriate conditions, and their optimal values are equal, that is, $\min(P) = \max(D)$. The third theorem is to show that if Ω and Ω^* are respectively the optimal solutions of (P) and (D), then $\Omega = \Omega^*$ and their optimal values coincide.

Now we construct two parameter-free dual models denoted respectively by $(D1)$ and $(D2)$ as follows:

$(D1)$ Maximize

$$\frac{u^\top F(\Lambda)+z^\top H(\Lambda)}{u^\top G(\Lambda)}$$

subject to $\Lambda \in \mathcal{S}$ and

$$\begin{aligned} 0 \quad \in \quad & u^\top G(\Lambda)\left[\partial\left(u^\top F\right)(\Lambda)+\partial\left(z^\top H\right)(\Lambda)\right] \\ & -\left[u^\top F(\Lambda)+z^\top H(\Lambda)\right]\partial\left(u^\top G\right)(\Lambda)+N_{\mathcal{S}}(\Lambda) \end{aligned}$$

where

$$u^\top F(\Lambda)+z^\top H(\Lambda)\geq 0 \text{ and } u^\top G(\Lambda)>0, \text{ and } u\in I\subset \mathcal{R}_+^n,\ z\in\mathcal{R}_+^m,$$

and

$(D2)$ Maximize

$$\frac{u^\top F(\Lambda)}{u^\top G(\Lambda)}$$

subject to $\Lambda \in \mathcal{S}$ and

$$0\in u^\top G(\Lambda)\left[\partial\left(u^\top F\right)(\Lambda)+\partial\left(z^\top H\right)(\Lambda)\right]-u^\top F(\Lambda)\,\partial\left(u^\top G\right)(\Lambda)+N_S(\Lambda)$$

with $z^\top H(\Lambda)\geq 0$, where $u\in I\subset\mathcal{R}_+^n,\ z\in\mathcal{R}_+^m,\ u^\top F(\Lambda)\geq 0$ and $u^\top G(\Lambda)>0$.

Another parametric dual model, denoted by $(D3)$, is as follows:

$(D3)$ Maximize $\lambda\,(\in\mathcal{R}_+)$ subject to $\Lambda\in\mathcal{S}$,

$$0\in\partial\left(u^\top F\right)(\Lambda)-\lambda\partial\left(u^\top G\right)(\Lambda)+\partial\left(z^\top H\right)(\Lambda)+N_S(\Lambda),$$

$$u^\top F(\Lambda)-\lambda u^\top G(\Lambda)\geq 0 \text{ and } z^\top H(\Lambda)\geq 0 \text{ where } u\in I\subset\mathcal{R}_+^n, z\in\mathcal{R}_+^m.$$

We denote the feasible solutions of $(D1)$, $(D2)$ and $(D3)$ by K_1, K_2 and K_3, respectively.

To each dual problem, we state the weak duality, strong duality, and strict duality theorems. (see [17], cf. also [15, 16]). First we handle the dual problem $(D1)$.

THEOREM 13 (Weak duality) (cf. [17, Theorem 4.1]). Let $\Omega\in\mathcal{F}_P$, $(\Lambda,u,z)\in K_1$ and set

$$D(\cdot)=u^\top G(\Omega)\left[u^\top F(\cdot)+z^\top H(\cdot)\right]-u^\top G(\cdot)\left[u^\top F(\Omega)+z^\top H(\Omega)\right].$$

Suppose $\mathcal{F}(\Omega,\Lambda,-\eta)\geq 0$ for each $\eta\in N_{\mathcal{S}}(\Lambda)$, and any one of the following conditions holds:

(1) $u^\top F$ is $(\mathcal{F},\rho_1,\theta)$-convex, $-u^\top G$ is $(\mathcal{F},\rho_2,\theta)$-convex, $z^\top H$ is $(\mathcal{F},\rho_3,\theta)$-convex and $u^\top G(\Lambda)\rho_1+\left[u^\top F(\Lambda)+z^\top H(\Lambda)\right]\rho_2+u^\top G(\Lambda)\rho_3\geq 0$;

(2) D is (F,ρ,θ)-pseudoconvex and $\rho\geq 0$;

(3) D is strictly (F, ρ, θ)-quasiconvex and $\rho > 0$.

Then

$$\varphi(\Omega) \equiv \max_{1 \le i \le n} \frac{F_i(\Omega)}{G_i(\Omega)} \ge \frac{u^\top F(\Lambda) + z^\top H(\Lambda)}{u^\top G(\Lambda)},$$

where $\varphi(\Omega)$ is the objective function of (P).

THEOREM 14 (Strong duality) [17, Theorem 4.2]. Suppose the assumptions of Theorem 13 hold. Assume further that the vector set functions F, $-G$ and H are convex (equivalently, $(\mathcal{F}, \rho, \theta)$-convex with $\mathcal{F}(\Omega, \Lambda; f) = \langle \chi_\Omega - \chi_\Lambda, f \rangle, f \in L^1$). If Ω^* is an optimal solution of (P) and u^*, z^* are as in Theorem 11, then (Ω^*, u^*, z^*) is an optimal solution of $(D1)$ and $\min(P) = \max(D1)$.

THEOREM 15 (Strict converse duality) [17, Theorem 4.3]. Let Ω_0 and (Λ, u_0, z_0) be optimal solutions of (P) and $(D1)$ respectively, and suppose the assumptions of Theorem 14 are fulfilled. If the set function

$$D(\cdot) = u_0^\top G(\Omega_0)\left[u_0^\top F(\cdot) + z_0^\top H(\cdot)\right] - u_0^\top G(\cdot)\left[u_0^\top F(\Omega_0) + z_0^\top H(\Omega_0)\right]$$

is strictly $(\mathcal{F}, \rho, \theta)$-pseudoconvex with $\rho \ge 0$, then $\Lambda = \Omega_0$ is an optimal solution of (P) and they have the same optimal values:

$$\varphi(\Omega_0) = \frac{u_0^\top F(\Omega_0)}{u_0^\top G(\Omega_0)} = \max_{1 \le i \le n} \frac{F_i(\Omega_0)}{G_i(\Omega_0)} = \frac{u_0^\top F(\Omega_0) + z_0^\top H(\Omega_0)}{u_0^\top G(\Omega_0)}$$

where $\varphi(\Omega)$ is the objective function of (P). That is, $\min(P) = \max(D1)$.

For $(D2)$, there are also three duality theorems similar to Theorems 13–15 which we state as follows.

THEOREM 16 (Weak duality) (cf. [17, Theorem 5.1]). Let $\Omega \in \mathcal{F}_P$, $(\Lambda, u, z) \in K_2$. Set $E(\cdot) \equiv u^\top G(\Lambda) u^\top F(\cdot) - u^\top F(\Lambda) u^\top G(\cdot)$, $L(\cdot) \equiv z^\top H(\cdot)$ and $J(\cdot) \equiv E(\cdot) + u^\top G(\Lambda) L(\cdot)$. Suppose $\mathcal{F}(\Omega, \Lambda, -\eta) \ge 0$ for each $\eta \in N_{\mathcal{S}}(\Lambda)$. Moreover suppose any one of the following conditions holds:

(1) $u^\top F$ is $(\mathcal{F}, \rho_1, \theta)$-convex, $-u^\top G$ is $(\mathcal{F}, \rho_2, \theta)$-convex, $z^\top H$ is $(\mathcal{F}, \rho_3, \theta)$-convex, and

$$u^\top G(\Lambda)\rho_1 + u^\top F(\Lambda)\rho_2 + u^\top G(\Lambda)\rho_3 \ge 0.$$

(2) E is $(\mathcal{F}, \rho_1, \theta)$-pseudoconvex, L is $(\mathcal{F}, \rho_2, \theta)$-quasiconvex, and

$$\rho_1 + u^\top G(\Lambda)\rho_2 \ge 0.$$

(3) E is $(\mathcal{F}, \rho_1, \theta)$-quasiconvex, L is strictly $(\mathcal{F}, \rho_2, \theta)$-pseudoconvex, and

$$\rho_1 + u^\top G(\Lambda)\rho_2 \ge 0.$$

(4) E is strictly $(\mathcal{F}, \rho_1, \theta)$-quasiconvex, L is $(\mathcal{F}, \rho_2, \theta)$-quasiconvex, and

$$\rho_1 + u^\top G(\Lambda)\rho_2 > 0.$$

(5) J is $(\mathcal{F},\rho,\theta)$-pseudoconvex, and $\rho \geq 0$.

(6) J is strictly $(\mathcal{F},\rho,\theta)$-quasiconvex and $\rho > 0$.

Then

$$\varphi(\Omega) \geq \frac{u^\top F(\Lambda)}{u^\top G(\Lambda)}.$$

THEOREM 17 (Strong duality) (cf. [17, Theorem 5.2]). Suppose the assumptions of Theorem 16 hold. Suppose further that the vector set functions F, $-G$ and H are convex. If Ω^* is an optimal solution of (P), then there exist $u^* \in I \subset \mathcal{R}^n_+$ and $z^* \in \mathcal{R}^m_+$ such that $(\Omega^*, u^*, z^*) \in K_2$. Furthermore if conditions of Theorem 16 hold for all feasible solutions of $(D2)$, then (Ω^*, u^*, z^*) is an optimal solution of $(D2)$ and $\min(P) = \max(D2)$.

THEOREM 18 (Strict converse duality) (cf. [17, Theorem 5.3]). Let Ω_0 and (Λ, u_0, z_0) be optimal solution of (P) and $(D2)$, respectively. Assume that the assumptions of Theorem 17 are fulfilled, and that $E(\cdot) = u_0^\top G(\Lambda) u_0^\top F(\cdot) - u_0^\top F(\Lambda) u_0^\top G(\cdot)$ is strictly $(\mathcal{F},\rho_1,\theta)$-pseudoconvex, $L(\cdot) = z_0^\top H(\cdot)$ is$(\mathcal{F},\rho_2,\theta)$-quasiconvex and $\rho_1 + \rho_2 \geq 0$. Then $\Lambda = \Omega_0$ and $\min(P) = \max(D2)$, that is

$$\varphi(\Omega_1) = \frac{u_0^\top F(\Lambda)}{u_0^\top G(\Lambda)}.$$

For the parametric dual model $(D3)$, the following three theorems have been established.

THEOREM 19 (Weak duality) (cf. [17, Theorem 6.1]). Let $\Omega \in \mathcal{F}_P$ and $(\Lambda, u, z, \lambda) \in K_3$. Set $Q(\cdot) = u^\top F(\cdot) - \lambda u^\top G(\cdot)$, $L(\cdot) = z^\top H(\cdot)$ and, $M(\cdot) = Q(\cdot) + L(\cdot)$. Suppose any one of the following conditions holds:

(1) $u^\top F$ is $(\mathcal{F},\rho_1,\theta)$-convex, $-u^\top G$ is $(\mathcal{F},\rho_2,\theta)$-convex, $z^\top H$ is $(\mathcal{F},\rho_3,\theta)$-convex, and $\rho_1 + \lambda\rho_2 + \rho_3 \geq 0$.

(2) Q is $(\mathcal{F},\rho_1,\theta)$-pseudoconvex, L is $(\mathcal{F},\rho_2,\theta)$-quasiconvex, and $\rho_1 + \rho_2 \geq 0$.

(3) Q is $(\mathcal{F},\rho_1,\theta)$-quasiconvex, L is strictly $(\mathcal{F},\rho_2,\theta)$-pseudoconvex, and $\rho_1 + \rho_2 \geq 0$.

(4) Q is strictly $(\mathcal{F},\rho_1,\theta)$-quasiconvex, L is $(\mathcal{F},\rho_2,\theta)$-quasiconvex, and $\rho_1 + \rho_2 > 0$.

(5) M is $(\mathcal{F},\rho,\theta)$-pseudoconvex and $\rho \geq 0$.

(6) M is strictly $(\mathcal{F},\rho,\theta)$-quasiconvex and $\rho > 0$.

Then $\varphi(\Omega) \geq \lambda$.

THEOREM 20 (Strong duality) (cf. [17, Theorem 6.2]). Let the vector set functions F, $-G$ and H be convex in $\mathcal{S}$. If Ω^* is an optimal solution of (P) satisfying conditions (22)-(25) in Theorem 11, then $(\Omega^*, u^*, z^*, \lambda^*) \in K_3$, a feasible solution of $(D3)$. Furthermore if the conditions of Theorem 19 hold for all feasible solutions of $(D3)$, then $(\Omega^*, u^*, z^*, \lambda^*)$ is an optimal solution of $(D3)$ and $\min(P) = \max(D3)$.

THEOREM 21 (Strict converse duality) (cf. [17, Theorem 6.3]). Let Ω_0 and $(\Lambda, u_0, z_0, \lambda_0)$ be optimal solutions of (P) and $(D3)$ respectively. Assume that the assumptions of Theorem 20 are fulfilled. Furthermore if $u_0 F(\cdot) - \lambda_0 u_0^\top G(\cdot)$ is strictly $(\mathcal{F}, \rho, \theta)$-pseudoconvex, $L(\cdot) = z_0^\top H(\cdot)$ is $(\mathcal{F}, \rho_2, \theta)$-quasiconvex, and $\rho_1 + \rho_2 \geq 0$. Then $\Lambda = \Omega_0$ and $\varphi(\Omega_0) = \lambda_0$, that is, $\min(P) = \max(D3)$.

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