Applied Mathematics E-Notes, 2(2002), 72-77 © ISSN 1607-2510
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A Bound for the Spectral Variation of Two Matrices *

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Received 25 September 2001

Abstract

We derive a new bound for the spectral variation of two matrices.

1 Introduction

Everywhere below, A and B are $n \times n$-matrices, $\lambda_j(A)$ and $\lambda_j(B)$, $j = 1, ..., n$, are the eigenvalues counting their multiplicities of A and B, respectively. The spectral variation of A and B is

$$v(A, B) \equiv \min_{\pi} \max_{i} |\lambda_{\pi(i)}(B) - \lambda_i(A)|,$$

where π is taken over all permutations of $\{1, 2, ..., n\}$. In addition, $\|.\|$ denotes the Euclidean norm and

$$q = \|A - B\|_2.$$

The norms for matrices here and below are understood in the sense of the operator norms.

A lot of papers and books are devoted to bounds for $v(A, B)$, cf. [1-4], [7], [9-11] and the references therein. One of the recent results is due to Bhatia *et al.* [2]. They prove that

$$v(A, B) \leq 2^{2-1/n} q^{1/n} (\|A\|_2 + \|B\|_2)^{1-1/n}. \tag{1}$$

Note that in [4] this inequality is improved in the case when both A and B are normal matrices with spectra on two intersecting lines.

In the present paper we improve inequality (1) under the condition (6) below.

The following quantity plays an essential role hereafter:

$$g(A) = (N^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2)^{1/2}.$$

Here $N^2(A)$ is the Frobenius norm: $N^2(A) = \text{Trace}(A^*A)$. The asterisk means the adjointness. As proved in [5, Corollary 1.2.7], the inequality

$$g^2(A) \leq N^2(A^* - A)/2 \tag{2}$$

*Mathematics Subject Classifications: 15A18, 15A09.

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holds. Moreover,

$$g(A) = 0 \text{ iff } A \text{ is normal: } A^*A = AA^*. \tag{3}$$

Let

$$\omega_k(A),\ k = 1, ..., m \le n$$

be *the different eigenvalues of* A: $\omega_k(A) \ne \omega_j(A)$ if $j \ne k;\ j, k = 1, ..., m$. Put

$$w(A) \equiv \min_{k \ne j;\ j,k=1,\dots,m} |\omega_k(A) - \omega_j(A)|/2.$$

So the minimum is taken over all the different eigenvalues. Consider the algebraic equation

$$z^n = q \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}} z^{n-k-1}. \tag{4}$$

It is equivalent to the equation

$$q \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}} z^{-k-1} = 1$$

Since the function in the left-hand side of this equation monotonically decreases as $z > 0$ increases, equation (4) has a unique positive root. Denote this root by $z(q, A)$.

Now we are in a position to formulate the main result of the paper.

THEOREM 1.1. For arbitrary $n \times n$-matrices A and B, the inequality

$$v(A, B) \le z(q, A) \tag{5}$$

is valid, provided

$$w(A) > z(q, A). \tag{6}$$

The proof of this theorem is presented in the next section.

COROLLARY 1.2. Let A be normal. Then

$$v(A, B) \le q \tag{7}$$

provided $w(A) > q$.

Indeed, according to (3), $z(q, A) = q$. Now the result is due to Theorem 1.1. In this corollary we do not assume that B is a normal matrix and that the spectra of A and B are on two intersecting lines.

We need the following simple result (see Lemma 4.3.2 from [5]): let z_0 be the unique positive root of the algebraic equation

$$z^n = \sum_{k=0}^{n-1} c_k z^{n-k-1},\ c_k \ge 0;\ k = 1, ..., n-1;\ c_0 > 0,$$

and let

$$P(1) = \sum_{k=0}^{n-1} c_k.$$

Then the relation

$$z_0 \le P^{1/n}(1) \ \text{ if } \ P(1) \le 1, \tag{8}$$

is valid.

Let us set

$$\delta_1(A) = q \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}}.$$

Due to (8) we have

$$z(q, A) \le [\delta_1(A)]^{1/n} \ \text{ if } \ \delta_1(A) \le 1.$$

Now Theorem 1.1 yields

COROLLARY 1.3. For arbitrary $n \times n$-matrices A and B, the inequality $v(A, B) \le [\delta_1(A)]^{1/n}$ is valid, provided $\delta_1(A) \le 1$ and $w(A) \ge [\delta_1(A)]^{1/n}$.

Substitute $z = g(A)x$ in (4). Then

$$x^n = qg^{-1}(A) \sum_{k=0}^{n-1} \frac{x^{n-k-1}}{\sqrt{k!}}.$$

Now put

$$\zeta_n(A) = \sum_{k=0}^{n-1} \frac{1}{\sqrt{k!}}.$$

By using inequality (8) once more, we see that

$$z(q, A) \le [qg^{1-n}(A)\zeta_n]^{1/n} \ \text{ if } q\zeta_n(A) \le g(A).$$

Hence, Theorem 1.1 yields

COROLLARY 1.4. For arbitrary $n \times n$-matrices A and B, the inequality

$$v(A, B) \le [qg^{1-n}(A)\zeta_n]^{1/n}$$

is valid, provided,

$$q\zeta_n(A) \le g(A) \text{ and } w(A) \ge [qg^{1-n}(A)\zeta_n]^{1/n}.$$

Theorem 1.1 and its corollaries under the corresponding restrictions are sharper than (1).

Indeed, let A be a normal matrix and $w(A) > q$. In view of (7),

$$q = \|A - B\|_2 \le \|A\|_2 + \|B\|_2$$

which improves inequality (1).

Note that due to (2), $g(A)$ in Theorem 1.1 and Corollaries 1.3 and 1.4 can be replaced by the easily calculated quantity $\sqrt{1/2}N(A^* - A)$.

2 Proof of Theorem 1.1

Let $\|.\|$ be an arbitrary norm in $\mathbf{C}^n$. Denote by $\rho(A,\lambda)$ the distance between the spectrum $\sigma(A)$ of A and a complex point λ, and suppose that the resolvent $R_\lambda(A) \equiv (A-\lambda I)^{-1}$, where I is the unit matrix, satisfies the inequality

$$\|R_\lambda(A)\| \le \phi\left(\frac{1}{\rho(A,\lambda)}\right) \text{ for all regular } \lambda, \tag{9}$$

where $\phi(x)$ is a continuous monotonically increasing positive function of a positive variable x with the properties $\phi(0)=0$ and $\phi(\infty)=\infty$. Let z_ϕ be the unique positive root of the equation

$$\|A-B\|\phi(1/z)=1. \tag{10}$$

Denote

$$E_k(z_\phi)=\{z\in\mathbf{C} : |z-\omega_k|\le z_\phi\},\ k=1,...,m.$$

LEMMA 2.1. Under condition (9), let

$$\min_{k=2,\ldots,n}|\omega_1-\omega_k|>2z_\phi. \tag{11}$$

Then the total algebraic multiplicity of the eigenvalues of B that lie in the set $E_1(z_\phi)$ is equal to the algebraic multiplicity of eigenvalue ω_1.

PROOF. Put $B_t\equiv(1-t)A+tB$ for $0\le t\le 1$. Let $\mu_t\in\sigma(B_t)$ for $0\le t\le 1$. Then either $\mu_t\in\sigma(A)$ or

$$1\le\|A-B\|\|R_{\mu_t}(A)\|\le\|A-B\|\phi\left(\frac{1}{\rho(A,\mu_t)}\right). \tag{12}$$

Indeed,

$$\|B_t-A\|=\|A-A(1-t)-tB\|=t\|B-A\|.$$

From the Hilbert identity

$$R_\lambda(B_t)-R_\lambda(A)=R_\lambda(A)(A-B_t)R_\lambda(B_t)$$

it follows that if $\|A-B_t\|\|R_\lambda(A)\|<1$, then λ is a regular point of B. Thus, if

$$t\|A-B\|\|R_\lambda(A)\|<1,$$

then λ is a regular point of B. This proves relation (12).

Since ϕ is monotone, from (12) and (9) the inequality

$$\rho(A,\mu_t)\equiv\inf_k|\omega_k(A)-\mu_t|\le z_\phi \tag{13}$$

follows. Furthermore, introduce the set

$$\Omega=\{z\in\mathbf{C} : z \text{ is an eigenvalue of } B_t \text{ for some } t\in[0,1]\}.$$

Let $\Omega_1 \subset \Omega$ be a connected component beginning at ω_1. If $\mu_t \in \Omega_1$, then due to the continuity of eigenvalues and the fact that $\mu_0 = \omega_1(A)$,

$$\inf_k |\omega_k(A) - \mu_t| = |\omega_1(A) - \mu_t| \tag{14}$$

for all sufficiently small positive t. Let us suppose that for some $s \in [0,1]$

$$\inf_k |\omega_k(A) - \mu_s| = |\omega_j(A) - \mu_s| \ \ (\mu_s \in \Omega_1) \tag{15}$$

with a $j \neq 1$. But due to the continuity of eigenvalues, for any $\epsilon > 0$ there is a $t_0 < s$, such that $|\mu_{t_0} - \mu_s| \leq \epsilon$. So

$$|\omega_1(A) - \mu_s| \leq |\omega_1(A) - \mu_{t_0}| + \epsilon.$$

Thus,

$$|\omega_1(A) - \omega_j(A)| \leq |\omega_1(A) - \mu_s| + |\omega_j(A) - \mu_s| \leq |\omega_1(A) - \mu_{t_0}| + |\omega_j(A) - \mu_s| + \epsilon.$$

Hence, due to (13), (14) and (15)

$$|\omega_1(A) - \omega_j(A)| \leq 2z_\phi + \epsilon.$$

Taking ϵ small enough we get the contradiction to (11), which proves that relation (15) is impossible. So (14) holds for all $t \in [0,1]$ and $\mu_t \in \Omega_1$. Thus, in view of (13) $|\omega_1(A) - \mu_t| \leq z_\phi$. That is, $\Omega_1 \subseteq E_1(r_\phi)$. This proves the result.

COROLLARY 2.2. Let the conditions (9) and $w(A) > z_\phi$ hold. Then

$$\sigma(B) \subset \cup_{k=1,\ldots,m} E_k(z_\phi).$$

Moreover, the total algebraic multiplicity of the eigenvalues of B that lie in set $E_k(z_\phi)$ is equal to the algebraic multiplicity of ω_k.

LEMMA 2.3. Let the conditions (9) and $w(A) > z_\phi$ hold. Then $v(A,B) \leq z_\phi$.

Indeed this result is due to Corollary 2.2.

We now turn to the proof of Theorem 1.1. As it is proved in [5, Corollary 1.2.4], for any an $n \times n$-matrix A, the inequality

$$\|(A - \lambda I)^{-1}\|_2 \leq \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A,\lambda)} \text{ for all regular } \lambda.$$

is valid (see also Lemma 8.1.2 from [6]). The required result now follows from Lemma 2.3.

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