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Oscillation for Nonlinear Neutral Difference Equations*

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Abstract

In this note, some oscillatory results for neutral difference equations are obtained.

Recently, Tang and Liu in [1] considered delay difference equations of the form

$$\Delta x_n + q_n x_{n-k}^{\alpha} = 0,\ n = 0, 1, 2, ..., \tag{1}$$

where $\{q_n\}$ is a sequence of nonnegative numbers, k is a positive integer, and $\alpha \in (0, \infty)$ is a quotient of odd positive integers. They obtained the following theorems.

THEOREM 1. Assume that $0 < \alpha < 1$. Then every solution of (1) oscillates if, and only if,

$$\sum_{n=0}^{\infty} q_n = \infty. \tag{2}$$

THEOREM 2. Assume that $\alpha > 1$. Suppose further that there exists a $\lambda > k^{-1} \ln \alpha$ such that

$$\liminf_{n\to\infty} q_n \exp\left(-e^{\lambda n}\right) > 0,$$

then every solution of (1) oscillates.

In this note, we will consider the neutral difference equations

$$\Delta \left(x_n - p x_{n-l}\right) + q_n x_{n-k}^{\alpha} = 0,\ n = 0, 1, 2, ..., \tag{3}$$

and

$$\Delta^2 \left(x_n + p x_{n-l}\right) + q_n x_{n-k}^{\alpha} = 0,\ n = 0, 1, 2, ..., \tag{4}$$

where α, k and $\{q_n\}$ are defined as before, l is a positive integer, and $0 \le p < 1$. It is clear that equation (1) is a particular case of (3). We will discuss the oscillation of (3) and (4) in two cases where $\alpha < 1$ and $\alpha > 1$. In the sequel, for convenience, when we write a functional inequality without specifying its domain of validity, we assume that it holds for all sufficiently large n.

THEOREM 3. Assume that $0 \le p < 1$ and $0 < \alpha < 1$. Then every solution of (3) oscillates if, and only if, (2) holds.

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PROOF. The fact that oscillation of (3) implies (2) can be found in [2]. Next, assume that $\{x_n\}$ is an eventually positive solution of (3). In view of Lemma 1 in [3], we get that $z_n = x_n - px_{n-l}$ is eventually positive. Thus,

$$x_{n-k} = z_{n-k} + px_{n-l-k} \geq z_{n-k} + pz_{n-l-k} \geq (1+p)\, z_n.$$

Substituting it into (3), we have

$$\Delta z_n + q_n\,(1+p)^{\alpha}\, z_n^{\alpha} \leq 0.$$

Thus

$$z_n^{-\alpha} \Delta z_n + q_n\,(1+p)^{\alpha} \leq 0. \tag{5}$$

We define $r(t) = z_m + (t-m)\,\Delta z_m$, $m \leq t \leq m+1$. Since $\Delta z_m \leq 0$, then $z_{m+1} \leq r(t) \leq z_m$ and

$$\frac{r'(t)}{r^{\alpha}(t)} \leq \frac{\Delta z_m}{z_m^{\alpha}}. \tag{6}$$

In view of (5), (6) and (2), we obtain

$$\int_{r(N)}^{r(\infty)} \frac{dr}{r^{\alpha}} = -\infty,$$

which contradicts the fact $\alpha \in (0,1)$. The proof is complete.

If $\alpha > 1$, $0 \leq p < 1$ and $\{x_n\}$ is an eventually positive solution of (3), then we have

$$\begin{aligned} x_{n-k} &= z_{n-k} + px_{n-l-k} = z_{n-k} + pz_{n-l-k} + p^2 x_{n-l-2k} \\ &= z_{n-k} + pz_{n-l-k} + \ldots + p^m z_{n-l-mk} + p^{m+1} x_{n-l-(m+1)k} \\ &\leq p^m z_{n-l-mk}. \end{aligned}$$

By (3), we have

$$\Delta z_n + q_n p^{m\alpha} z_{n-l-mk} \leq 0$$

for any $m \geq 0$. Note that when $p > 0$,

$$\liminf_{n\to\infty} q_n \exp\left(-e^{\lambda n}\right) > 0 \Leftrightarrow \liminf_{n\to\infty} p^{m\alpha} q_n \exp\left(-e^{\lambda n}\right) > 0.$$

In view of Theorem 2, we have the following result.

THEOREM 4. Assume that $\alpha > 1$ and $0 \leq p < 1$. Suppose further that for some nonnegative integer m there exists a $\lambda > (l+mk)^{-1} \ln \alpha$ (where $m = 0$ if $p = 0$) such that

$$\liminf_{n\to\infty} q_n \exp\left(-e^{\lambda n}\right) > 0,$$

then every solution of (3) oscillates.

To obtain oscillatory results of equation (4), we need the following lemma which can be found in [4].

LEMMA 1. An eventually concave sequence $\{x_n\}$(i.e. $\Delta^2 x_n \leq 0$ for all large n) is of constant sign eventually. If $x_n > 0$ and $\Delta^2 x_n \leq 0$ eventually and $\left\{\Delta^2 x_n\right\}$ has a negative subsequence, then $\{\Delta x_n\}$ is eventually positive. Furthermore, there is a number $\theta \in (0,1)$ such that $x_n \geq \theta n \Delta x_n$ for all large n.

Assume that $\{x_n\}$ is an eventually positive solution of equation (4). Clearly, we have $y_n = x_n + p x_{n-l} > \theta n \Delta y_n > 0$ and $\Delta^2 y_n \leq 0$. Thus,

$$\begin{aligned} x_{n-k} &= y_{n-k} - p y_{n-k-l} + p^2 x_{n-l-2k} \geq (1-p)\, y_{n-k-l} \\ &\geq (1-p)\,\theta\,(n-k-l)\,\Delta y_{n-k-l}. \end{aligned}$$

Substituting it into (4), we have

$$\Delta^2 y_n + q_n\,(1-p)^\alpha\,\theta^\alpha\,(n-k-l)^\alpha\,(\Delta y_{n-k-l})^\alpha \leq 0.$$

In view of Theorem 1, we have the following theorem.

THEOREM 5. Assume that $\alpha \in (0,1)$ and $p \in [0,1)$. Suppose further that

$$\sum_{n=0}^{\infty} q_n\,(n-k-l)^\alpha = \infty,$$

then every solution of (4) oscillates. While if $\alpha > 1$, $p \in [0,1)$, and there exists $\lambda > (k+l)^{-1} \ln \alpha$ such that

$$\liminf_{n\to\infty} q_n\,(n-k-l)^\alpha \exp\left(-e^{\lambda n}\right) > 0,$$

then all solutions of (4) oscillate.

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