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A Question of Gross and Weighted Sharing of a Finite Set by Meromorphic Functions *

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Abstract

We prove a uniqueness theorem for meromorphic functions sharing one finite set with weight two and this improves some results of Yi [11], Li and Yang [8] and Fang and Hua [2].

1 Introduction

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $S \subset \mathbb{C} \cup \{\infty\}$ we define by $E_f(S)$ the set

$$E_f(S) = \cup_{a \in S}\{z : f(z) - a = 0\},$$

where an a-point of multiplicity m is counted m times.

In 1976, Gross [3] proved that there exist three finite sets S_1, S_2, S_3 such that any two entire functions f, g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical. In the same paper Gross asked the following question: Can one find two (or even one) finite sets S_1 and S_2 such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

A set S for which two meromorphic functions f, g satisfying $E_f(S) = E_g(S)$ become identical is called a unique range set of meromorphic functions (cf. [4, 8]).

In 1982, Gross and Yang [4] proved the following theorem.

THEOREM A. Let $S = \{z : e^z + z = 0\}$. If two entire functions f, g satisfy $E_f(S) = E_g(S)$ then $f \equiv g$.

Since the set $S = \{z : e^z + z = 0\}$ contains infinitely many elements, the above result does not answer the question of Gross.

In 1994, Yi [10] exhibited a finite set S containing 15 elements which is a unique range set of entire functions and provided an affirmative answer to the question of Gross.

In 1995, Yi [11] and Li and Yang [8] independently proved the following result which gives a better answer to the question of Gross.

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THEOREM B. Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If two entire functions f, g satisfy $E_f(S) = E_g(S)$ then $f \equiv g$.

Extending Theorem B to meromorphic functions, recently Fang and Hua [2] proved the following theorem.

THEOREM C. Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If two meromorphic functions f, g are such that $\Theta(\infty; f) > 11/12$, $\Theta(\infty; g) > 11/12$ and $E_f(S) = E_g(S)$ then $f \equiv g$.

Here Θ is the ramification index which is defined below.

In [6, 7] the notion of weighted sharing is introduced which we explain in the following definition.

DEFINITION 1. Let k be a nonnegative integer or infinity. For $a \in C \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$, and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$, and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We say that f, g share (a, k) if f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p which satisfies $0 \leq p < k$. Also we note that f, g share a value a IM (ignoring multiplicity) or CM (counting multiplicity) if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

DEFINITION 2. For $S \subset \mathbb{C} \cup \{\infty\}$, we define $E_f(S, k)$ as $E_f(S, k) = \cup_{a \in S} E_k(a; f)$, where k is a nonnegative integer or infinity.

The above definition is in [6]. Clearly $E_f(S) = E_f(S, \infty)$.

DEFINITION 3. A set S for which two meromorphic functions f, g satisfying $E_f(S, k) = E_g(S, k)$ becomes identical is called a unique range set of weight k for meromorphic functions.

Unless stated otherwise, throughout the paper f and g are two nonconstant meromorphic functions. We now explain some basic definitions and notations of the value distribution theory (see e.g. [5]). We denote by $n(r, f)$ the number of poles of f in $|z| \leq r$, where a pole is counted according to its multiplicity, and by $\overline{n}(r, f)$ the number of distinct poles of f in $|z| \leq r$. Also we put

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

and

$$\overline{N}(r, f) = \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt + \overline{n}(0, f) \log r.$$

The quantities $N(r, f)$, $\overline{N}(r, f)$ are called respectively the counting function and reduced counting function of poles of f. Let

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \mid f(re^{i\theta}) \mid d\theta,$$

where $\log^+ x = \log x$ if $x \geq 1$ and $\log^+ x = 0$ if $0 \leq x < 1$. We call $m(r, f)$ the proximity function of f. The sum $T(r, f) = m(r, f) + N(r, f)$ is called the Nevanlinna characteristic function of f. If a is a finite complex number, we put

$$m(r, a; f) = m\left(r, \frac{1}{f-a}\right),\ N(r, a; f) = N\left(r, \frac{1}{f-a}\right),\ \overline{N}(r, a; f) = \overline{N}\left(r, \frac{1}{f-a}\right).$$

The quantity

$$\Theta(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}$$

is called the ramification index, where $a \in \mathbb{C} \cup \{\infty\}$ and $\overline{N}(r, \infty; f) = \overline{N}(r, f)$. By the second fundamental theorem we know that the set $\{a : a \in \mathbb{C} \cup \{\infty\},\ \Theta(a; f) > 0\}$ is countable and $\sum_a \Theta(a; f) \leq 2$. Finally we denote by $N_2(r, a; f)$ the counting function of a-points of f where an a-point of multiplicity m is counted m times if $m \leq 2$ and is counted twice if $m > 2$ (see e.g. [1]).

In this paper we prove the following theorem which improves Theorem B and Theorem C.

THEOREM 1. Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If f and g satisfy $\Theta(\infty; f) + \Theta(\infty; g) > 3/2$ and $E_f(S, 2) = E_g(S, 2)$, then $f \equiv g$.

2 Preparatory Lemmas

In this section we present some lemmas which will be required to prove our main Theorem. The first one is in [9].

LEMMA 1. Let $P(f) = \sum_{j=0}^{n} a_j f^j$, where $a_0, a_1, \ldots, a_n (\not\equiv 0)$ are such that $T(r, a_j) = S(r, f)$ for $j = 0, 1, \ldots, n$. Then $T(r, P(f)) = nT(r, f) + S(r, f)$.

LEMMA 2. If $\Theta(\infty; f) + \Theta(\infty; g) > 3/2$, then for $n \geq 3$, $f^{n-1}(f-1)g^{n-1}(g-1) \not\equiv 1$.

PROOF. Assume to the contrary that

$$f^{n-1}(f-1)g^{n-1}(g-1) \equiv 1. \tag{1}$$

Suppose f does not have any pole. Then from (1) it follows that g has no zero nor 1-point. So by the deficiency relation we get $\Theta(\infty; g) = 0$, which contradicts the given condition. So the lemma is proved in this case. Similarly we can prove the lemma when g does not have any pole. Now we suppose that f and g have poles. From (1), we see that if z_0 is a zero of f with multiplicity p then z_0 is a pole of g with multiplicity q such that $p(n-1) = nq$, i.e., $p = qn/(n-1)$. Since n, p, q are all positive integers, it follows that $p \geq n$. Hence $\Theta(0; f) \geq 1 - 1/n$. Again from (1), we see that if z_0 is an 1-point of f with multiplicity p then z_0 is a pole of g with multiplicity q such that $p = nq$ and so $p \geq n$. Hence $\Theta(1; f) \geq 1 - 1/n$. Similarly we can prove that $\Theta(0; g) \geq 1 - 1/n$ and $\Theta(1; g) \geq 1 - 1/n$. So by the deficiency relation we get

$$\Theta(0; f) + \Theta(1; f) + \Theta(0; g) + \Theta(1; g) + \Theta(\infty; f) + \Theta(\infty; g) \leq 4,$$

or,

$$4(1 - \frac{1}{n}) + \frac{3}{2} \leq 4,$$

or $n \le 8/3$, a contradiction. This proves the lemma.

LEMMA 3. If $\Theta(\infty; f) + \Theta(\infty; g) > 3/2$, then for $n \ge 4$, $f^{n-1}(f-1) \equiv g^{n-1}(g-1)$ implies $f \equiv g$.

PROOF. Let

$$f^{n-1}(f-1) \equiv g^{n-1}(g-1). \tag{2}$$

Assume to the contrary that $f \not\equiv g$. Then from (2) we get

$$f \equiv 1 - \frac{y^{n-1}}{1 + y + y^2 + \cdots + y^{n-1}}, \tag{3}$$

where $y = g/f$. If y is constant then $y \ne 1$. Also from (2) we see that $y^n \ne 1$ and $y^{n-1} \ne 1$ and so (2) implies

$$f \equiv \frac{1 - y^{n-1}}{1 - y^n}$$

which is a contradiction because f is nonconstant. Let y be nonconstant. From (3) we get by the first fundamental theorem and Lemma 1 that

$$\begin{aligned} T(r,f) &= T(r, \sum_{j=0}^{n-1} \frac{1}{y^j}) + S(r,y) = (n-1)T(r, \frac{1}{y}) + S(r,y) \\ &= (n-1)T(r,y) + S(r,y). \end{aligned}$$

Now we note that any pole of y is not a pole of $1 - y^{n-1}/\sum_{j=1}^{n-1} y^j$. So from (3) it follows that

$$\sum_{k=1}^{n-1} \overline{N}(r, u_k; y) \le \overline{N}(r, \infty; f),$$

where $u_k = exp(2k\pi i/n)$ for $k = 1, 2, \ldots, n-1$. By the second fundamental theorem we get

$$\begin{aligned} (n-3)T(r,y) &\le \sum_{k=1}^{n-1} \overline{N}(r, u_k; y) + S(r,y) \\ &\le \overline{N}(r, \infty; f) + S(r,y) \\ &< (1 - \Theta(\infty; f) + \varepsilon)T(r,f) + S(r,y) \\ &= (n-1)(1 - \Theta(\infty; f) + \varepsilon)T(r,y) + S(r,y), \quad (4) \end{aligned}$$

where $\varepsilon > 0$.

Again putting $y_1 = 1/y$, noting that $T(r,y) = T(r,y_1) + O(1)$ and proceeding as above we get

$$(n-3)T(r,y) \le (n-1)(1 - \Theta(\infty; g) + \varepsilon)T(r,y) + S(r,y), \tag{5}$$

where $\varepsilon > 0$. From (4) and (5) we get in view of the given condition,

$$\begin{aligned} & 2(n-3)T(r,y) \\ \le\ & (n-1)(2 - \Theta(\infty; f) - \Theta(\infty; g) + 2\varepsilon)T(r,y) + S(r,y) \\ <\ & (n-1)(\frac{1}{2} + 2\varepsilon)T(r,y) + S(r,y), \end{aligned}$$

which implies a contradiction for all sufficiently small positive ε due to the assumption that $n \geq 4$. Hence $f \equiv g$.This completes the proof.

LEMMA 4. If f, g share $(1,2)$, then one of the following holds: (i) $T(r) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r,f)+S(r,g)$, where $T(r) = \max\{T(r,f),T(r,g)\}$, (ii) $fg \equiv 1$, or, (iii) $f \equiv g$.

The proof can be found in [7].

3 Proof of Theorem

Let $F = f^6(f-1)$ and $G = g^6(g-1)$. Since $E_f(S,2) = E_f(S,2)$, it follows that F, G share $(1,2)$. Also by Lemma 1, we see that $T(r,F) = 7T(r,f) + S(r,f)$ and $T(r,G) = 7T(r,g) + S(r,g)$. Now

$$\begin{aligned} & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;G) + N_2(r,\infty;G) + S(r,F) + S(r,G) \\ \leq\ & 2\overline{N}(r,0;f) + N_2(r,0;f-1) + 2\overline{N}(r,0;g) \\ & +N_2(r,0;g-1) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\ \leq\ & \{6 + 2(2 - \Theta(\infty;f) - \Theta(\infty;g) + \varepsilon)\}T(r) + S(r,f) + S(r,g) \\ =\ & (10 - 2\Theta(\infty;f) - 2\Theta(\infty;g) + 2\varepsilon)T(r) + S(r,f) + S(r,g), \end{aligned} \tag{6}$$

where $\varepsilon > 0$. Also we see that

$$\max\{T(r,F),T(r,G)\} = 7T(r) + S(r,f) + S(r,g). \tag{7}$$

From (6) and (7), we see that

$$\begin{aligned} & \max\{T(r,F),T(r,G)\} \\ \leq\ & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,F) + S(r,G) \end{aligned}$$

if

$$7T(r) \leq (10 - 2\Theta(\infty;f) - 2\Theta(\infty;g) + 2\varepsilon)T(r) + S(r,f) + S(r,g)$$

i.e., if

$$(2\Theta(\infty;f) + 2\Theta(\infty;g) - 3 - 2\varepsilon)T(r) \leq S(r,f) + S(r,g).$$

Then a contradiction is reached for sufficiently small positive ε because $\Theta(\infty;f) + \Theta(\infty;g) > 3/2$. By Lemma 2, we see that $FG \not\equiv 1$ because $\Theta(\infty;f) + \Theta(\infty;g) > 3/2$. Hence applying Lemma 4, we see that $F \equiv G$ and so by Lemma 3, we get $f \equiv g$. This completes the proof.

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