Applied Mathematics E-Notes, 2(2002), 147-154 © ISSN 1607-2510
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Common Solutions of a Pair of Matrix Equations *

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Received 21 October 2001

Abstract

Solvability condition and common solution of the pair of linear matrix equations $AX + XB = M$ and $AXB = C$ are determined by making use of ranks and generalized inverses of matrices. Some of their applications to generalized inverses of matrices are also presented.

1 Introduction

We consider in this article common solutions of the pair of simultaneous matrix equations

$$\begin{aligned} AX + XB &= M, \\ AXB &= C, \end{aligned} \tag{1}$$

and present some of their applications to generalized inverses of matrices. The first equation in (1) is called the Sylvester equation in the literature and is widely studied, see [3] and the references therein for its history and applications. The second equation in (1) is also well known in the literature, see [1, 2, 9].

A direct motivation for us to consider the common solutions of the pair of matrix equations in (1) arises from characterizing various commutativity for generalized inverses of matrices, such as, $AA^- = A^-A$, $A^kA^- = A^-A^k$, and $A^DA^- = A^-A^D$, $BAA^- = A^-AB$ and so on, as well as factorizations of matrix with the form $M = AA^- - A^-A$ or $M = A^kA^- - A^-A^k$ and so on. Note that generalized inverse (inner inverse) A^- is a solution to the matrix equation $AXA = A$. Hence the equalities mentioned above can be regarded as special cases of (1).

Throughout, $\mathbf{C}$ denotes the field of complex numbers. $\mathcal{R}(A)$, $r(A)$, A^* and A^- as usual denote the range (column space), the rank, the conjugate transpose, and a generalized inverse of matrix A, respectively. Moreover, we denote $E_A = I - AA^-$ and $F_A = I - A^-A$ for any A^-.

The following rank formulas are due to Marsaglia and Styan [6, Theorem 19].

LEMMA 1.1. Let $A \in \mathbf{C}^{m\times n}$, $B\in \mathbf{C}^{m\times k}$ and $C \in \mathbf{C}^{l\times n}$ be given. Then
(a) $r[\, A, B\,] = r(A) + r(E_AB) = r(B) + r(E_BA)$.
(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C)$.

*Mathematics Subject Classifications: 15A09, 15A24, 15A27.

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(c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C)$.

From Lemma 1.1(c), we obtain

$$r\begin{bmatrix} A & BF_{B_1} \\ E_{C_1}C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & C_1 \\ 0 & B_1 & 0 \end{bmatrix} - r(B_1) - r(C_1). \tag{2}$$

Lemma 1.1 and (2) are quite useful in simplifying various rank equalities involving generalized inverses of matrices.

The following result on the matrix equation $AXB = C$ is also well known, see [1, 2, 9].

LEMMA 1.2. The following five statements are equivalent:

(a) The matrix equation $AXB = C$ is consistent.

(b) $AA^-C = C$ and $CB^-B = C$.

(c) $AA^-CB^-B = C$.

(d) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$.

(e) $r[\,A, C\,] = r(A)$ and $r\begin{bmatrix} B \\ C \end{bmatrix} = r(B)$.

In case one of the five statements in Lemma 1.2 holds, the general solution of $AXB = C$ can be expressed in the form $X = A^-CB^- + V - A^-AVBB^-$, or $X = A^-CB^- + F_AV_1 + V_2E_B$, where V, V_1 and V_2 are arbitrary matrices.

LEMMA 1.3. Let $A \in \mathbf{C}^{m\times p}, B \in \mathbf{C}^{q\times n}, C \in \mathbf{C}^{m\times r}, D \in \mathbf{C}^{s\times n}$ and $N \in \mathbf{C}^{m\times n}$ be given. Then

(a) The matrix equation

$$AXB + CYD = N \tag{3}$$

is solvable if and only if the following four rank equalities hold [8]

$$r[\,A, C, N\,] = r[\,A, C\,],\ \ r\begin{bmatrix} B \\ D \\ N \end{bmatrix} = r\begin{bmatrix} B \\ D \end{bmatrix}, \tag{4}$$

$$r\begin{bmatrix} N & A \\ D & 0 \end{bmatrix} = r(A) + r(D),\ \ r\begin{bmatrix} N & C \\ B & 0 \end{bmatrix} = r(B) + r(C). \tag{5}$$

(b) In case (3) is solvable, the general solution of Eq.(3) can be expressed in the form [11, 12]

$$X = X_0 + X_1X_2 + X_3 \text{ and } Y = Y_0 + Y_1Y_2 + Y_3, \tag{6}$$

where X_0 and Y_0 are two special solutions of Eq.(3), X_1, X_2, X_3 and Y_1, Y_2, Y_3 are the general solutions of the following four homogeneous matrix equations

$$\begin{aligned} &AX_1 - CY_1 = 0, \\ &X_2B + Y_2D = 0, \\ &AX_3B = 0, \\ &CY_3D = 0, \end{aligned} \tag{7}$$

or explicitly

$$X = X_0 + [\, I_p, 0\,] F_G U E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} + F_A V_1 + V_2 E_B, \tag{8}$$

$$Y = Y_0 + [\, 0, I_r\,] F_G U E_H \begin{bmatrix} 0 \\ I_s \end{bmatrix} + F_C W_1 + W_2 E_D, \tag{9}$$

where X_0 and Y_0 are two special solutions of Eq.(3), $G = [\, A, -C\,]$, $H = \begin{bmatrix} B \\ D \end{bmatrix}$, U, V_1, V_2, W_1 and W_2 are arbitrary.

Some expressions of special solutions of (3) were given in [1] and [8]. But we only need (8) and (9).

2 Main Results

Our first main result is as follows.

THEOREM 2.1. Let $A \in \mathbf{C}^{m \times m}, B \in \mathbf{C}^{n \times n}$ and $C,\ M \in \mathbf{C}^{m \times n}$ be given. Then

(a) The matrix equations

$$AX + XB = M, \ \ AXB = C \tag{10}$$

have a common solution X if and only if A, B, C, M satisfy the following six conditions

$$\mathcal{R}(C) \subseteq \mathcal{R}(A), \ \ \mathcal{R}(C^*) \subseteq \mathcal{R}(B^*), \ \ r \begin{bmatrix} M & A \\ B & 0 \end{bmatrix} = r(A) + r(B), \tag{11}$$

$$AC + CB = AMB, \ \ \mathcal{R}(\, C - AM\,) \subseteq \mathcal{R}(A^2), \ \ \mathcal{R}[(\, C - MB\,)^*] \subseteq \mathcal{R}[(B^2)^*]. \tag{12}$$

(b) In case (11) and (12) hold, the general common solution of (10) can be expressed in the form

$$X = X_0 + [\, F_A, 0\,] F_G U E_H \begin{bmatrix} I_n \\ 0 \end{bmatrix} + [\, 0, I_m\,] F_G U E_H \begin{bmatrix} 0 \\ E_B \end{bmatrix} + F_A S E_B, \tag{13}$$

where X_0 is a special solution of Eq.(10), $G = [\, F_A, -A\,]$, $H = \begin{bmatrix} B \\ E_B \end{bmatrix}$, U and S are arbitrary.

(c) The equations in (10) have a unique common solution if and only if A and B are nonsingular and $AC + CB = AMB$. In this case, the unique common solution is $X = A^{-1}CB^{-1}$.

PROOF. Suppose first that (10) has a common solution. This implies that $AX + YB = C$ and $AXB = C$ are solvable respectively. Thus (11) follows directly from Lemmas 1.2 and 1.3. Pre- and post-multiplying A and B of the both sides of $AX + XB = M$, respectively, yield $A^2X = AM - C$ and $XB^2 = MB - C$, which imply the two range inclusions in (12). Consequently, pre- and post-multiplying A and B on the both sides of $AX + XB = M$ produces the first equality in (12).

We next show that under (11) and (12), the two equations in (10) has a common solution and their general common solution can be written as (13). By Lemma 1.2, the general solution of $AXB = C$ under (11) is

$$X = A^-CB^- + F_AV_1 + V_2E_B, \tag{14}$$

where V_1 and V_2 are arbitrary. Substituting it into $AX + XB = M$ yields

$$AV_2E_B + F_AV_1B = M - CB^- - A^-C. \tag{15}$$

By Lemma 1.3, this equation is solvable if and only if it satisfies the following four rank equalities

$$r[\, A, F_A, N\,] = r[\, A, F_A\,], \ r\begin{bmatrix} B \\ E_B \\ N \end{bmatrix} = r\begin{bmatrix} B \\ E_B \end{bmatrix}, \tag{16}$$

and

$$r\begin{bmatrix} N & A \\ B & 0 \end{bmatrix} = r(A) + r(B), \ r\begin{bmatrix} N & F_A \\ E_B & 0 \end{bmatrix} = r(F_A) + r(E_B), \tag{17}$$

where $N = M - CB^- - A^-C$. Simplifying them by Lemma 1.1 and (2), we find that

$$r[\, A, F_A, N\,] = r\begin{bmatrix} A & I_m & M - A^-C \\ 0 & A & 0 \end{bmatrix} - r(A) = r[\, A^2, C - AM\,] + m - r(A),$$

$$r[\, A, F_A\,] = r\begin{bmatrix} A & I_m \\ 0 & A \end{bmatrix} - r(A) = r(A^2) + m - r(A),$$

$$r\begin{bmatrix} B \\ E_B \\ N \end{bmatrix} = r\begin{bmatrix} B & 0 \\ I_n & B \\ M - CB^- & 0 \end{bmatrix} - r(B) = r\begin{bmatrix} B^2 \\ C - MB \end{bmatrix} + n - r(B),$$

$$r\begin{bmatrix} B \\ E_B \end{bmatrix} = r\begin{bmatrix} B & 0 \\ I_n & B \end{bmatrix} - r(B) = r(B^2) + n - r(B),$$

$$r\begin{bmatrix} N & A \\ B & 0 \end{bmatrix} = r\begin{bmatrix} M - CB^- - A^-C & A \\ B & 0 \end{bmatrix} = r\begin{bmatrix} M & A \\ B & 0 \end{bmatrix},$$

$$\begin{aligned} r\begin{bmatrix} N & F_A \\ E_B & 0 \end{bmatrix} &= r\begin{bmatrix} M - CB^- - A^-C & I_m & 0 \\ I_n & 0 & B \\ 0 & A & 0 \end{bmatrix} - r(A) - r(B) \\ &= r\begin{bmatrix} 0 & I_m & 0 \\ I_n & 0 & 0 \\ 0 & 0 & AMB - AC - CB \end{bmatrix} - r(A) - r(B) \\ &= m + n + r(AMB - AC - CB) - r(A) - r(B), \end{aligned}$$

and

$$r(F_A) + r(E_B) = m + n - r(A) - r(B).$$

Substituting them into (16) and (17) yields the results in (11) and (12). This fact implies that under (11) and (12), the equation (15) is solvable. Solving for V_1 and V_2 in (15) by Lemma 1.3, we obtain their general solutions

$$\begin{aligned} V_1 &= V_{10} + A^-AS_1 + S_2E_B \\ &\quad + [\,I_m, 0\,](\,I - [\,F_A, -A\,]^-[\,F_A, -A\,]\,)U\left(I - \begin{bmatrix} B \\ E_B \end{bmatrix}\begin{bmatrix} B \\ E_B \end{bmatrix}^-\right)\begin{bmatrix} I_n \\ 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} V_2 &= V_{20} + F_AT_1 + T_2BB^- \\ &\quad + [\,0, I_m\,](\,I - [\,F_A, -A\,]^-[\,F_A, -A\,]\,)U\left(I - \begin{bmatrix} B \\ E_B \end{bmatrix}\begin{bmatrix} B \\ E_B \end{bmatrix}^-\right)\begin{bmatrix} 0 \\ I_n \end{bmatrix}, \end{aligned}$$

where V_{10} and V_{20} are two special solutions of (15), $U,\ S_1,\ S_2,\ T_1$ and T_2 are arbitrary. Substituting them into (14) yields

$$\begin{aligned} X &= A^-CB^- + F_AV_{10} + V_{20}E_B + [F_A,\ 0\,]F_GUE_H\begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ &\quad + F_AS_2E_B + [\,0, I_m\,]F_GUE_H\begin{bmatrix} 0 \\ E_B \end{bmatrix} + F_AT_1E_B, \end{aligned}$$

which can also be written in the form of (13). The proof is complete.

Some direct consequences can be derived from the above theorem. Here are some of them.

COROLLARY 2.2. Let $A, M \in \mathbf{C}^{m\times m}$ be given. Then

(a) There is A^- such that

$$M = AA^- - A^-A \tag{18}$$

if and only if A and M satisfy the following four conditions

$$AMA = 0,\ \mathcal{R}(A + AM) \subseteq \mathcal{R}(A^2),$$

$$\mathcal{R}[(A - MA)^*] \subseteq \mathcal{R}[(A^2)^*],\ r\begin{bmatrix} M & A \\ A & 0 \end{bmatrix} = 2r(A).$$

(b) Under $r(A^2) = r(A)$, there is A^- such that Eq.(18) holds if and only if

$$AMA = 0 \text{ and } r\begin{bmatrix} M & A \\ A & 0 \end{bmatrix} = 2r(A).$$

(c) There is A^- such that

$$M = AA^- + A^-A \tag{19}$$

if and only if A and M satisfy the following four conditions

$$2A^2 = AMA,\ \mathcal{R}(\,A - AM\,) \subseteq \mathcal{R}(A^2),$$

$$\mathcal{R}[(\,A - MA\,)^*] \subseteq \mathcal{R}[(A^2)^*],\ r\begin{bmatrix} M & A \\ A & 0 \end{bmatrix} = 2r(A).$$

(d) Under $r(A^2) = r(A)$, there is A^- such that Eq.(19) holds if and only if

$$2A^2 = AMA \text{ and } r\begin{bmatrix} M & A \\ A & 0 \end{bmatrix} = 2r(A).$$

(e) There is A^- such that $AA^- = A^-A$ holds if and only if $r(A^2) = r(A)$ [13].

Indeed, applying Theorem 2.1 to the system $AX - XA = M$ and $AXA = A$ yields the results in the corollary.

An extension of Theorem 2.1 is given below. Its proof is similar to that of Theorem 2.1 and is therefore omitted.

THEOREM 2.3. Let $A \in \mathbf{C}^{m\times k}, B \in \mathbf{C}^{l\times n}, A_1 \in \mathbf{C}^{k\times k}, B_1 \in \mathbf{C}^{l\times l}, C \in \mathbf{C}^{m\times n}, M \in \mathbf{C}^{k\times l}$ and suppose that

$$\mathcal{R}(A_1^*) \subseteq \mathcal{R}(A^*) \text{ and } \mathcal{R}(B_1) \subseteq \mathcal{R}(B). \tag{20}$$

Then the following matrix equations

$$A_1X + XB_1 = M,\ AXB = C \tag{21}$$

have a common solution X if and only if the following conditions are satisfied:

$$\mathcal{R}(C) \subseteq \mathcal{R}(A),\ \mathcal{R}(C^*) \subseteq \mathcal{R}(B^*), \tag{22}$$

$$r\begin{bmatrix} M & A_1 \\ B_1 & 0 \end{bmatrix} = r(A_1) + r(B_1),\ AA_1A^-C + CB^-B_1B = AMB, \tag{23}$$

$$\mathcal{R}(\,AM - CB^-B_1\,) \subseteq \mathcal{R}(AA_1),\ \mathcal{R}[(\,MB - A_1A^-C\,)^*] \subseteq \mathcal{R}[(B_1B)^*]. \tag{24}$$

COROLLARY 2.4. Let $A,\ M \in \mathbf{C}^{m\times m}$ be given. Then there is A^- such that

$$M = A^kA^- - A^-A^k \tag{25}$$

if and only if A and M satisfy the following four conditions

$$AMA = 0,\ \mathcal{R}(\,A^k + AM\,) \subseteq \mathcal{R}(A^{k+1}),\ \mathcal{R}[(A^k - MA)^*] \subseteq \mathcal{R}[\,(A^{k+1})^*\,], \tag{26}$$

$$r\begin{bmatrix} M & A^k \\ A^k & 0 \end{bmatrix} = 2r(A^k). \tag{27}$$

In particular, there is A^- such that $A^kA^- = A^-A^k$ holds if and only if $r(A^{k+1}) = r(A^k)$ [13].

PROOF. In fact, (25) is equivalent to $A^kX - XA^k = M$ and $AXA = A$. Thus (25)–(27) follow from (20)–(24).

COROLLARY 2.5. Let $A \in \mathbf{C}^{m\times m}$ be given. Then there exists A^- such that $A^DA^- = A^-A^D$, where A^D is the Drazin inverse of A.

PROOF. It is obvious that $A^D A^- = A^- A^D$ is equivalent to

$$A^D X = X A^D \text{ and } AXA = A. \tag{28}$$

Note that $\mathcal{R}(A^D) \subseteq \mathcal{R}(A)$ and $\mathcal{R}[(A^D)^*] \subseteq \mathcal{R}(A^*)$. Then applying Theorem 2.3 to (28) yields the desired result.

COROLLARY 2.6. Let $A,\ B \in \mathbf{C}^{m\times m}$ be given. Then there is A^- such that

$$BAA^- = A^- AB \tag{29}$$

if and only if

$$r(ABA) = r(AB) = r(BA). \tag{30}$$

PROOF. The equality (29) is equivalent to the pair of matrix equations

$$BAX = XAB \quad \text{and} \quad AXA = A.$$

Thus (30) is derived from Theorem 2.3.

In particular when B is taken such that $r(ABA) = r(A)$, there exists A^- satisfying (29). In this case, this generalized inverse is called the commutative generalized inverse of A with respect to B and is denoted by A^-_B, which was examined in Khatri [5]. Note that $r(AA^*A) = r(AA^*) = r(A^*A) = r(A)$. Thus any square matrix A has a commutative generalized inverse $A^-_{A^*}$. In fact, the Moore-Penrose inverse $A^\dagger$ is a special case of the commutative generalized inverse $A^-_{A^*}$.

Acknowledgments. The author would like to thank the referee for valuable suggestions. The research of the author was supported in part by the Natural Sciences and Engineering Research Council of Canada.

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