A theorem on S-shaped bifurcation curve for a positone problem with convex–concave nonlinearity and its applications to the perturbed Gelfand problem

Kuo-Chih Hung, Shin-Hwa Wang

Department of Mathematics, National Tsing Hua University, Hsinchu, 300, Taiwan, ROC

Abstract

We study the bifurcation curve and exact multiplicity of positive solutions of the positone problem

\[ \begin{align*}
    u''(x) + \lambda f(u) &= 0, & -1 < x < 1, \\
    u(-1) &= u(1) = 0,
\end{align*} \]

where \( \lambda > 0 \) is a bifurcation parameter, \( f \in C^2[0, \infty) \) satisfies \( f(0) > 0 \) and \( f(u) > 0 \) for \( u > 0 \), and \( f \) is convex–concave on \((0, \infty)\). Under a mild condition, we prove that the bifurcation curve is S-shaped on the \( (\lambda, \|u\|_{\infty}) \)-plane. We give an application to the perturbed Gelfand problem

\[ \begin{align*}
    u''(x) + \lambda \exp\left(\frac{au}{a + u}\right) &= 0, & -1 < x < 1, \\
    u(-1) &= u(1) = 0,
\end{align*} \]

where \( a > 0 \) is the activation energy parameter. We prove that, if \( a \geq a^* \approx 4.166 \), the bifurcation curve is S-shaped on the \( (\lambda, \|u\|_{\infty}) \)-plane. Our results improve those in [S.-H. Wang, On S-shaped bifurcation curves, Nonlinear Anal. 22 (1994) 1475–1485] and [P. Korman, Y. Li, On the exactness of an S-shaped bifurcation curve, Proc. Amer. Math. Soc. 127 (1999) 1011–1020].

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1. Introduction

We study the bifurcation curve and exact multiplicity of positive solutions of the two-point boundary value problem

\[
\begin{cases}
  u''(x) + \lambda f(u) = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0,
\end{cases}
\]

where \( \lambda > 0 \) is a bifurcation parameter. We assume that nonlinearity \( f \in C^2[0, \infty) \) satisfies hypotheses (H1)–(H3) as follows:

(H1) \( f(0) > 0 \) (positone) and \( f(u) > 0 \) on \( (0, \infty) \).
(H2) \( f \) is convex–concave on \( (0, \infty) \); that is, there exists a number \( \gamma > 0 \) such that

\[
  f''(u)
  \begin{cases}
    > 0 & \text{on } [0, \gamma), \\
    = 0 & \text{when } u = \gamma, \\
    < 0 & \text{on } (\gamma, \infty).
  \end{cases}
\]

(H3) \( f \) is asymptotic sublinear; that is, \( \lim_{u \to \infty} \frac{f(u)}{u} = 0 \).

We define the bifurcation curve of positive solutions of (1.1),

\[
S = \{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)} \}.
\]

We say that, on the \( (\lambda, \|u\|_\infty) \)-plane, the bifurcation curve \( S \) is S-shaped if \( S \) has exactly two turning points at some points \( (\lambda^+, \|u_{\lambda^+}\|_\infty) \) and \( (\lambda^-, \|u_{\lambda^-}\|_\infty) \) where \( \lambda^- < \lambda^+ \) are two positive numbers such that

(i) \( \|u_{\lambda^+}\|_\infty < \|u_{\lambda^-}\|_\infty \)

(ii) at \( (\lambda^+, \|u_{\lambda^+}\|_\infty) \) the bifurcation curve \( S \) turns to the left,

(iii) at \( (\lambda^-, \|u_{\lambda^-}\|_\infty) \) the bifurcation curve \( S \) turns to the right.

See Fig. 1(i) for example.

A motivating prominent example of this study is the famous perturbed Gelfand problem in one space variable

\[
\begin{cases}
  u''(x) + \lambda \exp\left(\frac{au}{a+u}\right) = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0,
\end{cases}
\]

where \( \lambda > 0 \) is the Frank–Kamenetskii parameter or ignition parameter, \( a > 0 \) is the activation energy parameter, \( u(x) \) is the dimensionless temperature, and the reaction term \( \exp(\frac{au}{a+u}) \) is the temperature dependence obeying the simple Arrhenius reaction-rate law in irreversible chemical reaction kinetics, see, e.g., Boddington et al. [2]. This is the one-dimensional case of a problem arising in the study of (steady state) solid fuel ignition models in thermal combustion theory. The derivation of the problem and more background can refer to Bebernes and Eberly [1].

For (1.2), it has been a long-standing conjecture [3,9,10,13–15,17,18] that, there exists a critical bifurcation value \( a_0 > 0 \) such that the bifurcation curve \( S \) is S-shaped for \( a > a_0 \) and is monotone increasing for \( 0 < a \leq a_0 \); in particular, when \( a = a_0 \), there is a turning point, see Fig. 1(i)–(iii).

This kind of global bifurcation result is useful in understanding the profiles of the solutions to the full exothermic reaction–diffusion system, see [12] for details.) Korman, Li and Ouyang [10] gave a computer-assisted proof of this conjecture. We note that it is also conjectured [13,17] that, at this
critical bifurcation value \( a = a_0 \), two “simple” turning points coalesce into a single “double” turning point \((\tilde{\lambda}, \tilde{u})\), see Fig. 1(i)–(ii).

For (1.2), it is easy to show that the bifurcation curve \( S \) is a monotone increasing curve without turning points for \( 0 < a \leq 4 \), and hence \( a_0 > 4 \), see [3, p. 482] and Fig. 1(iii). The study of S-shaped bifurcation curve \( S \) of (1.2) and its generalization to higher dimensions has been extensively investigated by many authors, see, e.g. [3–10,14,15,18]. Brown et al. [3] showed that the bifurcation curve \( S \) is S-like shaped (i.e., \( S \) has at least two turning points) for \( a \geq \tilde{a} \approx 4.25 \) for some constant \( \tilde{a} \) by applying next Theorem 1.1(ii). Hastings and McLeod [7], using quadratures, proved that the bifurcation curve is S-shaped for \( a \) large enough. Wang [18, Theorem 1], using quadrature method (time-map method), proved that the bifurcation curve is S-shaped for \( a > \bar{a} \approx 4.4967 \) for some constant \( \bar{a} \). This upper bound of \( a_0 \) was improved to \( \tilde{a} \approx 4.35 \) for some constant \( \tilde{a} \) by Korman and Li [9, Theorem 3.1] by using the bifurcation approach. The above upper bound results of \( a_0 \) are based upon next Theorem 1.1(i) proved by Wang [18, Theorem 3] and by Korman and Li [9, Theorem 2.1] independently. The following Theorem 1.1(ii) is due to Brown et al. [3, Theorem 2.2(b)]. Define

\[
\theta(u) = 2 \int_0^u f(t) \, dt - uf(u) \quad \text{for } u \geq 0.
\] (1.3)

**Theorem 1.1.** Consider (1.1). Assume \( f \in C^2[0, \infty) \) satisfies (H1)–(H3). Then the following assertions (i) and (ii) hold:

(i) ([18, Theorem 3] and [9, Theorem 2.1]) If

\[
\theta(\gamma) < 0,
\] (1.4)

then the bifurcation curve \( S \) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane. More precisely, there exist two positive numbers \( \lambda_* < \lambda^* \) such that (1.1) has exactly three positive solutions for \( \lambda_* < \lambda < \lambda^* \), exactly two positive solutions for \( \lambda = \lambda_* \) and \( \lambda = \lambda^* \), and exactly one positive solution for \( 0 < \lambda < \lambda_* \) and \( \lambda > \lambda^* \). Moreover, let \((\lambda^*, \|u_{\lambda^*}\|_\infty)\) and \((\lambda_*, \|u_{\lambda_*}\|_\infty)\) be the two turning points of the bifurcation curve \( S \) satisfying \( \lambda_* < \lambda^* \) and \( \|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty \), then \( p_1 < \|u_{\lambda^*}\|_\infty < \gamma < p_2 < \|u_{\lambda_*}\|_\infty \) for some positive numbers \( p_1 < \gamma < p_2 \) satisfying \( f(u) - uf'(u) = 0 \).

(ii) [3, Theorem 2.2(b)] If

\[
\theta(u_0) \leq 0 \quad \text{for some } u_0 > 0.
\] (1.5)
then the bifurcation curve $S$ is S-like shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist two positive numbers $\lambda_+ < \lambda^*$ such that (1.1) has at least three positive solutions for $\lambda_+ < \lambda < \lambda^*$, at least two positive solutions for $\lambda = \lambda_+$ and $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda_+$ and $\lambda > \lambda^*$.

2. Main results

The main results in this paper are next Theorems 2.1 and 2.2. In Theorem 2.1, we improve the results in Theorem 1.1; that is, we weaken conditions (1.4) and (1.5) as (2.2) and (2.4), respectively (see Proposition 3.4 below). In Theorem 2.2, by applying Theorem 2.1, we prove that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve of (1.2) is S-shaped for $a > a^* \approx 4.166$ and is S-like shaped for $a > a^{**} \approx 4.107$ > $a_0$, where $a^* \approx 4.166$ is the constant defined in (3.22) and (4.107) \( a^{**} \in (4, a^*) \) is the constant defined in (3.23). So we give a much sharper upper bound $a^* \approx 4.166$ of the conjectured critical bifurcation value $a_0$ for the perturbed Gelfand problem (1.2). Note that, numerical simulations show that $a_0 \approx 4.07$, see [3, p. 482] and [2, p. 441].

We first define the following auxiliary function

$$H(u) = 3 \int_0^u tf(t) \, dt - u^2 f(u) \quad \text{for } u \geq 0. \quad (2.1)$$

**Theorem 2.1.** (See Fig. 1(i).) Consider (1.1). Assume $f \in C^2[0, \infty)$ satisfies (H1)–(H3). Then the following assertions (i) and (ii) hold:

(i) If

$$H(\gamma) \leq 0, \quad (2.2)$$

then the bifurcation curve $S$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist two positive numbers $\lambda_+ < \lambda^*$ such that (1.1) has exactly three positive solutions for $\lambda_+ < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_+$ and $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda_+$ and $\lambda > \lambda^*$. Moreover, let $(\lambda^*, \|u_{\lambda^*}\|_{\infty})$ and $(\lambda_+, \|u_{\lambda_+}\|_{\infty})$ be the two turning points of the bifurcation curve $S$ satisfying $\lambda_+ < \lambda^*$ and $\|u_{\lambda^*}\|_{\infty} < \|u_{\lambda_+}\|_{\infty}$, then

$$p_1 < \|u_{\lambda^*}\|_{\infty} < \gamma < p_2 < \|u_{\lambda_+}\|_{\infty} \quad (2.3)$$

for some positive numbers $p_1 < \gamma < p_2$ satisfying $f(u) - uf'(u) = 0$.

(ii) If

$$H(u_0) \leq 0 \quad \text{for some } u_0 > 0, \quad (2.4)$$

then the bifurcation curve $S$ is S-like shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist two positive numbers $\lambda_+ < \lambda^*$ such that (1.1) has at least three positive solutions for $\lambda_+ < \lambda < \lambda^*$, at least two positive solutions for $\lambda = \lambda_+$ and $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda_+$ and $\lambda > \lambda^*$.

For (1.2) with $a > 4$, in (2.3),

$$p_1 = \frac{a^2 - 2a}{2} - \frac{a\sqrt{a^2 - 4a}}{2} < \gamma = \frac{a^2 - 2a}{2} < p_2 = \frac{a^2 - 2a}{2} + \frac{a\sqrt{a^2 - 4a}}{2}. \quad (2.5)$$

Notice that $p_1 > 0$ and $p_1 + p_2 = 2\gamma$. 
Theorem 2.2. (See Fig. 1(i).) Consider (1.2). Then the following assertions (i) and (ii) hold:

(i) If \( a \geq a^* \approx 4.166 \), then the bifurcation curve \( S \) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane. More precisely, there exist two positive numbers \( \lambda_* < \lambda^* \) such that (1.2) has exactly three positive solutions for \( \lambda_* < \lambda < \lambda^* \), exactly two positive solutions for \( \lambda = \lambda_* \) and \( \lambda = \lambda^* \), and exactly one positive solution for \( 0 < \lambda < \lambda_* \) and \( \lambda > \lambda^* \). Moreover, let \( (\lambda^*, \|u_{\lambda^*}\|_\infty) \) and \( (\lambda^*, \|u_{\lambda_*}\|_\infty) \) be the two turning points of the bifurcation curve \( S \) satisfying \( \lambda_* < \lambda^* \) and \( \|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty \), then

\[
\frac{a^2 - 2a}{2} - \frac{a\sqrt{a^2 - 4a}}{2} < \|u_{\lambda^*}\|_\infty < \frac{a^2 - 2a}{2} + \frac{a\sqrt{a^2 - 4a}}{2} < \|u_{\lambda_*}\|_\infty. \tag{2.6}
\]

(ii) If \( a^* > a \geq a^{**} \approx 4.107 \), then the bifurcation curve \( S \) is S-like shaped on the \((\lambda, \|u\|_\infty)\)-plane. More precisely, there exist two positive numbers \( \lambda_* < \lambda^* \) such that (1.2) has at least three positive solutions for \( \lambda_* < \lambda < \lambda^* \), at least two positive solutions for \( \lambda = \lambda_* \) and \( \lambda = \lambda^* \), and exactly one positive solution for \( 0 < \lambda < \lambda_* \) and \( \lambda > \lambda^* \).

Remark 1. In Theorem 2.2(i), assertion (2.6) implies that \( \lim_{a \to \infty} \|u_{\lambda_*}\|_\infty = \infty \) and

\[
\liminf_{a \to \infty} \|u_{\lambda_*}\|_\infty \geq \lim_{a \to \infty} \left( \frac{a^2 - 2a}{2} - \frac{a\sqrt{a^2 - 4a}}{2} \right) = \lim_{a \to \infty} \frac{2a}{a - 2 + \sqrt{a^2 - 4a}} = 1.
\]

3. Proofs of main results

In this paper, to prove our main results (i.e., Theorems 2.1 and 2.2), we modify the time-map techniques used in Smoller and Wasserman [16] and Laetsch [11]. The time-map formula which we apply to study (1.1) takes the form as follows:

\[
\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha \left[ F(u) - F(u) \right]^{-1/2} du = T(\alpha) \quad \text{for } 0 < \alpha < \infty, \tag{3.1}
\]

where \( F(u) \equiv \int_0^u f(t) dt \); see Laetsch [11]. So positive solutions \( u \) of (1.1) correspond to

\[
\|u\|_\infty = \alpha \quad \text{and} \quad T(\alpha) = \sqrt{\lambda}. \tag{3.2}
\]

Thus, studying of the exact number of positive solutions of (1.1) is equivalent to studying the shape of the time map \( T(\alpha) \) on \((0, \infty)\). Also, proving that the bifurcation curve \( S \) is S-shaped (resp. S-like shaped) on the \((\lambda, \|u\|_\infty)\)-plane is equivalent to proving that \( T(\alpha) \) has exactly two (resp. at least two) critical points, a local maximum at some \( \alpha_* \) and a local minimum at some \( \alpha^* > \alpha_* \), on \((0, \infty)\). See Fig. 1(i).

3.1. Proof of Theorem 2.1

The following lemma contains some basic properties of the time map \( T(\alpha) \), which follows from Laetsch [11, Theorems 2.6, 2.9 and 3.2].

Lemma 3.1. Consider (1.1). Assume that \( f \in C^2[0, \infty) \) satisfies (H1)-(H3). Then

\[
\lim_{\alpha \to 0^+} T(\alpha) = 0, \quad \lim_{\alpha \to \infty} T(\alpha) = \infty, \tag{3.3}
\]

and \( T(\alpha) \) has at most one critical point, a local maximum at some \( \alpha_* \), on \((0, \gamma)\).
Fig. 2. Three possible graphs of $\theta(u)$. (i) $\theta(\gamma) \leq 0$. (ii) $\theta(\gamma) > 0 \geq \theta(p_2)$. (iii) $\theta(\gamma) > \theta(p_2) \geq 0$.

For $T(\alpha)$ in (3.1), we compute that

$$T'(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{[F(\alpha) - F(u)]^{3/2}} du,$$

(3.4)

where $\theta(u) = 2F(u) - uf(u) = 2 \int_0^u f(t) dt - uf(u)$ is defined in (1.3).

By (1.3), (2.1) and (H2), we obtain that

$$\theta'(u) = f(u) - uf'(u) \quad \text{for } u \geq 0,$$

(3.5)

$$\theta''(u) = -uf''(u) \begin{cases} < 0 & \text{on } (0, \gamma), \\ = 0 & \text{when } u = \gamma, \\ > 0 & \text{on } (\gamma, \infty) \end{cases}$$

(3.6)

and

$$H'(u) = uf(u) - u^2 f'(u) = u\theta'(u) \quad \text{for } u \geq 0.$$  

(3.7)

In (1.1), if $f$ satisfies (H1)-(H3) and $H(u_0) \leq 0$ for some $u_0 > 0$, then by (1.3) and (3.3)-(3.7), we obtain that $\theta(0) = 0$, $\theta'(0) = f(0) > 0$, and there exist positive numbers $p_1 < \gamma < p_2 < p_3$ such that

$$\theta'(u) \begin{cases} > 0 & \text{on } [0, p_1) \cup (p_2, \infty), \\ = 0 & \text{when } u = p_1, p_2, \\ < 0 & \text{on } (p_1, p_2) \end{cases}$$

(3.8)

and

$$\theta(p_1) = \theta(p_3) > 0.$$  

(3.9)

The graph of $\theta(u)$ can be depicted in Fig. 2.

We thus notice that

$$H'(u) = uf(u) - u^2 f'(u) = u\theta'(u) \begin{cases} > 0 & \text{on } (0, p_1) \cup (p_2, \infty), \\ = 0 & \text{when } u = p_1, p_2, \\ < 0 & \text{on } (p_1, p_2). \end{cases}$$

(3.10)

The following lemma is a key lemma in the proof of Theorem 2.1.
Lemma 3.2. Consider (1.1). Assume that $f \in C^2[0, \infty)$ satisfies (H1)–(H3) and $H(\gamma) \leq 0$. Then $T'(\gamma) < 0$. Moreover, $T(\alpha)$ has exactly one critical point, a local minimum at some $\alpha^*$, on $(\gamma, \infty)$.

Proof. Observe that (3.8)–(3.10) hold by the assumptions. First, we show $T'(\gamma) < 0$.

Case (A). Suppose $\theta(\gamma) \leq 0$ (see Fig. 2(i)). So $\theta(u) > \theta(\gamma)$ for all $u \in (0, \gamma)$ by (3.8), and hence $T'(\gamma) < 0$ by (3.4).

Case (B). Suppose $\theta(\gamma) > 0$ (see Fig. 2(ii)–(iii)). Since $\theta(0) = 0$, $f(u) > 0$ on $(0, \infty)$, and by (3.8), there exists a unique positive number $\tilde{\gamma} < p_1 (< \gamma)$ such that

$$\theta(\gamma) - \theta(u) \begin{cases} > 0 & \text{on } (0, \tilde{\gamma}), \\ = 0 & \text{when } u = \tilde{\gamma}, \\ < 0 & \text{on } (\tilde{\gamma}, \gamma) \end{cases}$$

and

$$F(\gamma) - F(u) = \int_{u}^{\gamma} f(t) \, dt \begin{cases} > F(\gamma) - F(\tilde{\gamma}) & \text{on } (0, \tilde{\gamma}), \\ < F(\gamma) - F(\tilde{\gamma}) & \text{on } (\tilde{\gamma}, \gamma). \end{cases}$$

Thus by (3.4), (3.10) and the assumption $H(\gamma) \leq 0$,

$$T'(\gamma) = \frac{1}{2\sqrt{2} \gamma} \int_{0}^{\gamma} \frac{\theta(\gamma) - \theta(u)}{[F(\gamma) - F(u)]^{3/2}} \, du$$

$$= \frac{1}{2\sqrt{2} \gamma} \int_{0}^{\tilde{\gamma}} \frac{\theta(\gamma) - \theta(u)}{[F(\gamma) - F(u)]^{3/2}} \, du + \frac{1}{2\sqrt{2} \gamma} \int_{\tilde{\gamma}}^{\gamma} \frac{\theta(\gamma) - \theta(u)}{[F(\gamma) - F(u)]^{3/2}} \, du$$

$$< \frac{1}{2\sqrt{2} \gamma} \int_{0}^{\tilde{\gamma}} \frac{\theta(\gamma) - \theta(u)}{[F(\gamma) - F(\tilde{\gamma})]^{3/2}} \, du + \frac{1}{2\sqrt{2} \gamma} \int_{\tilde{\gamma}}^{\gamma} \frac{\theta(\gamma) - \theta(u)}{[F(\gamma) - F(\tilde{\gamma})]^{3/2}} \, du$$

$$= \frac{1}{2\sqrt{2} \gamma \sqrt{[F(\gamma) - F(\tilde{\gamma})]^{3/2}}} \int_{0}^{\gamma} \left[\theta(\gamma) - \theta(u)\right] \, du$$

$$= \frac{1}{2\sqrt{2} \gamma \sqrt{[F(\gamma) - F(\tilde{\gamma})]^{3/2}}} \left[\gamma \theta(\gamma) - \int_{0}^{\gamma} \theta(u) \, du\right]$$

$$= \frac{1}{2\sqrt{2} \gamma \sqrt{[F(\gamma) - F(\tilde{\gamma})]^{3/2}}} \int_{0}^{\gamma} u \theta'(u) \, du$$

$$= \frac{1}{2\sqrt{2} \gamma \sqrt{[F(\gamma) - F(\tilde{\gamma})]^{3/2}}} H(\gamma) \leq 0. \quad (3.11)$$

So $T'(\gamma) < 0$.

We next study the monotonicity and number of critical points of $T(\alpha)$ on $(\gamma, \infty)$. We divide the interval $(\gamma, \infty)$ into three subintervals $(\gamma, p_2]$, $(p_2, p_3)$, $[p_3, \infty)$.
Fig. 3. Two possible graphs of $\phi(u)$. (i) $\phi(\alpha) \geq 0$. (ii) $\phi(\alpha) < 0$.

(I) $\alpha \in (\gamma, p_2]$. For any $\alpha \in (\gamma, p_2]$ (see Fig. 2(i)–(iii)), (3.10) implies

$$H(\alpha) < H(\gamma) \leq 0.$$  

Moreover, if $\theta(\alpha) \leq 0$, then $\theta(u) > \theta(\alpha)$ for all $u \in (0, \alpha)$ by (3.8). So $T'(\alpha) < 0$ by (3.4). If $\theta(\alpha) > 0$, then $H(\alpha) < 0$, and $T'(\alpha) < 0$ by applying the same arguments in (3.11).

(II) $\alpha \in [p_3, \infty)$. For any $\alpha \in [p_3, \infty)$ (see Fig. 2(i)–(iii)), then $\theta(p_1) \leq \theta(\alpha)$ and $\theta(u) < \theta(\alpha)$ for all $u \in (0, p_1) \cup (p_1, \alpha)$ by (3.8) and (3.9). So $T'(\alpha) > 0$ on $[p_3, \infty)$ by (3.4).

By above (I) and (II), to complete the proof, we need only to prove that $T(\alpha)$ has exactly one critical point, a local minimum at some $\alpha^*$, on $(p_2, p_3)$.

(III) $\alpha \in (p_2, p_3)$. By (3.4), we compute that

$$T''(\alpha) = \frac{1}{2\sqrt{2}\alpha^2} \int_0^\alpha \frac{-\frac{3}{2}[\theta(\alpha) - \theta(u)][\phi(\alpha) - \phi(u)]}{[F(\alpha) - F(u)]^{5/2}} du$$

and

$$T''(\alpha) + \frac{2}{\alpha} T'(\alpha) \geq \frac{1}{2\sqrt{2}\alpha^2} \int_0^\alpha \frac{\phi(\alpha) - \phi(u)}{[F(\alpha) - F(u)]^{3/2}} du,$$  

(3.12)

where $\phi(u) = u\theta'(u) - \theta(u)$, see Smoller and Wasserman [16, pp. 272–275]. Then $\phi(0) = 0$, and

$$\phi'(u) = u\theta''(u) \begin{cases} < 0 & \text{on } (0, \gamma), \\ = 0 & \text{when } u = \gamma, \\ > 0 & \text{on } (\gamma, \infty) \end{cases}$$  

(3.13)

by (3.6). The graph of $\phi(u)$ can be depicted in Fig. 3.

Now for any $\alpha \in (p_2, p_3)$, we next prove

$$T''(\alpha) + \frac{2}{\alpha} T'(\alpha) > 0.$$  

(3.14)
Case (A). Suppose $\phi(\alpha) \geq 0$ (see Fig. 3(i)). Then $\phi(u) < \phi(\alpha)$ for all $u \in (0, \alpha)$ by (3.13). So (3.14) holds by (3.12).

Case (B). Suppose $\phi(\alpha) < 0$ (see Fig. 3(ii)). Since $\phi(0) = 0$ and $f(u) > 0$ on $(0, \infty)$, and by (3.13), there exists a unique positive number $\tilde{\alpha} < \gamma < \alpha$ such that

$$\phi(\alpha) - \phi(u) \begin{cases} < 0 & \text{on } (0, \tilde{\alpha}), \\ = 0 & \text{when } u = \tilde{\alpha}, \\ > 0 & \text{on } (\tilde{\alpha}, \alpha), \end{cases}$$

and

$$F(\alpha) - F(u) = \int_{u}^{\alpha} f(t) \, dt \begin{cases} > F(\alpha) - F(\tilde{\alpha}) & \text{on } (0, \tilde{\alpha}), \\ < F(\alpha) - F(\tilde{\alpha}) & \text{on } (\tilde{\alpha}, \alpha). \end{cases}$$

Thus by (3.12),

$$T''(\alpha) + \frac{2}{\alpha} T'(\alpha) \geq \frac{1}{2 \sqrt{2} \alpha^2} \int_{0}^{\alpha} \frac{\phi(\alpha) - \phi(u)}{[F(\alpha) - F(u)]^{3/2}} \, du$$

$$= \frac{1}{2 \sqrt{2} \alpha^2} \int_{0}^{\tilde{\alpha}} \frac{\phi(\alpha) - \phi(u)}{[F(\alpha) - F(u)]^{3/2}} \, du + \frac{1}{2 \sqrt{2} \alpha^2} \int_{\tilde{\alpha}}^{\alpha} \frac{\phi(\alpha) - \phi(u)}{[F(\alpha) - F(u)]^{3/2}} \, du$$

$$> \frac{1}{2 \sqrt{2} \alpha^2} \int_{0}^{\tilde{\alpha}} \frac{\phi(\alpha) - \phi(u)}{[F(\alpha) - F(\tilde{\alpha})]^{3/2}} \, du + \frac{1}{2 \sqrt{2} \alpha^2} \int_{\tilde{\alpha}}^{\alpha} \frac{\phi(\alpha) - \phi(u)}{[F(\alpha) - F(\tilde{\alpha})]^{3/2}} \, du$$

$$= \frac{1}{2 \sqrt{2} \alpha^2 [F(\alpha) - F(\tilde{\alpha})]^{3/2}} \int_{0}^{\alpha} (\phi(\alpha) - \phi(u)) \, du$$

$$= \frac{1}{2 \sqrt{2} \alpha^2 [F(\alpha) - F(\tilde{\alpha})]^{3/2}} \left[ \alpha \phi(\alpha) - \int_{0}^{\alpha} \phi(u) \, du \right]$$

$$= \frac{1}{2 \sqrt{2} \alpha^2 [F(\alpha) - F(\tilde{\alpha})]^{3/2}} \int_{0}^{\alpha} u \phi'(u) \, du. \quad (3.15)$$

Since $p_2 \in (\gamma, \alpha)$, by (3.8), (3.10) and (3.13), we obtain that

$$\int_{0}^{\alpha} u \phi'(u) \, du > \int_{0}^{p_2} u \phi'(u) \, du = \int_{0}^{p_2} u^2 \theta''(u) \, du$$

$$= u^2 \theta'(u)_{u=0}^{p_2} - 2 \int_{0}^{p_2} u \theta'(u) \, du = -2H(p_2) > -2H(\gamma) \geq 0. \quad (3.16)$$

Hence (3.15) and (3.16) imply that
Proof. Observe that (3.8)–(3.10) hold by the assumptions. By (3.10), there exists a critical point, a local minimum, on at least two critical points, a local maximum at some $\alpha$ for any $\alpha \neq 0$.

Proposition 3.4. Note that Proposition 3.4 is also needed in the proof of Theorem 2.2. Proposition 3.4 proves Theorem 1.1; that is, conditions (1.4) and (1.5) can be weakened as (2.2) and (2.4), respectively.

Proof of Theorem 2.1. We first prove part (ii) as follows:

Lemma 3.3. Consider (1.1). Assume $f \in C^2[0, \infty)$ satisfies (H1)–(H3) and $H(u_0) \leq 0$ for some $u_0 > 0$. Then $T(\alpha)$ has exactly one critical point, a local minimum, on $(p_2, p_3)$.

Proof. Observe that (3.8)–(3.10) hold by the assumptions. By (3.10), there exists a $u_1 \in (p_1, p_2]$ satisfying $H(u_1) = H(u_0) \leq 0$. Applying similar arguments in the proof of Lemma 3.2, we obtain that

$$T'(\alpha) < 0 \quad \text{on } [u_1, p_2], \quad T'(\alpha) > 0 \quad \text{on } [p_3, \infty)$$

and

$$T''(\alpha) + \frac{2}{\alpha} T'(\alpha) > 0 \quad \text{on } (p_2, p_3),$$

and hence $T(\alpha)$ has exactly one critical point, a local minimum, on $(p_2, p_3)$. The proof of Lemma 3.3 is complete.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. (i) Suppose $H(\gamma) \leq 0$. Then by Lemmas 3.1 and 3.2, $T'(\gamma) < 0$, $T(\alpha)$ satisfies (3.3) and has exactly two critical points, a local maximum at some $\alpha_* \in (0, \gamma)$ and a local minimum at some $\alpha^* \in (\gamma, \infty)$, on $(0, \infty)$. So we obtain immediately the exact multiplicity result of positive solutions of (1.1) by (3.2) and (3.3). Also (2.3) follows by $T'(p_1) > 0$, $T'(\gamma) < 0$ and $T'(p_2) < 0$. Hence Theorem 2.1(i) holds.

(ii) Suppose $H(u_0) \leq 0$ for some $u_0 > 0$. Then by Lemmas 3.1 and 3.3, $T(\alpha)$ satisfies (3.3) and has at least two critical points, a local maximum at some $\alpha_*$ and a local minimum at some $\alpha^* > \alpha_*$, on $(0, \infty)$. So we obtain immediately the multiplicity result of positive solutions of (1.1) by (3.2) and (3.3). Hence Theorem 2.1(ii) holds.

The proof of Theorem 2.1 is complete.

We end this subsection by giving the following Proposition 3.4 which shows that Theorem 2.1 improves Theorem 1.1; that is, conditions (1.4) and (1.5) can be weakened as (2.2) and (2.4), respectively. Note that Proposition 3.4 is also needed in the proof of Theorem 2.2.

Proposition 3.4. Consider (1.1). Assume $f \in C^2[0, \infty)$ satisfies (H1)–(H3). Then:

(i) If $\theta(\gamma) \leq 0$, then $H(\gamma) < 0$.

(ii) If $\theta(u_0) \leq 0$ for some $u_0 > 0$, then $H(u_1) < 0$ for some $u_1 > 0$.

Proof. We first prove part (ii) as follows:

Observe that (3.8)–(3.10) hold by the assumptions. Also, $u_0 > p_1$ by $\theta(u_0) < 0$ for some $u_0 > 0$ (see Fig. 2(i)–(ii)).

Case (i). $u_0 \leq p_2$. By (3.8) and (3.10),
\[ H(u_0) = \int_0^{u_0} t\theta'(t) \, dt \]
\[ = \int_0^{p_1} t\theta'(t) \, dt + \int_{p_1}^{u_0} t\theta'(t) \, dt \]
\[ < \int_0^{p_1} p_1\theta'(t) \, dt + \int_{p_1}^{u_0} p_1\theta'(t) \, dt \]
\[ = p_1 \int_0^{u_0} \theta'(t) \, dt \]
\[ = p_1 \theta(u_0) \leq 0. \]

We then take \( u_1 = u_0 \) in this Case (I).

Case (II). \( u_0 > p_2 \). There exists \( u_1 \in (p_1, p_2) \) such that \( \theta(u_1) = \theta(u_0) < 0 \). Hence

\[ H(u_1) < p_1 \theta(u_1) \leq 0 \]

by the same arguments in Case (I).

Part (i) follows by the same arguments in the proof of Case (I) of part (ii) as \( \gamma < p_2 \); we omit the proof.

This completes the proof of Proposition 3.4. \( \square \)

### 3.2. Proof of Theorem 2.2

In this subsection we apply Theorem 2.1 to prove Theorem 2.2 for the perturbed Gelfand problem (1.2). Here we always assume

\[ f(u) = \exp \left( \frac{au}{a + u} \right) \quad \text{with } a \geq 4. \]  \( (3.17) \)

So \( f \) satisfies that \( f(0) > 0, f(u) > 0 \) on \( (0, \infty) \),

\[ f'(u) = f(u) \frac{a^2}{(a + u)^2}, \]

and

\[ f''(u) = f(u) \frac{a^2}{(a + u)^4} [a^2 - 2a - 2u] \begin{cases} > 0 & \text{on } [0, \gamma(a)), \\ = 0 & \text{when } u = \gamma(a), \\ < 0 & \text{on } (\gamma(a), \infty), \end{cases} \]  \( (3.18) \)

where

\[ \gamma(a) \equiv \frac{a^2 - 2a}{2} > 0. \]  \( (3.19) \)
By (3.18) and (3.19), we know that $f$ is convex–concave on $(0, \infty)$ for $a \geq 4$. In addition, it is easy to check $f$ is asymptotic sublinear to 0. So, $f = \exp(\frac{a u}{a + u})$ satisfies (H1)–(H3) for $a \geq 4$. For

$$H(u) = 3 \int_0^u tf(t) \, dt - u^2 f(u)$$

in (2.1) with $a \geq 4$, we compute that

$$H'(u) = uf(u) - u^2 f'(u)$$

$$= f(u) \frac{u}{(a + u)^2} \left[ u^2 - (a^2 - 2a)u + a^2 \right]$$

$$\begin{cases} \geq 0 & \text{on } (0, p_1(a)) \cup (p_2(a), \infty), \\ = 0 & \text{when } u = p_1(a), p_2(a), \\ < 0 & \text{on } (p_1(a), p_2(a)). \end{cases}$$

(3.20)

where

$$p_1(a) \equiv \frac{\sqrt{a^2 - 2a} - a}{2} \leq \frac{\sqrt{a^2 - 2a} - a}{2} + \frac{a}{\sqrt{a^2 - 2a} - a} \equiv p_2(a).$$

(3.21)

Notice that, in above, the equality holds only for $a = 4$.

Theorem 2.2 follows by Theorem 2.1, (2.3), (2.5), and the following Lemma 3.5. We prove Lemma 3.5 by modifying the proof of Shivaji [15, Lemma 2.1].

**Lemma 3.5.** Consider (1.2). Then the following assertions (i) and (ii) hold:

(i) There exists a constant $a^* > 4$ satisfying

$$H(\gamma(a^*)) = 0 \quad \text{and} \quad H(\gamma(a)) < 0 \quad \text{for all } a > a^*.$$  

(Numerical simulation shows that $a^* \approx 4.166$.)

(ii) There exists a constant $a^{**} \in (4, a^*)$ satisfying

$$H(p_2(a^{**})) = 0 \quad \text{and} \quad H(p_2(a)) < 0 \quad \text{for all } a > a^{**}.$$  

(Numerical simulation shows that $a^{**} \approx 4.107$.)

**Proof.** For $H(u) = 3 \int_0^u tf(t) \, dt - u^2 f(u)$ with $f(u) = \exp(\frac{a u}{a + u})$, to specify the dependence of function $H(u)$ on the parameter $a$, in this proof, we sometimes write $H(u) = H(a, u)$.

First, we show that $\frac{\partial}{\partial a} H(p_2(a)) < 0$ for $a \geq 4$.

By (2.1), (3.17), and (3.20), $H(p_2(a)) = H(a, p_2(a))$, $\frac{\partial}{\partial u} H(a, p_2(a)) = 0$. In addition, we compute that

$$\frac{\partial}{\partial a} H(p_2(a)) = \frac{\partial}{\partial a} H(a, p_2(a)) + \frac{\partial}{\partial u} H(a, p_2(a)) \frac{dp_2(a)}{da}$$

$$= \frac{\partial}{\partial a} H(a, p_2(a))$$

$$= \left[ \frac{\partial}{\partial a} \left( 3 \int_0^u \frac{at}{a + t} \, dt - u^2 \exp(\frac{a u}{a + u}) \right) \right]_{u=p_2(a)}"
\[
= \left[ 3 \int_0^u tf(t) \frac{t^2}{(a+t)^2} \, dt - u^2 f(u) \frac{u^2}{(a+u)^2} \right]_{u=p_2(a)} \\
= \left[ 3 \int_0^u t \tilde{f}(t) \, dt - u^2 \tilde{f}(u) \right]_{u=p_2(a)} \\
= \tilde{H}(p_2(a)),
\]
(3.24)

where
\[
\tilde{H}(u) \equiv 3 \int_0^u t \tilde{f}(t) \, dt - u^2 \tilde{f}(u) 
\]
(3.25)

and
\[
\tilde{f}(u) \equiv f(u) \frac{u^2}{(a+u)^2}.
\]

We compute that
\[
\tilde{H}(0) = 0 
\]
and
\[
\tilde{H}(u) < 0 \text{ for } 0 < u < \frac{a}{2} (a + \sqrt{a^2 + 4}).
\]
(3.26)

Observe that
\[
0 < p_2(a) = \frac{a}{2} (a - 2 + \sqrt{a^2 - 4a}) < \frac{a}{2} (a + \sqrt{a^2 + 4}) \text{ for } a \geq 4.
\]
(3.27)

Thus by (3.24), (3.26) and (3.27),
\[ \frac{\partial}{\partial a} H(p_2(a)) = \dot{H}(p_2(a)) < 0 \quad \text{for } a \geq 4, \quad (3.28) \]

and hence \( H(p_2(a)) \) is a strictly decreasing function of \( a \geq 4 \).

Similarly, by (3.17), (3.19)–(3.20), and (3.24)–(3.26), we obtain that \( H(\gamma(a)) = H(a, \gamma(a)) \), \( \frac{\partial}{\partial a} H(a, \gamma(a)) \leq 0 \), and

\[
\frac{\partial}{\partial a} H(\gamma(a)) = \frac{\partial}{\partial a} H(a, \gamma(a)) + \frac{\partial}{\partial u} H(a, \gamma(a)) \frac{d\gamma(a)}{da} \\
= \frac{\partial}{\partial a} H(a, \gamma(a)) + \frac{\partial}{\partial u} H(a, \gamma(a))(a - 1) \\
\leq \frac{\partial}{\partial a} H(a, \gamma(a)) \\
= H(\gamma(a)) < 0 \quad \text{for } a \geq 4. \quad (3.29)
\]

So \( H(\gamma(a)) \) is also a strictly decreasing function of \( a \geq 4 \).

It is easy to check \( \theta(\gamma(a)) < 0 \) for large \( a \), cf. Korman and Li [9, Lemma 3.1]. Then by (3.20) and Proposition 3.4(i),

\[ H(p_2(a)) < H(\gamma(a)) < \theta(\gamma(a)) < 0 \quad \text{for large } a. \quad (3.30) \]

Moreover,

\[ H(p_1(4)) = H(\gamma(4)) = H(p_2(4)) > 0 \quad (3.31) \]

by (3.19)–(3.21). So, (3.28)–(3.31) imply that there exist two constants \( a^* > a^{**} > 4 \) satisfying (3.22) and (3.23), separately.

The proof of Lemma 3.5 is now complete. \( \Box \)

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**References**