Applied Mathematics E-Notes, 2(2002), 117-124 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Inequalities Involving Khatri-Rao Products of Positive Semi-definite Matrices *

Xian Zhang†, Zhong-peng Yang‡, Chong-guang Cao§

Received 20 August 2001

Abstract

Several inequalities involving the Khatri-Rao products of two four-block positive definite real symmetric matrices are established by Liu in [1]. We extend these to two general positive semi-definite real symmetric block matrices and necessary and sufficient conditions under which these inequalities become equalities are presented.

Let $S(m)$ be the set of all real symmetric matrices of order m. Consider matrices $M \in S(m)$ and $N \in S(p)$ which are partitioned as follows

$$M = \begin{pmatrix} M_{11} & \dots & M_{1t} \\ \dots & \dots & \dots \\ M'_{1t} & \dots & M_{tt} \end{pmatrix}, \ N = \begin{pmatrix} N_{11} & \dots & N_{1t} \\ \dots & \dots & \dots \\ N'_{1t} & \dots & N_{tt} \end{pmatrix}, \quad (1)$$

where $M_{ii} \in S(m_i)$ and $N_{ii} \in S(p_i)$ for $i = 1, 2, \cdots, t$. Obviously,

$$\sum_{i=1}^{t} m_i = m, \ \sum_{i=1}^{t} p_i = p.$$

We denote by

$$M * N = (M_{ij} \otimes N_{ij})_{ij}$$

and

$$M \odot N = (M_{ij} \odot N)_{ij} = \left((M_{ij} \otimes N_{kl})_{kl}\right)_{ij}$$

the Khatri-Rao and Tracy-Singh products of M and N respectively, where $\otimes$ represents the Kronecker product. Obviously,

$$M \odot N \in S(mp), \ M * N \in S\left(\sum_{i=1}^{t} m_i p_i\right).$$

*Mathematics Subject Classifications: 15A45, 15A69.

†Department of Mathematics, Heilongjiang University, Harbin, Heilongjiang, 150080, P. R. China.
‡Department of Mathematics, Putian College, Putian, Fujian, 351100, P. R. China.
§Department of Mathematics, Heilongjiang University, Harbin, Heilongjiang 150080, P. R. China

When M and N are positive definite real symmetric matrices and $t = 2$, the following inequalities are obtained by Liu in [1]:

$$(M * N)^{-1} \leq M^{-1} * N^{-1}; \tag{2}$$

$$M^{-1} * N^{-1} \leq \frac{(\lambda_1 + \lambda_{mp})^2}{4\lambda_1 \lambda_{mp}} (M * N)^{-1}; \tag{3}$$

$$M * N - (M^{-1} * N^{-1})^{-1} \leq (\sqrt{\lambda_1} - \sqrt{\lambda_{mp}})^2 I; \tag{4}$$

$$M^2 * N^2 \leq \frac{(\lambda_1 + \lambda_{mp})^2}{4\lambda_1 \lambda_{mp}} (M * N)^2; \tag{5}$$

$$M^2 * N^2 - (M * N)^2 \leq \frac{1}{4}(\lambda_1 - \lambda_{mp})^2 I; \tag{6}$$

$$(M^2 * N^2)^{1/2} \leq \frac{\lambda_1 + \lambda_{mp}}{2\sqrt{\lambda_1 \lambda_{mp}}} (M * N); \tag{7}$$

$$(M^2 * N^2)^{1/2} - M * N \leq \frac{(\lambda_1 - \lambda_{mp})^2}{4(\lambda_1 + \lambda_{mp})} I, \tag{8}$$

where λ_1 and λ_{mp} are the largest and smallest eigenvalue of $M \odot N$ respectively, and $A \geq B$ (or $B \leq A$) means that $A - B$ is positive semi-definite. We remark that the inequality (6) is erroneously printed as $(M * N)^2 - M^2 * N^2 \leq \frac{1}{4}(\lambda_1 - \lambda_{mp})^2 I$ in [1, Theorem 8]). We remark further that conditions for equalities in (2)-(8) are not known.

The purpose of this paper is to extend these inequalities for general block matrices. We also find necessary and sufficient conditions for equalities to hold. Liu [1, p.269] also shows that the Khatri-Rao product can be viewed as a generalization of the Hadamard product. Therefore, our results can also be viewed as a generalization of those corresponding inequalities involving the Hadamard product, see e.g., [3, (1.4), (1.5), (2.14), (2.15), (2.19), (2.20)].

For a matrix $A \in S(m)$, we denote by $\lambda(A)$ and $\tau(A)$ the largest and smallest nonzero eigenvalue of A respectively. Let $R(A)$ be the column space of matrix A. We denote the $n \times n$ identity matrix by I_n, or by I when the order of matrix is clear. Let $S^+(m)$ and $S_0^+(m)$ be the set of all positive definite and semi-definite real symmetric matrices of order m respectively.

LEMMA 1. ([1, Theorem 1 (a)(b)]) If A and B are compatibly partitioned, then

$$(A \odot B)(C \odot D) = (AC) \odot (BD) \tag{9}$$

and

$$(A \odot B)^+ = A^+ \odot B^+, \tag{10}$$

where A^+ is the Moore-Penrose inverse of A.

LEMMA 2. Let A and B be compatibly partitioned matrices, then $(A \odot B)' = A' \odot B'$.

Indeed,

$$\begin{aligned}(A \odot B)' &= \left((A_{ij} \odot B)_{ij}\right)' = \left(((A_{ij} \otimes B_{kl})_{kl})_{ij}\right)' \\ &= \left(((A_{ij} \otimes B_{kl})_{kl})'\right)_{ji} = (((A_{ij} \otimes B_{kl})')_{lk})_{ji} \\ &= \left((A'_{ij} \otimes B'_{kl})_{lk}\right)_{ji} = (A'_{ij} \odot B')_{ji} \\ &= A' \odot B'.\end{aligned}$$

LEMMA 3. Suppose $A \in S(m)$ and $B \in S(p)$. Then

i) $A \odot B \in S(mp)$, $\lambda(A \odot B) = \lambda(A)\lambda(B)$, $\tau(A \odot B) = \tau(A)\tau(B)$, and $(A \odot B)^n = A^n \odot B^n$ for any positive integer number n;

ii) $A \odot B \in S_0^+(mp)$ if $A \in S_0^+(m)$ and $B \in S_0^+(p)$;

iii) $A \odot B \in S^+(mp)$ if $A \in S^+(m)$ and $B \in S^+(p)$.

PROOF. Let $A = U'_A D_A U_A$ and $B = U'_B D_B U_B$ be the spectral decompositions of A and B respectively. Then using (9) and Lemma 2,

$$\begin{aligned}A \odot B &= (U'_A D_A U_A) \odot (U'_B D_B U_B) \\ &= (U'_A \odot U'_B)(D_A \odot D_B)(U_A \odot U_B) \\ &= (U_A \odot U_B)'(D_A \odot D_B)(U_A \odot U_B) \qquad (11)\end{aligned}$$

and

$$\begin{aligned}(U_A \odot U_B)'(U_A \odot U_B) &= (U'_A \odot U'_B)(U_A \odot U_B) \\ &= (U'_A U_A) \odot (U'_B U_B). \qquad (12)\end{aligned}$$

Substituting $U'_A U_A = I_m$ and $U'_B U_B = I_p$ into (12), we see that

$$(U_A \odot U_B)'(U_A \odot U_B) = I_{mp}. \qquad (13)$$

Combining (13) and (11) completes the proof.

THEOREM 1. There exists a real matrix Z of order $mp \times \sum_{i=1}^{t} m_i p_i$ such that $Z'Z = I$ and

$$A * B = Z'(A \odot B)Z \qquad (14)$$

for any $A \in S(m)$ and $B \in S(p)$ which are partitioned as in (1).

PROOF. Let

$$Z_i = \begin{pmatrix} O_{i1} & \dots & Q_{i,i-1} & I_{m_i p_i} & O_{i,i+1} & \dots & Q_{it} \end{pmatrix}', \; i = 1, 2, ..., t, \qquad (15)$$

where O_{ik} is the zero matrix of order $m_i p_i \times m_i p_k$ for any $k \neq i$. Then $Z'_i Z_i = I$ and

$$Z'_i(A_{ij} \odot B)Z_i = Z'_i(A_{ij} \otimes B_{kl})_{kl} Z_j = A_{ij} \otimes B_{ij}, \; i, j = 1, 2, ..., t.$$

Letting

$$Z = \begin{pmatrix} Z_1 & & \\ & \ddots & \\ & & Z_t \end{pmatrix}, \tag{16}$$

the result then follows by a direct computation.

THEOREM 2. Suppose Z is defined as in Theorem 1, $A \in S_0^+(mp)$, $W = \lambda(A)$, $w = \tau(A)$, and

$$R(Z) \subseteq R(A). \tag{17}$$

Then the following conclusions hold.

(i) $(Z'AZ)^+ \leq Z'A^+Z$, and the equality holds if, and only if, $R(Z) = R(AZ)$.

(ii) $Z'A^+Z \leq \frac{(W+w)^2}{4Ww}(Z'AZ)^+$, and the equality holds if, and only if, $Z'AZ = \frac{W+w}{2}I$ and $Z'A^+Z = \frac{W+w}{2Ww}I$.

(iii) $Z'AZ - (Z'A^+Z)^+ \leq (\sqrt{W} - \sqrt{w})^2 I$, and the equality holds if, and only if, $W = w$ or

$$Z'AZ = (W + w - \sqrt{Ww})I,\ Z'A^+Z = \frac{1}{\sqrt{Ww}}I. \tag{18}$$

(iv) $Z'A^2Z \leq \frac{(W+w)^2}{4Ww}(Z'AZ)^2$, and the equality holds if, and only if, $Z'AZ = \frac{2Ww}{W+w}I$ and $Z'A^2Z = WwI$.

(v) $Z'A^2Z - (Z'AZ)^2 \leq \frac{1}{4}(W-w)^2 I$, and the equality holds if, and only if, $W = w$ or

$$Z'AZ = \frac{W+w}{2}I,\ Z'A^2Z = \frac{W^2+w^2}{2}I. \tag{19}$$

(vi) $(Z'A^2Z)^{1/2} - Z'AZ \leq \frac{(W-w)^2}{4(W+w)}I$, and the equality holds if, and only if, $W = w$ or

$$Z'A^2Z = \frac{(W+w)^2}{4}I,\ Z'AZ = \frac{W^2+w^2+6Ww}{4(W+w)}I. \tag{20}$$

PROOF. It follows from (15) and (16) that $Z^+ = Z'$ and $ZZ' \leq I$. This, together with [2, (23), and Propositions 3.1, 3.2, 3.3 and 3.4], yields the conclusions (i)–(v). Now we prove (vi). Combining [2, (4)] and $Z'Z = I$ yields

$$\begin{aligned} & (Z'A^2Z)^{1/2} - Z'AZ \\ \leq\ & (Z'AZ)^{1/2} - \frac{1}{W+w}Z'A^2Z - \frac{Ww}{W+w}I \\ =\ & \frac{(W-w)^2}{4(W+w)}I - \left[\frac{1}{\sqrt{W+w}}(Z'A^2Z)^{1/2} - \frac{\sqrt{W+w}}{2}I\right]^2 \\ \leq\ & \frac{(W-w)^2}{4(W+w)}I, \end{aligned}$$

and the above inequalities become equalities if and only if $W = w$ or (20).

THEOREM 3. Suppose $M \in S_0^+(m)$ and $N \in S_0^+(p)$ are partitioned as in (1), Z is defined as in Theorem 1, $W = \lambda(M)\lambda(N)$, $w = \tau(M)\tau(N)$, and

$$R(Z) \subseteq R(M \odot N). \tag{21}$$

Then the following conclusions hold.

(i) $(M * N)^+ \leq M^+ * N^+$, and the equality holds if, and only if, $R(Z) = R((M \odot N)Z)$.

(ii) $M^+ * N^+ \leq \frac{(W+w)^2}{4Ww}(M * N)^+$, and the equality holds if, and only if, $M * N = \frac{W+w}{2} I$ and $M^+ * N^+ = \frac{W+w}{2Ww} I$.

(iii) $M * N - (M^+ * N^+)^+ \leq (\sqrt{W} - \sqrt{w})^2 I$, and the equality holds if, and only if, $W = w$ or

$$M * N = (W + w - \sqrt{Ww})I, \qquad M^+ * N^+ = \frac{1}{\sqrt{Ww}} I. \tag{22}$$

(iv) $M^2 * N^2 \leq \frac{(W+w)}{4Ww}^2$, and the equality holds if, and only if, $M * N = \frac{2Ww}{W+w} I$ and $M^2 * N^2 = WwI$.

(v) $M^2 * N^2 - (M * N)^2 \leq \frac{1}{4}(W - w)^2 I$, and the equality holds if, and only if, $W = w$ or

$$M * N = \frac{W+w}{2} I, \qquad M^2 * N^2 = \frac{W^2 + w^2}{2} I. \tag{23}$$

(vi) $(M^2 * N^2)^{1/2} - M * N \leq \frac{(W-w)^2}{4(W+w)} I$, and the equality holds if, and only if, $W = w$ or

$$M^2 * N^2 = \frac{(W+w)^2}{4} I, \qquad M * N = \frac{W^2 + w^2 + 6Ww}{4(W+w)} I. \tag{24}$$

(vii) $(M^2 * N^2)^{1/2} \leq \frac{W+w}{2\sqrt{Ww}}(M * N)$, and the equality holds if, and only if, $M * N = \frac{W+w}{2} I$ and $M^2 * N^2 = WwI$.

PROOF. Noting i) and ii) of Lemma 3 and (10) and substituting A by $M \odot N$ in Theorem 2, we can obtain the conclusions (i)–(vi). Furthermore, the conclusion (vii) follows from (iv) and [2, Proposition 2.3].

REMARK 1. If $t = 2$, $M \in S^+(m)$ and $N \in S^+(p)$, then M and N automatically satisfy the assumptions of Theorem 3 by applying iii) of Lemma 3, and hence Theorem 3 extends the inequalities (2)–(8).

REMARK 2. We have shown that the condition (21) is sufficient for the inequalities stated in Theorem 3. The following two examples will show that this is not a necessary condition. It is still an open problem to determine a sufficient and necessary condition under which these inequalities stated in Theorem 3 hold.

EXAMPLE 1. Consider matrices

$$M = \begin{pmatrix} 7 & 1 & 0 \\ 1 & 2 & 4 \\ 0 & 4 & 9 \end{pmatrix}, \; N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{25}$$

It is easy to verify that $M, N \in S_0^+(3)$. According (15) and (16), we can obtain

$$Z' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Furthermore, we can easily show that matrices M, N and Z do not satisfy the condition (21) and the inequalities stated in Theorem 3. Indeed, $W = 21.7204$ and $w = 7.0747$. Furthermore,

$$M^+ = \begin{pmatrix} 0.4000 & -1.8000 & 0.8000 \\ -1.8000 & 12.6000 & -5.6000 \\ 0.8000 & -5.6000 & 2.6000 \end{pmatrix}, \; N^+ = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0 & 0.2500 & 0.2500 \\ 0 & 0.2500 & 0.2500 \end{pmatrix},$$

$$M^+ * N^+ = \begin{pmatrix} 0.4000 & 0 & 0 \\ 0 & 3.1500 & -1.4000 \\ 0 & -1.4000 & 0.6500 \end{pmatrix},$$

$$(M * N)^+ = \begin{pmatrix} 0.1429 & 0 & 0 \\ 0 & 4.5000 & -2.0000 \\ 0 & -2.0000 & 1.0000 \end{pmatrix},$$

$$M^2 = \begin{pmatrix} 50 & 9 & 4 \\ 9 & 21 & 44 \\ 4 & 44 & 97 \end{pmatrix}, \; N^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix},$$

$$M^2 * N^2 = \begin{pmatrix} 50 & 0 & 0 \\ 0 & 42 & 88 \\ 0 & 88 & 194 \end{pmatrix}, \; (M^2 * N^2)^{1/2} = \begin{pmatrix} 7.0711 & 0 & 0 \\ 0 & 3.7366 & 5.2951 \\ 0 & 5.2951 & 12.8826 \end{pmatrix},$$

$$(M^+ * N^+)^+ = \begin{pmatrix} 2.5000 & 0 & 0 \\ 0 & 7.4286 & 16.0000 \\ 0 & 16.0000 & 36.0000 \end{pmatrix},$$

$$M^+ * N^+ - (M * N)^+ = \begin{pmatrix} 0.2571 & 0 & 0 \\ 0 & -1.3500 & 0.6000 \\ 0 & 0.6000 & -0.3500 \end{pmatrix} \ngeq O,$$

$$\frac{(W+w)^2}{4Ww}(M * N)^+ - M^+ * N^+ = \begin{pmatrix} -0.2073 & 0 & 0 \\ 0 & 2.9204 & -1.2979 \\ 0 & -1.2979 & 0.6990 \end{pmatrix} \ngeq O,$$

$$(\sqrt{W} - \sqrt{w})^2 I - M * N - (M^+ * N^+)^+ = \begin{pmatrix} -0.4973 & 0 & 0 \\ 0 & 9.4313 & 12.0000 \\ 0 & 12.0000 & 31.0027 \end{pmatrix} \ngeq O,$$

$$\frac{(W+w)^2}{4Ww}(M * N)^2 - M^2 * N^2 = \begin{pmatrix} 16.0994 & 0 & 0 \\ 0 & -15.0207 & -28.6455 \\ 0 & -28.6455 & -63.1502 \end{pmatrix} \ngeq O,$$

$$\frac{(W-w)^2}{4(W+w)} I - (M^2 * N^2)^{1/2} - M * N = \begin{pmatrix} 1.7912 & 0 & 0 \\ 0 & 0.1257 & -1.2951 \\ 0 & -1.2951 & -2.0204 \end{pmatrix} \ngeq O,$$

$$\frac{1}{4}(W-w)^2 I - M^2 * N^2 - (M * N)^2 = \begin{pmatrix} 52.6243 & 0 & 0 \\ 0 & 31.6243 & -44.0000 \\ 0 & -44.0000 & -43.3757 \end{pmatrix} \ngeq O,$$

and

$$\frac{W+w}{2\sqrt{Ww}}(M*N)-(M^2*N^2)^{1/2}=\begin{pmatrix} -3.0060 & 0 & 0 \\ 0 & -2.5752 & -2.9722 \\ 0 & -2.9722 & -7.6561 \end{pmatrix} \ngeq O.$$

EXAMPLE 2. Consider matrices

$$M=\begin{pmatrix} 3 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & 9 \end{pmatrix},\ N=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{27}$$

It is easy to verify that $M, N \in S_0^+(3)$. According to (15) and (16), we can see that the matrix Z possesses the form (26). Furthermore, we can easily show that matrices M, N and Z do not satisfy the condition (21), but they satisfy the inequalities stated in Theorem 3. Indeed, $W = 23.6619$ and $w = 0.1690$. Furthermore,

$$M^+=\begin{pmatrix} 4.5000 & 0 & -2.5000 \\ 0 & 0 & 0 \\ -2.5000 & 0 & 1.5000 \end{pmatrix},\ \text{and } N^+=\begin{pmatrix} 1.0000 & 0 & 0 \\ 0 & 0.2500 & 0.2500 \\ 0 & 0.2500 & 0.2500 \end{pmatrix}.$$

Then

$$M^+*N^+=\begin{pmatrix} 4.5000 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.3750 \end{pmatrix},\ (M*N)^+=\begin{pmatrix} 0.3333 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1111 \end{pmatrix},$$

$$M^2=\begin{pmatrix} 34 & 0 & 60 \\ 0 & 0 & 0 \\ 60 & 0 & 106 \end{pmatrix},\ N^2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix},$$

$$M^2*N^2=\begin{pmatrix} 34 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 212 \end{pmatrix},\ (M^2*N^2)^{1/2}=\begin{pmatrix} 5.8310 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14.5602 \end{pmatrix},$$

$$(M^+*N^+)^+=\begin{pmatrix} 0.2222 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2.6667 \end{pmatrix},$$

$$M^+*N^+-(M*N)^+=\begin{pmatrix} 4.1667 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.2639 \end{pmatrix} \geq O,$$

$$\frac{(W+w)^2}{4Ww}(M*N)^+-M^+*N^+=\begin{pmatrix} 7.3315 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3.5688 \end{pmatrix} \geq O,$$

$$(\sqrt{W}-\sqrt{w})^2I-M*N-(M^+*N^+)^+=\begin{pmatrix} 17.0532 & 0 & 0 \\ 0 & 19.8310 & 0 \\ 0 & 0 & 13.4976 \end{pmatrix} > O,$$

$$\frac{(W+w)^2}{4Ww}(M*N)^2 - M^2*N^2 = 10^{-3}\begin{pmatrix} 0.2855 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2.6631 \end{pmatrix} \geq O,$$

$$\frac{(W-w)^2}{4(W+w)}I - (M^2*N^2)^{1/2} - M*N = \begin{pmatrix} 2.9589 & 0 & 0 \\ 0 & 5.7899 & 0 \\ 0 & 0 & 0.2297 \end{pmatrix} > O,$$

$$\frac{1}{4}(W-w)^2 I - M^2*N^2 - (M*N)^2 = \begin{pmatrix} 112.9786 & 0 & 0 \\ 0 & 137.9786 & 0 \\ 0 & 0 & 6.9786 \end{pmatrix} > O,$$

and

$$\frac{W+w}{2\sqrt{Ww}} - (M^2*N^2)^{1/2} = \begin{pmatrix} 3.1057 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12.2496 \end{pmatrix} \geq O.$$

REMARK 3. Since Theorem 3 can be obtained by substituting A with $M \odot N$ in Theorem 2, the condition (17) is not necessary for the inequalities stated in Theorem 2 to hold by choosing $A = M \odot N$, where M and N are defined as in Examples 1 and 2 respectively. It is also an open problem to determine a sufficient and necessary condition under which these inequalities stated in Theorem 2 hold.

ACKNOWLEDGMENT. We wish to thank the referee for his helpful comments. The first and third authors are partially supported by the NSF of Heilongjiang (No. A01-07) and the NSF of Heilongjiang Education Committee (No. 15011014). The second author is partially supported by the NSF of Fujian Education Committee.

References

[1] S. Z. Liu, Matrix results on the Khatri-Rao and Tracy-Singh products, Lin. Alg. Appl., 289(1999), 266-277.

[2] S. Liu and W. Polasek, Equality conditions for matrix Kantorovich-Type inequalities, J. Math. Anal. Appl., 212(1997), 517-528.

[3] B. Mond and J. E. Pecaric, Inequalities for the Hadamard product of matrix, SIAM. J. Matrix Anal. Appl., 19(1)(1998), 66-70.