Exact multiplicity results for a \( p \)-Laplacian problem with concave–convex–concave nonlinearities

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Received 18 October 2001; accepted 11 March 2002

Abstract

We study the exact number of positive solutions of a two-point Dirichlet boundary-value problem involving the \( p \)-Laplacian operator. We consider the case \( p = 2 \) as well as the case \( p > 1 \), when the nonlinearity \( f \) satisfies \( f(0) = 0 \) and has two distinct simple positive zeros and such that \( f'' \) changes sign exactly twice on \((0, \infty)\). Note that we may allow that \( f'' \) changes sign more than twice on \((0, \infty)\). Some interesting examples of quartic polynomials are given. In particular, for \( f(u) = -u^2(u-1)(u-2) \), we study the evolution of the bifurcation curves of the \( p \)-Laplacian problem as \( p \) increases from 1 to infinity, and hence are able to determine the exact multiplicity of positive solutions for each \( p > 1 \).

Keywords: Exact multiplicity result; \( p \)-Laplacian; Concave–convex–concave nonlinearity; Positive solution; Dead-core solution; Bifurcation; Time-map

1. Introduction

In this paper we present exact multiplicity results of positive solutions for the nonlinear two-point Dirichlet boundary-value problem

\[
-(\varphi_p(u'(x)))' = \lambda f(u(x)), \quad -1 < x < 1,
\]

\[
u(-1) = u(1) = 0, \quad (1.1)
\]

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\textsuperscript{1}Work partially supported by the National Science Council, Republic of China.
where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$ and $(\varphi_p(u'))'$ is the one-dimensional $p$-Laplacian, $\lambda > 0$ and $f \in C^2[0, \infty)$ satisfies $f(0) = 0$ and has two distinct simple positive zeros $b < c$. For $p = 2$, $(\varphi_p(u'))' = u''$, and problem (1.1) reduces to

$$-u''(x) = \lambda f(u(x)), \quad -1 < x < 1,$$

$$u(-1) = u(1) = 0.$$  \hspace{1cm} (1.2)

When the nonlinearity $f(u) = -u(u - b)(u - c)$ is a cubic polynomial satisfying $0 < b < c/2$, Smoller and Wasserman [14] first gave exact multiplicity results of (classical) positive solutions of problem (1.2) by applying the time-map analysis. For problem (1.2), they showed the existence of a critical $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ problem (1.2) has no positive solution, it has exactly one positive solution for $\lambda = \lambda_0$, and exactly two positive solutions for $\lambda > \lambda_0$. Consequently, Wang and Kazarinoff [16] studied problem (1.2) when $f$ is a cubic-like nonlinearity. Let

$$F(u) = \int_0^u f(t) \, dt.$$

Theorem 1.1 (Wang and Kazarinoff [16, Theorem 1 and Remark 2]). Suppose $f \in C^2[0, \infty)$ and there exist $0 < b < c$ such that the following conditions are satisfied:

$$f(0) = f(b) = f(c) = 0,$$  \hspace{1cm} (1.3)

$$f(u) < 0 \quad \text{for} \quad u \in (0, b) \cup (c, \infty) \quad \text{and} \quad f(u) > 0 \quad \text{for} \quad u \in (b, c),$$  \hspace{1cm} (1.4)

$$\int_0^c f(u) \, du > 0,$$  \hspace{1cm} (1.5)

there exists $r \in (0, c)$ such that $f''(u) > 0$ for $0 < u < r$

and $f''(u) < 0$ for $r < u < \infty$.  \hspace{1cm} (1.6)

Then there exists $\lambda_0 > 0$ such that

(a) for $0 < \lambda < \lambda_0$, problem (1.2) has no positive solution,
(b) for $\lambda = \lambda_0$, problem (1.2) has exactly one positive solution $u_1$ satisfying $\beta < \|u_1\| < c$,
(c) for $\lambda > \lambda_0$, problem (1.2) has exactly two positive solutions $u_1 < u_2$ satisfying $\beta < \|u_1\| < \|u_2\| < c$.

Remark 1. If $f \in C[0, \infty)$ satisfies (1.3)–(1.5), then it can be shown that

(i) By the maximum principle, every classical positive solution $u$ of (1.2) satisfies $\beta < \|u\|_\infty < c$. 

(ii) Any two distinct positive solutions of (1.2) are strictly ordered. That is, let \( u \) and \( \hat{u} \) be any two distinct positive solutions of (1.2) with \( 0 < \|u\|_\infty < \|\hat{u}\|_\infty \), then \( u < \hat{u} \), see e.g. [16, Lemma 1].

For \( f \) a cubic-like nonlinearity and for problem (1.2), the same exact multiplicity results when \( f'' \) changes sign exactly once were proved by Korman et al. [6, Theorem 2.3]. Recently, Korman and Shi [10, Theorem 5] proved an exact multiplicity result which generalizes theorem [6, Theorem 2.3] by slightly weakening the convexity assumptions on \( f \). We also note that Korman et al. [7] and Ouyang and Shi [12,13] have extended similar results on an \( n \)-dimensional ball (\( n \geq 2 \)), see [7,12,13] for details. The main tool used in [6,7,10,12,13] is a bifurcation theorem of Crandall and Rabinowitz [3]. We note that the case where \( f'' \) changes sign exactly twice has not been studied yet.

Other related results are available in the literature; see for instance, Korman [5], Korman and Ouyang [8,9], and Maya and Shivaji [11].

For problem (1.1) with \( p \neq 2 \), when \( f \) is a cubic-like nonlinearity satisfying (1.3)–(1.5), little is known. We refer to Addou [2] in which \( f'' \) is assumed to change sign exactly once.

2. Classification of positive solutions and the time-map method

To state the main results, we first give a classification of positive solutions of problem (1.1) in \( C^1[-1,1] \). By a positive solution to problem (1.1) we mean a positive function \( u \in C^1[-1,1] \) with \( \varphi_p(u') \in C^1[-1,1] \) satisfying (1.1). Let

\[ Z = \{ x \in [-1,1] : u'(x) = 0 \}. \]

We note that it is easy to show that, if \( f \in C \) and \( u \) is a positive solution of problem (1.1), then \( u \in C^2[-1,1] \) if \( 1 < p \leq 2 \) and \( u \in C^2([-1,1]-Z) \) if \( p > 2 \). For the proof we refer to Addou [1, Lemma 6].

Let \( A^+_0 \) be the subset of \( C^1[-1,1] \) consisting of the functions \( u \) satisfying

(i) \( u(x) > 0 \) for all \( x \in (-1,1) \), \( u(-1) = u(1) = 0 < u'(-1) \).
(ii) \( u \) is symmetrical with respect to 0.
(iii) The derivative of \( u \) vanishes once and only once.

Let \( A^+_1 \) be the subset of \( C^1[-1,1] \) consisting of the functions \( u \) satisfying

(i) \( u(x) > 0 \) for all \( x \in (-1,1) \), \( u(-1) = u(1) = 0 < u'(-1) \).
(ii) \( u \) is symmetrical with respect to 0.
(iii) There exists \( k \in (0,1) \) such that for all \( x \in (-1,1) \), \( u'(x) = 0 \) if and only if \( -k \leq x \leq k \).

Note that if a solution \( u \in A^+_1 \), then it is usually called a dead-core solution of problem (1.1).
Let $B_0^+$ be the subset of $C^1[-1,1]$ consisting of the functions $u$ satisfying

(i) $u(x) > 0$ for all $x \in (-1,1)$, $u(-1) = u(1) = 0 = u'(-1)$.
(ii) $u$ is symmetrical with respect to 0.
(iii) The derivative of $u$ vanishes once and only once.

It is easy to derive an energy relation of solutions $u$ of problem (1.1), see e.g. [4, p. 421]. Denote by $p' = p/(p-1)$ the conjugate exponent of $p$.

**Lemma 2.1** (Energy relation). Let $p > 1$ and assume that $u$ is a positive solution of problem (1.1), then

$$(|u'(x)|^p + p'\lambda F(u(x)))' = 0 \quad \text{for all } x \in [-1,1].$$

**Remark 2.** By Lemma 2.1, it can be shown that if $u$ is a positive solution of problem (1.1), then $u$ is symmetrical with respect to 0. In addition, $u \in A_0^+ \cup A_1^+ \cup B_0^+$ if $f$ satisfies (1.3)–(1.5) for some numbers $0 < b < c$. We omit the proof, see e.g. [2, Lemma 6].

To study problem (1.1), we make use of the time-map method (quadrature method). Suppose $f \in C[0,\infty)$ satisfies (1.3)–(1.5) for some numbers $0 < b < c$. For any $E \geq 0$ and $s > 0$, let $G(E,s) := E^p - p'\lambda F(s)$ and

$$X(E) = \{s > 0 : s \in \text{dom } G(E, \cdot) \text{ and } G(E,u) > 0 \text{ for all } u \in (0,s)\},$$

$$r(E) = \begin{cases} 0 & \text{if } X(E) = \emptyset, \\ \sup(X(E)) & \text{otherwise.} \end{cases}$$

Let

$$\hat{D} = \{E \geq 0 : r(E) \in \text{dom } G(E, \cdot), G(E,r(E)) = 0 \text{ and } \int_0^{r(E)} (E^p - p'\lambda F(t))^{-1/p} dt < \infty \}.$$

Define the time-map

$$T(E) = \int_0^{r(E)} (E^p - p'\lambda F(t))^{-1/p} dt, \quad E \in \hat{D}.$$

By Lemma 2.1 and arguments in [4], we have the following theorem. Note that in this paper, by Remark 2, we restrict ourself on positive solutions $u \in A_0^+ \cup A_1^+ \cup B_0^+$.

**Theorem 2.2.** Consider problem (1.1). Suppose $f \in C[0,\infty)$ satisfies (1.3)–(1.5) for some numbers $0 < b < c$. Let $E \geq 0$. Then $T$ is a continuous function of $E \in \hat{D}$.
Moreover,

(i) Problem (1.1) has a solution \(u \in A^+\) satisfying \(u(-1) = E > 0\) if and only if \(E \in \hat{D} - \{0\}\) and \(T(E) = 1\), and in this case the solution is unique.

(ii) Problem (1.1) has a solution \(u \in A^+_1\) satisfying \(u(-1) = E > 0\) if and only if \(E \in \hat{D} - \{0\}\) and \(T(E) < 1\), and in this case the solution is unique.

(iii) Problem (1.1) has a solution \(u \in B^+_0\) satisfying \(u(-1) = 0\) if and only if \(0 \in \hat{D}\) and \(T(0) = 1\), and in this case the solution is unique.

**Remark 3.** In practice, we first study the variations of the real-valued function \(G(E, \cdot)\), then compute \(X(E)\) and deduce \(r(E)\). Next, we compute \(\hat{D}\). For this, we first compute the set

\[ D = \{ E > 0 : r(E) \in \text{dom } G(E, \cdot), G(E, r(E)) = 0 \text{ and } f(r(E)) > 0 \} \]

and then we deduce \(\hat{D}\) by observing that \(D \subset \hat{D} - \{0\} \subset \hat{D}\); we omit the proof. (Note that \(\hat{D}\) is the closure of \(D\).) After that, we define the time-map \(T\) on \(\hat{D}\) and then compute its limits at the boundary points of \(\hat{D}\). Next, we study the variations of \(T\) on \(\hat{D}\). We achieve our study by discussing the number of solutions to

(i) Equation \(T(E) = 1\) for \(E \in \hat{D} - \{0\}\) in case of looking for solutions in \(A^+_0\).

(ii) Inequality \(T(E) < 1\) for \(E \in \hat{D} - \{0\}\) in case of looking for solutions in \(A^+_1\).

(iii) Equation \(T(0) = 1\) in case where \(0 \in \hat{D}\) and when we are looking for solutions in \(B^+_0\).

3. **Main results**

We determine the exact multiplicity of positive solutions of problem (1.1) for each \(\lambda > 0\). We assume that \(f\) satisfies condition (f4) below, which implies for the particular case \(p = 2\), that \(f''\) changes sign exactly twice on \((0, \infty)\), i.e., \(f\) is concave–convex–concave on \((0, \infty)\). Note that we may allow that \(f''\) changes sign more than twice on \((0, \infty)\), i.e., we may allow that \(f\) is concave–convex–concave–convex on \((0, \infty)\); see Remark 6.

For \(f\), recalling that \(F(u) = \int_0^u f(t) \, dt\), we let

\[ \theta_p(u) := pF(u) - uf(u), \]

\[ \Psi_p(u) := u\theta_p(u) - \theta_p(u) = pf(u) + u^2 f'(u) - pF(u), \]

\[ v_p := \left\{ \int_0^c (F(c) - F(u)')^{-1/p} \, du \right\}^{1/p} / p' \in (0, \infty], \]

\[ \lambda_p := \left\{ \int_0^\beta (F(\beta) - F(u))^{-1/p} \, du \right\}^{1/p} / p' \in (0, \infty], \]

\( (3.1) \)

\( (3.2) \)

\( (3.3) \)
$$\mu_p := \inf_{\beta \leq \xi \leq c} \left\{ \int_0^\xi (F(\xi) - F(u))^{-1/p} \, du \right\}^{p/p'},$$  \hspace{1cm} (3.4)$$

where $0 < \beta < c$ are defined behind. We shall show that $0 < \mu_p < \infty$ for $p > 1$. For all $\lambda > 0$, we denote $S_\lambda$ the positive solution set of (1.1).

For fixed $p > 1$, suppose $f \in C^2[0, \infty)$ and there exist $0 < b < c$ such that the following conditions are satisfied:

(f1) $f(0) = 0$,

(f2) $f(u) < 0$ for $0 < u < b$,

(f3) $f(u) > 0$ for $b < u < c$,

(f4) $f(u) < 0$ for $u > c$,

(f5) $\int_0^c f(u) \, du > 0$ (then there exists a unique $\beta \in (b, c)$ satisfying $\int_0^\beta f(u) \, du = 0$),

(f6) there exist $0 < r_p < s_p < c$ such that

$$(p - 2)f'(u) - uf''(u) > 0 \text{ for } 0 < u < r_p,$$

$$(p - 2)f'(u) - uf''(u) < 0 \text{ for } r_p < u < s_p,$$

$$(p - 2)f'(u) - uf''(u) > 0 \text{ for } s_p < u < c$$

(it can be weakened later, see Remark 6),

(f5) there exists a unique $\sigma_p \in (s_p, c)$ satisfying $(p - 1)f(\sigma_p) - \sigma_pf'(\sigma_p) = 0$ and such that $\Psi_p(\sigma_p) \geq \Psi_p(r_p)$.

**Remark 4.** The geometric meaning of (f5) is as follows: On the $xy$-plane, consider the graph of $y = \theta_p(x)$, by (3.1), the $y$-intercept of the tangent line through the point $(r_p, \theta_p(r_p))$ is greater than or equal to the $y$-intercept of the tangent line through the point $(\sigma_p, \theta_p(\sigma_p))$.

**Remark 5** (cf. Remark 1). If $f \in C[0, \infty)$ satisfies (f1)--(f3), then by applying Lemma 2.1, it can be shown that

(i) Every positive solution $u$ of problem (1.1) satisfies $\beta \leq \|u\|_\infty \leq c$.

(ii) Any two distinct positive solutions of problem (1.1) are strictly ordered. That is, let $u$ and $\hat{u}$ be any two distinct positive solutions of problem (1.1) with $0 < \|u\|_\infty < \|\hat{u}\|_\infty$, then $u < \hat{u}$.

The next theorem gives complete description of the set $S_\lambda$ for $p$ and $f$ satisfying certain conditions.
Theorem 3.1 (S₁, see Fig. 1). Assume that \( p > 1 \) and \( f \in C^2[0, \infty) \) satisfies (f1)–(f5).

(a) If \( 1 < p \leq 2 \), then:

(i) For \( 0 < \lambda < \mu_p \), \( S_\lambda = \emptyset \).

(ii) For \( \lambda = \mu_p \), there exists \( u_\lambda \in A^+_0 \) such that \( S_\lambda = \{u_\lambda\} \). Moreover, \( \beta < \|u_\lambda\|_\infty < c \).

(iii) For \( \lambda > \mu_p \), there exist \( u_\lambda, v_\lambda \in A^+_0 \) such that \( u_\lambda < v_\lambda \) and \( S_\lambda = \{u_\lambda, v_\lambda\} \). Moreover, \( \beta < \|u_\lambda\|_\infty < \|v_\lambda\|_\infty < c \).

(b) If \( f'(0) = \cdots = f^{(m-1)}(0) = 0 \) and \( f^{(m)}(0) < 0 \) for some integer \( m \geq 2 \) (we allow \( m = \infty \)), and \( 2 < p \leq m + 1 \) then \( 0 < \mu_p < v_p < \infty \). Moreover:

(iv) For \( 0 < \lambda < \mu_p \), \( S_\lambda = \emptyset \).

(v) For \( \lambda = \mu_p \), there exists \( u_\lambda \in A^+_0 \) such that \( S_\lambda = \{u_\lambda\} \). Moreover, \( \beta < \|u_\lambda\|_\infty < c \).

(vi) For \( \mu_p < \lambda < v_p \), there exist \( u_\lambda, v_\lambda \in A^+_0 \) such that \( u_\lambda < v_\lambda \) and \( S_\lambda = \{u_\lambda, v_\lambda\} \). Moreover, \( \beta < \|u_\lambda\|_\infty < \|v_\lambda\|_\infty \leq c \) and \( \|v_\lambda\|_\infty = c \) if and only if \( \lambda = v_p \).

(vii) For \( \lambda > v_p \), there exist \( u_\lambda \in A^+_0 \) and \( v_\lambda \in A^+_1 \) such that \( u_\lambda < v_\lambda \) and \( S_\lambda = \{u_\lambda, v_\lambda\} \). Moreover, \( \beta < \|u_\lambda\|_\infty < \|v_\lambda\|_\infty = c \).

In what follows, we consider the remaining cases:

(C1) \( f'(0) < 0 \) and \( p > 2 \).

(C2) \( f'(0) = \cdots = f^{(m-1)}(0) = 0 \) and \( f^{(m)}(0) < 0 \) for some integer \( m \geq 2 \), and \( p > m + 1 \).

By Remark 2, \( S_\lambda \subset A^+_0 \cup A^+_1 \cup B^+_0 \), then

\[ S_\lambda = (S_\lambda \cap A^+_0) \cup (S_\lambda \cap A^+_1) \cup (S_\lambda \cap B^+_0). \]

Theorem 3.2 (resp. Theorems 3.3 and 3.4) gives complete description of the set \( S_\lambda \cap A^+_0 \) (resp. \( S_\lambda \cap A^+_1, S_\lambda \cap B^+_0 \)). We shall show that, for \( p > 2 \), \( 0 < \mu_p < \lambda_p < \infty \) and \( 0 < \mu_p < v_p \), see Lemmas 4.7 and 4.5.
Fig. 2. Bifurcation diagrams of problem (1.1) with \( f(0) = 0, \ p > 2, \) and \( f \) satisfying either (C1) or (C2).

(a) \( \mu_p < v_p < \lambda_p; \) (b) \( \mu_p < v_p = \lambda_p; \) (c) \( \mu_p < \lambda_p < v_p. \)

**Theorem 3.2** \((S_{\lambda} \cap A_{0}^+, \text{ see Fig. } 2). \) Assume that \( p > 2. \) In addition to (f1)–(f5), suppose \( f \in C^2[0,\infty) \) satisfies either (C1) or (C2) above. Then \( 0 < \mu_p < \lambda_p < \infty \) and \( 0 < \mu_p < v_p < \infty, \) and each eventual solution \( u_{\lambda} \) of (1.1) in \( A_{0}^+ \) satisfies \( \beta < \|u_{\lambda}\|_{\infty} \leq c. \) Moreover,

(a) If \( \mu_p < v_p < \lambda_p, \) then:

(i) For \( 0 < \lambda < \mu_p, \) \( S_{\lambda} \cap A_{0}^+ = \emptyset. \)

(ii) For \( \lambda = \mu_p, \) there exists \( u_{\lambda} \in A_{0}^+ \) such that \( S_{\lambda} \cap A_{0}^+ = \{u_{\lambda}\}. \) Moreover, \( \beta < \|u_{\lambda}\|_{\infty} < c. \)

(iii) For \( \mu_p < \lambda \leq v_p, \) there exist \( u_{\lambda}, \ v_{\lambda} \in A_{0}^+ \) such that \( u_{\lambda} < v_{\lambda} \) and \( S_{\lambda} \cap A_{0}^+ = \{u_{\lambda}, v_{\lambda}\}. \) Moreover, \( \beta < \|u_{\lambda}\|_{\infty} < \|v_{\lambda}\|_{\infty} \leq c, \) and \( \|v_{\lambda}\|_{\infty} = c \) if and only if \( \lambda = v_p. \)

(iv) For \( v_p < \lambda \leq \lambda_p, \) there exists \( u_{\lambda} \in A_{0}^+ \) such that \( S_{\lambda} \cap A_{0}^+ = \{u_{\lambda}\}. \) Moreover, \( \beta < \|u_{\lambda}\|_{\infty} < c. \)

(v) For \( \lambda > \lambda_p, S_{\lambda} \cap A_{0}^+ = \emptyset. \)

(b) If \( \mu_p < v_p = \lambda_p, \) then:

(vi) For \( 0 < \lambda < \mu_p, \) \( S_{\lambda} \cap A_{0}^+ = \emptyset. \)

(vii) For \( \lambda = \mu_p, \) there exists \( u_{\lambda} \in A_{0}^+ \) such that \( S_{\lambda} \cap A_{0}^+ = \{u_{\lambda}\}. \) Moreover, \( \beta < \|u_{\lambda}\|_{\infty} < c. \)

(viii) For \( \mu_p < \lambda < \lambda_p = v_p, \) there exist \( u_{\lambda}, \ v_{\lambda} \in A_{0}^+ \) such that \( u_{\lambda} < v_{\lambda} \) and \( S_{\lambda} \cap A_{0}^+ = \{u_{\lambda}, v_{\lambda}\}. \) Moreover, \( \beta < \|u_{\lambda}\|_{\infty} < \|v_{\lambda}\|_{\infty} < c. \)

(ix) For \( \lambda = \lambda_p = v_p, \) there exists \( u_{\lambda} \in A_{0}^+ \) such that \( S_{\lambda} \cap A_{0}^+ = \{u_{\lambda}\}. \) Moreover, \( \|u_{\lambda}\|_{\infty} = c. \)

(x) For \( \lambda > \lambda_p, S_{\lambda} \cap A_{0}^+ = \emptyset. \)

(c) If \( \mu_p < \lambda_p < v_p, \) then:

(xi) For \( 0 < \lambda < \mu_p, S_{\lambda} \cap A_{0}^+ = \emptyset. \)

(xii) For \( \lambda = \mu_p, \) there exists \( u_{\lambda} \in A_{0}^+ \) such that \( S_{\lambda} \cap A_{0}^+ = \{u_{\lambda}\}. \) Moreover, \( \beta < \|u_{\lambda}\|_{\infty} < c. \)

(xiii) For \( \mu_p < \lambda < \lambda_p, \) there exist \( u_{\lambda}, \ v_{\lambda} \in A_{0}^+ \) such that \( u_{\lambda} < v_{\lambda} \) and \( S_{\lambda} \cap A_{0}^+ = \{u_{\lambda}, v_{\lambda}\}. \) Moreover, \( \beta < \|u_{\lambda}\|_{\infty} < \|v_{\lambda}\|_{\infty} < c. \)
(xiv) For $\lambda_p \leq \lambda \leq \nu_p$, there exists $u_\lambda \in A_0^+$ such that $S_{\lambda} \cap A_0^+ = \{u_\lambda\}$. Moreover, $\beta < \|u_{\lambda}\|_{\infty} \leq c$, and $\|u_{\lambda}\|_{\infty} = c$ if and only if $\lambda = \nu_p$.

(xv) For $\lambda > \nu_p$, $S_{\lambda} \cap A_0^+ = \emptyset$.

**Theorem 3.3** ($S_{\lambda} \cap A_1^+$, see Fig. 2). Assume that $p > 2$. In addition to (f1)–(f5), suppose $f \in C[2][0, \infty)$ satisfies either (C1) or (C2) above. Then each eventual solution $u_\lambda$ of (1.1) in $A_1^+$ satisfies $\|u_{\lambda}\|_{\infty} = c$. Moreover,

(i) For $0 < \lambda \leq \nu_p$, $S_{\lambda} \cap A_1^+ = \emptyset$.

(ii) For $\lambda > \nu_p$, there exists $u_\lambda \in A_1^+$ such that $S_{\lambda} \cap A_1^+ = \{u_\lambda\}$.

**Theorem 3.4** ($S_{\lambda} \cap B_0^+$, see Fig. 2). Assume that $p > 2$. In addition to (f1)–(f5), suppose $f \in C[2][0, \infty)$ satisfies either (C1) or (C2) above. Then each eventual solution $u_\lambda$ of (1.1) in $B_0^+$ satisfies $\|u_{\lambda}\|_{\infty} = \beta$. Moreover,

(i) For $\lambda = \lambda_p$, there exists $u_\lambda \in B_0^+$ such that $S_{\lambda} \cap B_0^+ = \{u_\lambda\}$.

(ii) For $0 < \lambda \neq \lambda_p$, $S_{\lambda} \cap B_0^+ = \emptyset$.

We note that, for $p > 2$ and for each $\lambda > \lambda_p$, it can be proved that there do not exist dead-core solutions $u_\lambda$ of problem (1.1) satisfying $\|u_{\lambda}\|_{\infty} = \beta$ since $f(\beta) > 0$ and by applying Lemma 2.1.

**Remark 6.** As one can check the proof of Theorems 3.1–3.4, the “convexity” assumption of $\theta_p(u) = pF(u) - uf(u)$ on $(0, c)$ in (f4) in Theorems 3.1–3.4 can actually be weakened as follows:

(f4') There exist $0 \leq r_p < s_p < c$ such that

$$(p-2)f'(u) - uf''(u) > 0 \quad \text{for } 0 < u < r_p \quad \text{(it is not necessary if } r_p = 0),$$

$$(p-2)f'(u) - uf''(u) < 0 \quad \text{for } r_p < u < s_p,$$

$$(p-2)f'(u) - uf''(u) > 0 \quad \text{for } s_p < u < c \quad \text{(it can be weakened below)}.$$ 

See Section 3.1 for examples. We note that in (f4') if $r_p = 0$ then (f5) is automatically satisfied since it can be easily shown that, for $p > 1$, there exists a unique $\sigma_p \in (s_p, c)$ satisfying $(p-1)f(\sigma_p) - \sigma pf'(\sigma_p) = 0$ and such that $\Psi_p(\sigma_p) > 0 = \Psi_p(r_p)$. We also note that in (f4') the assumption

$$(p-2)f'(u) - uf''(u) > 0 \quad \text{for } s_p < u < c$$

can actually be weakened as

$$\theta_p'(u) = (p-1)f(u) - uf'(u) \begin{cases} \leq 0 & \text{for } s_p < u < \sigma_p, \\ \geq 0 & \text{for } d \leq u \leq c, \end{cases} \quad (3.5)$$

$$\theta_p''(u) = (p-2)f'(u) - uf''(u) \geq 0 \quad \text{for } \sigma_p \leq u < d, \quad (3.6)$$
where $d \in (\sigma_p, c]$ is defined by

$$d := \begin{cases} 
    c, & \text{if } \theta_p(c) \leq \theta_p(t_p), \\
    \inf \{u \in (\sigma_p, c] : \theta_p(\xi) > \theta_p(t_p) \text{ for all } \xi \in (u, c]\}, & \text{otherwise,}
\end{cases} \quad (3.7)$$

where $t_p$ is the unique zero of $\theta_p'(u)$ on $(r_p, s_p)$.

We summarize that (f4′) can be weakened as follows:

(f4′′) There exist $0 < r_p < s_p < \sigma_p < d < c$ such that

$$\begin{align*}
    (p - 2)f'(u) - uf''(u) > 0 & \quad \text{for } 0 < u < r_p \quad \text{(it is not necessary if } r_p = 0), \\
    (p - 2)f'(u) - uf''(u) < 0 & \quad \text{for } r_p < u < s_p, \\
    \theta_p'(u) = (p - 1)f(u) - uf'(u) \begin{cases} 
        \leq 0 & \text{for } s_p < u < \sigma_p, \\
        \geq 0 & \text{for } d \leq u \leq c,
    \end{cases} \\
    \theta_p''(u) = (p - 2)f''(u) - uf''(u) \geq 0 & \quad \text{for } \sigma_p \leq u < d,
\end{align*}$$

where $d$ is defined in (3.7).

Note that if (3.5) and (3.6) are satisfied instead of the third condition in (f4) and (f4′) then the result in Lemma 4.2 stated behind remains valid; that is, $f''(c) < 0$. Also note that if $d < c$ then $S_p'(x) > 0$ for $d < x < c$ and Lemma 4.6 stated behind should read as follows:

$$S_p''(x) + \frac{p - 1}{p} S_p'(x) > 0 \quad \text{for all } x \in (\max\{\sigma_p, \beta\}, d) \quad \text{and } \quad p > 1.$$

An example to Remark 6. Let $p = 2$ and $f = f_1(u) = u^2(u - 1)(u - 3)^2$. The function $f_1$ satisfies $f_1(0) = 0$ and $f_1''(u) = 20u^3 - 84u^2 + 90u - 18$ changes sign exactly thrice on $(0, \infty)$ with its zeros at

$$r \approx 0.259, \quad s_1 \approx 1.336, \quad s_2 \approx 2.605.$$ 

It can be easily checked that $f_1$ satisfies (f1)–(f5) in Theorem 3.1 except (f4). Nevertheless, it can be easily checked that $f_1$ satisfies (f4′′). Thus, for $p = 2$, the bifurcation diagram of problem (1.1) is depicted in Fig. 1(a).

3.1. Some examples

For the sake of simplicity, we mainly restrict ourselves to the case $p = 2$ and first give two examples of quartic polynomials to Theorems 3.1 and 1.1. In Theorem 3.5, $f(u) = -u^2(u - b)(u - c)$ satisfies $f(0) = f'(0) = 0$, $f''(0) = -2bc < 0$ and $f''$ changes sign exactly twice on $(0, \infty)$. In Theorem 3.6, $f(u) = -u(u - \hat{a})(u - b)(u - c)$ satisfies $f(0) = 0$, $f'(0) < 0$, and $f''$ changes sign exactly twice on $(0, \infty)$ if $\hat{a}b + bc + c\hat{a} = \frac{1}{2} f''(0) > 0$,

$$f''$$ changes sign exactly once on $(0, \infty)$ if $\hat{a}b + bc + c\hat{a} = \frac{1}{2} f''(0) \leq 0.$
Theorem 3.5. Let $p = 2$. Let $f(u) = -u^2(u - b)(u - c)$ satisfy
\[ c > \frac{5}{3} b > 0 \quad \left( \iff \int_0^c f(u) \, du > 0 \right). \tag{3.8} \]

Then $f$ satisfies (f1)–(f5) in Theorem 3.1. Thus the bifurcation diagram of problem (1.1) is depicted in Fig. 1(a).

An example to Theorems 3.5, 3.1, 3.2 and Remark 6. We study the evolution of the bifurcation curves as $p$ increases from 1 to infinity for $f = f_2(u) = -u^2(u-1)(u-2) = -2u^2 + 3u^3 - u^4$ satisfying $f_2(0) = f'_2(0) = 0$ and $f''_2(0) = -4$. First, for $p = 2$, $f_2$ satisfies condition (3.8) in Theorem 3.5 and hence (f1)–(f5) in Theorem 3.1. While for all $p > 1$, it can be easily checked that
(i) For $1 < p < 3$, $f_2$ satisfies
\[
0 < r_p = \frac{36 - 9p - \sqrt{336 - 136p + 17p^2}}{40 - 8p} < s_p = \frac{36 - 9p + \sqrt{336 - 136p + 17p^2}}{40 - 8p} < \sigma_p = \frac{12 - 3p + \sqrt{24 - 8p + p^2}}{10 - 2p} < c = 2,
\]
we omit the proof. Thus the bifurcation diagram of problem (1.1) is depicted in Fig. 1(a) for $1 < p \leq 2$, and in Fig. 1(b) for $2 < p < 3$.

(ii) For $p = 3$, $f_2$ satisfies (f1)–(f3), (f5) and (f4') with the definition of $r_p = 0$ (since the function $f''_2(u) - uf'_2(u)$ has exactly one positive zero on $(0, 2)$) and $0 < s_p = \frac{9}{8} < \sigma_p = \frac{3}{2} < c = 2$. Thus the bifurcation diagram of problem (1.1) is depicted in Fig. 1(b) for $p = 3$.

(iii) For $p > 3$, $f_2$ satisfies (f1)–(f3). Moreover, it can be easily shown that:

(A) For $3 < p \leq 10$, $f_2$ satisfies (f4') with the definition of $r_p = 0$ (since the function $f''_2(u) - uf'_2(u)$ has exactly one positive zero on $(0, 2)$); we omit the proof. In this case (f5) is automatically satisfied, see Remark 6.

(B) For $p > 10$, $f_2$ satisfies (f4'') with the definition of $r_p = 0$ (since the function $f''_2(u) - uf'_2(u)$ has exactly one positive zero on $(0, 2)$); we omit the proof. In this case (f5) is automatically satisfied, see Remark 6.

When $p > 3$, we compute that
\[
0 < s_p = \frac{36 - 9p + \sqrt{336 - 136p + 17p^2}}{40 - 8p} < \sigma_p = \frac{12 - 3p + \sqrt{24 - 8p + p^2}}{10 - 2p} < c = 2.
\]
Numerical simulation (see Fig. 3) shows that there exists a number $p^* \approx 7.649$ such that

(a) for $3 < p < p^*$, $0 < \nu_p < \lambda_p < \infty$. So the bifurcation diagram of problem (1.1) is depicted in Fig. 2(a).

(b) for $p = p^*$, $0 < \nu_p = \lambda_p < \infty$. So the bifurcation diagram of problem (1.1) is depicted in Fig. 2(b).

(c) for $p^* < p \leq 10$, $0 < \lambda_p < \nu_p < \infty$. So the bifurcation diagram of problem (1.1) is depicted in Fig. 2(c).

It follows immediately by the above analysis that, for all $p > 1$, problem (1.1) has at most two positive solutions for each $\lambda > 0$.

**Theorem 3.6.** Let $p = 2$. Let $f(u) = -u(u - \hat{a})(u - b)(u - c)$ satisfy

\[
\hat{a} < 0 < b < c
\]  

and

\[
c > \frac{1}{6} (5\hat{a} + 5b + \sqrt{(5\hat{a} + 5b)^2 - 120\hat{a}b}) \left( \leftrightarrow \int_{0}^{c} f(u) \, du > 0 \right).
\]  

Then

(i) If $\hat{a} b + bc + c\hat{a} = -\frac{1}{2} q''(0) \leq 0$, then $f$ satisfies all conditions (1.3)–(1.6) in Theorem 1.1.

(ii) If $\hat{a} b + bc + c\hat{a} = -\frac{1}{2} q''(0) > 0$, then $f$ satisfies all conditions (f1)–(f5) in Theorem 3.1.

Therefore, in either case (i) or (ii), the bifurcation diagram of problem (1.1) is depicted in Fig. 1(a).
An example to Theorem 3.6. Let $p = 2$. The function $f = f_3(u) = -u(u + 1/2)(u - 1)$ $(u - 3)$ satisfies conditions (3.9) and (3.10) in Theorem 3.6 with $f''_3(0) = -2 < 0$.

4. Proofs of main results

First, we have the next lemma which follows easily for function $f \in C[0, \infty)$ satisfying (f1)–(f3). We omit the proof.

**Lemma 4.1.** Assume that $f \in C[0, \infty)$ satisfies (f1)–(f3). Consider the function defined on $(0, \infty)$ by

$$s \mapsto G(\lambda, E, s) := E^p - p' F(s),$$

where $p > 1$, $E \geq 0$ and $\lambda > 0$ are real parameters. Then

(i) If $E > E_c := (p' F(c))^{1/p} > 0$, the function $G(\lambda, E, \cdot)$ is strictly positive on $(0, \infty)$.

(ii) If $E = E_c$, the function $G(\lambda, E_c, \cdot)$ is strictly positive on $(0, c)$ and vanishes at $c$.

(iii) If $0 < E < E_c$, the function $G(\lambda, E, \cdot)$ has on the open interval $(\beta, c)$ a unique zero $\hat{s}(\lambda, E)$ and is strictly positive on the open interval $(0, \hat{s}(\lambda, E))$.

Moreover,

(a) The function $E \mapsto \hat{s}(\lambda, E)$ is $C^1$ on $(0, E_c)$ and,

$$\frac{\partial \hat{s}}{\partial E}(\lambda, E) = \frac{(p - 1)E^{p-1}}{\hat{\lambda} f(\hat{s}(\lambda, E))} > 0 \quad \text{for all } E \in (0, E_c).$$

(b) $\lim_{E \to 0^+} \hat{s}(\lambda, E) = \beta$ and $\lim_{E \to E_c^-} \hat{s}(\lambda, E) = c$.

(iv) If $E = 0$, the function $G(\lambda, 0, s)$

$$\begin{cases} > 0 & \text{for } 0 < s < \beta, \\ = 0 & \text{for } s = \beta, \\ < 0 & \text{for } \beta < s \leq c. \end{cases}$$

Now, for $p > 1$, $\lambda > 0$ and $E \geq 0$, we let

$$X(\lambda, E) := \{s \in \text{dom } G(\lambda, E, \cdot) = (0, \infty) : G(\lambda, E, u) > 0 \text{ for all } u \in (0, s)\}$$

$$= \begin{cases} (0, \infty) & \text{if } E > E_c, \\ (0, c] & \text{if } E = E_c, \\ (0, \hat{s}(\lambda, E)] & \text{if } 0 < E < E_c, \\ (0, \beta] & \text{if } E = 0, \end{cases}$$
by Lemma 4.1. Therefore,

\[ r(\lambda, E) := \sup X(\lambda, E) = \begin{cases} 
\infty & \text{if } E > E_c, \\
\beta & \text{if } E = 0,
\end{cases} \]

\[ \hat{s}(\lambda, E) \] if \( 0 < E < E_c, \]

\[ \hat{s}(\lambda, E) \] if \( 0 < E < E_c, \]

\[ D_p(\lambda) := \{ E > 0 : r(\lambda, E) \in \text{dom } G(\lambda, E, \cdot) = (0, \infty), \]

\[ G(\lambda, E, r(\lambda, E)) = 0 \text{ and } f(r(\lambda, E)) > 0 \} \]

\[ = (0, E_c). \]

Notice that the definition domain of the time-map \( T_p \) is given by

\[ \tilde{D}_p(\lambda) := \left\{ E \geq 0 : r(\lambda, E) \in \text{dom } G(\lambda, E, \cdot) = (0, \infty), G(\lambda, E, r(\lambda, E)) = 0, \right\} \]

and

\[ \int_0^{r(\lambda, E)} (G(\lambda, E, u))^{-1/p} \, du < \infty. \]

In the present case, \((0, E_c) \subset \tilde{D}_p(\lambda) \subset [0, E_c]. \) We define, for \( E \in \tilde{D}_p(\lambda), \) the time-map

\[ T_p(\lambda, E) := \int_0^{r(\lambda, E)} (G(\lambda, E, u))^{-1/p} \, du \]

\[ = \int_0^{r(\lambda, E)} (E^p - p'\lambda F(u))^{-1/p} \, du \]

\[ = (p'\lambda)^{-1/p} \int_0^{r(\lambda, E)} (F(\lambda, E)) - F(u))^{-1/p} \, du, \]

since \( G(\lambda, E, r(\lambda, E)) = E^p - p'\lambda F(r(\lambda, E)) = 0. \) For \( p > 1, \) we define

\[ S_p(x) := \int_0^{x} (F(x) - F(u))^{-1/p} \, du \quad \text{for all } x \in [\beta, c]. \quad (4.1) \]

Thus our time-map may be written as

\[ T_p(\lambda, E) = (p'\lambda)^{-1/p} S_p(r(\lambda, E)). \]

We observe that (f2) implies that \( f'(c) \leq 0. \) Actually, it can be proved that \( f'(c) < 0 \) by using (f2) and (f4). We omit the proof.

**Lemma 4.2.** Suppose \( f \in C^2[0, \infty) \) satisfies (f1), (f2) and (f4). Then \( f'(c) < 0. \)

**Lemma 4.3.** \( S_p(c) = \infty \) if and only if \( 1 < p \leq 2. \)
Proof. By Lemma 4.2, \( f'(c) < 0 \). Then
\[
(F(c) - F(u))^{-1/p} \approx \left( \frac{-f'(c)}{2} \right)^{-1/p} (c - u)^{-2/p} \quad \text{near } c^-.
\]
Hence, by (4.1), easy computation shows that \( S_p(c) = \infty \) if and only if \( 1 < p \leq 2 \).

Lemma 4.4. (i) If \( f'(0) < 0 \), then \( S_p(\beta) = \infty \) if and only if \( 1 < p \leq 2 \).
(ii) If \( f'(0) = 0 \) and \( f''(0) < 0 \), then \( S_p(\beta) = \infty \) if and only if \( 1 < p \leq 3 \).
(iii) If \( f'(0) = \cdots = f^{(m-1)}(0) = 0 \) and \( f^{(m)}(0) < 0 \) for some integer \( m \geq 2 \), then \( S_p(\beta) = \infty \) if and only if \( 1 < p \leq m + 1 \).
(iv) If \( f^{(k)}(0) = 0 \) for any integer \( k = 1, 2, \ldots \), then \( S_p(\beta) = \infty \) for all \( p > 1 \).

Proof. By (4.1), the integral which represents \( S_p(\beta) \) presents two singularities, at 0 and \( \beta \). By the fact that \( f(\beta) > 0 \), it follows \( (F(\beta) - F(u))^{-1/p} \approx (-f(u))^{-1/p}(\beta - u)^{-1/p} \) near \( \beta^+ \). Since \( p > 1 \), it follows that \( \int_{\beta-\epsilon}^{\beta} (\beta - u)^{-1/p} du < \infty \) for any \( \epsilon > 0 \), small enough. Therefore \( \int_{\beta-\epsilon}^{\beta} (F(\beta) - F(u))^{-1/p} du < \infty \).

We study the singularity at 0 in the following:
(i) Assume that \( f''(0) < 0 \). Since \( F(\beta) = F(0) = 0 \), by using l’Hôpital’s rule twice, we obtain
\[
(F(\beta) - F(u))^{-1/p} = (-F(u))^{-1/p} = \left( \frac{-F(u)}{u^2} \right)^{-1/p} u^{-2/p}
\]
\[
\approx \left( \frac{-f''(0)}{2} \right)^{-1/p} u^{-2/p} \quad \text{near } 0^+.
\]
Hence, by (4.1), easy computation shows that \( S_p(\beta) = \infty \) if and only if \( 1 < p \leq 2 \).
(ii) Assume that \( f''(0) = 0 \) and \( f'''(0) < 0 \). Similarly, using l’Hôpital’s rule three times, we obtain
\[
(F(\beta) - F(u))^{-1/p} = (-F(u))^{-1/p} = \left( \frac{-F(u)}{u^3} \right)^{-1/p} u^{-3/p}
\]
\[
\approx \left( \frac{-f'''(0)}{6} \right)^{-1/p} u^{-3/p} \quad \text{near } 0^+.
\]
Hence, by (4.1), easy computation shows that \( S_p(\beta) = \infty \) if and only if \( 1 < p \leq 3 \).
(iii) Suppose \( f''(0) = \cdots = f^{(m-1)}(0) = 0 \) and \( f^{(m)}(0) < 0 \) for some integer \( m \geq 2 \). By applying similar arguments in above, we can show that \( S_p(\beta) = \infty \) if and only if \( 1 < p \leq m + 1 \).
(iii) Suppose \( f^{(k)}(0) = 0 \) for any integer \( k = 1, 2, \ldots \). Let \( p > 1 \). By applying l’Hôpital’s rule \( p \) times it follows:
\[
\lim_{u \to 0^+} \frac{-F(u)}{u^p} = \lim_{u \to 0^+} \frac{-f(u)}{pu^{p-1}} = \lim_{u \to 0^+} \frac{-f'(u)}{p(p-1)u^{p-2}} = \cdots = \lim_{u \to 0^+} \frac{-f^{(p-1)}(u)}{p!} = 0
\]
then there exists \( \varepsilon = \varepsilon_p > 0 \) such that \( 0 < -F(u) < u^p \) for \( 0 < u < \varepsilon_p \). Since \( F(\beta) = F(0) = 0 \),

\[
(F(\beta) - F(u))^{-\frac{1}{p}} > u^{-1} \quad \text{for} \quad 0 < u < \varepsilon_p.
\]

Hence, by (4.1), easy computation shows that \( S_p(\beta) = \infty \) for all \( p > 1 \).

The proof of Lemma 4.4 is complete. \( \Box \)

Now, after computing the limits of the time-map \( T_p(\lambda, \cdot) \), we study its exact variations in the interior of its definition domain; that is, \((0, E_c)\). Notice that, to study the variations of \( T_p(\lambda, \cdot) \) in \((0, E_c)\), it suffices to study those of \( S_p(\cdot) \) in the interior of the range of \( r(\lambda, \cdot) \); that is, \((\beta, c)\). One has

\[
S_p'(x) = \frac{1}{p^2} \int_0^x \frac{\theta_p(x) - \theta_p(u)}{(F(x) - F(u))^{\frac{1}{p}}} \, du \quad \text{for} \quad x \in (\beta, c), \tag{4.2}
\]

where

\[
\theta_p(u) := pF(u) - uf(u) \tag{4.3}
\]

for \( u \in [0, c], \ p > 1 \). By (4.3), it follows that

\[
\begin{align*}
\theta_p'(u) &= (p - 1)f(u) - uf'(u), \\
\theta_p''(u) &= (p - 2)f'(u) - uf''(u). \tag{4.4}
\end{align*}
\]

Thus by (f1) and (f4),

\[
\theta_p(0) = 0, \tag{4.6}
\]

\[
\begin{align*}
\theta_p'(0) &= (p - 1)f(0) = 0, \\
\theta_p''(u) &> 0 \quad \text{for} \quad 0 < u < r_p, \\
\theta_p''(u) &< 0 \quad \text{for} \quad r_p < u < s_p, \\
\theta_p''(u) &> 0 \quad \text{for} \quad s_p < u < c.
\end{align*}
\]

In addition, by (f3), (f2) and Lemma 4.2,

\[
\begin{align*}
\theta_p(\beta) &= -\beta f(\beta) < 0, \\
\theta_p(c) &= pF(c) - cf(c) = pF(c) > 0, \\
\theta_p'(c) &= (p - 1)f(c) - cf'(c) = -cf'(c) > 0.
\end{align*}
\]

Hence there exist \( t_p \in (r_p, s_p) \) and \( \sigma_p \in (s_p, c) \) such that

\[
\begin{align*}
\theta_p &\text{ is strictly increasing on } (0, t_p), \tag{4.7} \\
\theta_p &\text{ is strictly decreasing on } (t_p, \sigma_p), \tag{4.8} \\
\theta_p &\text{ is strictly increasing on } (\sigma_p, c). \tag{4.9}
\end{align*}
\]
In addition, there exist \( \delta_p \in (t_p, \sigma_p) \) and \( \gamma_p \in (\sigma_p, c) \) such that

\[
\theta_p(\delta_p) = \theta_p(\gamma_p) = 0.
\]  

(4.10)

The typical graph of \( \theta_p(u) \) on \([0, c]\) is depicted in Fig. 4.

**Lemma 4.5.** Assume that \( p > 1 \). Then

(i) If \( p > 2 \) then \( S'_p(c) = \infty \).

(ii) If \( \beta < \sigma_p \) then \( S'_p(\alpha) < 0 \) for all \( \alpha \in (\beta, \sigma_p] \).

(iii) If \( \beta = \sigma_p \) and \( S_p(\beta) < \infty \) then \( -\infty \leq S'_p(\beta) < 0 \).

(iv) If \( \beta > \sigma_p \) and \( S_p(\beta) < \infty \) then \( S'_p(\beta) = -\infty \).

**Proof.** Recall (4.2),

\[
S'_p(\alpha) = \frac{1}{p\alpha} \int_0^\alpha \frac{\theta_p(\alpha) - \theta_p(u)}{(F(\alpha) - F(u))^{1/p}} du \quad \text{for} \quad \alpha \in [\beta, c].
\]

(i) Assume that \( p > 2 \). By Lemma 4.2, \( f'(c) < 0 \). This implies that

\[
\frac{\theta_p(c) - \theta_p(u)}{(F(c) - F(u))^{1+1/p}} \approx \frac{2^{1+1/p}c(-f'(c))^{-1/p}}{(c - u)^{1+2/p}} \quad \text{near} \quad c^-.
\]

Thus easy computation shows that \( S'_p(c) = \infty \).

(ii) Assume that \( (\delta_p < \beta < \sigma_p) \). By (4.6)–(4.8) and (4.10), it follows that \( \theta_p(\alpha) - \theta_p(u) < 0 \) for all \( \alpha \in (\beta, \sigma_p] \) and for all \( u \in (0, \alpha) \). So \( S'_p(\alpha) < 0 \) for all \( \alpha \in (\beta, \sigma_p] \).

(iii) If \( \beta = \sigma_p \), then \( \theta_p(\beta) - \theta_p(u) = \theta_p(\sigma_p) - \theta_p(u) < 0 \) for all \( u \in (0, \beta) \). So \( -\infty \leq S'_p(\beta) < 0 \).

(iv) Since \( F(\beta) = F(0) = 0 \), it follows that the integral representing \( S'_p(\beta) \) has two singularities; one at 0 and the other at \( \beta \). So we write

\[
S'_p(\beta) = \frac{1}{p\beta} (I_0 + I_\beta),
\]
where
\[ I_0 := \int_0^{\delta_p} \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{1+1/p}} \, du \quad \text{and} \quad I_{\beta} := \int_{\delta_p}^{\beta} \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{1+1/p}} \, du. \]

We next show \( I_{\beta} < \infty \) and \( I_0 = -\infty \).
First we show \( I_{\beta} < \infty \). By \((c >) \beta > \sigma_p\), it follows that \( \theta'(\beta) > 0 \) and therefore,
\[ \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{1+1/p}} \approx \frac{\theta_p(\beta)}{(f(\beta))^{1+1/p}} \frac{1}{(\beta - u)^{1/p}} \quad \text{near } \beta^- . \]
Since \( 1/p < 1 \) and \( \theta'(\beta)(f(\beta))^{-1-1/p} > 0 \), easy computation shows that \( I_{\beta} < \infty \).
Secondly we show \( I_0 = -\infty \). We distinguish two cases:
(\text{I}) Assume that \( f'(0) < 0 \). Note that
\[ \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{1+1/p}} \approx \frac{\theta_p(\beta)}{(-f'(0)/2)^{1+1/p}} \frac{1}{u^{3(1+1/p)}} \quad \text{near } 0^+. \]
Since \( 2(1 + 1/p) > 1 \) and \( \theta_p(\beta)(-f'(0)/2)^{-1-1/p} < 0 \), easy computation shows that \( I_0 = -\infty \).
(\text{II}) Assume that \( f'(0) = 0 \). It is easy to see that \( f''(0) \leq 0 \) by (f2). Using l’Hôpital’s rule three times, it follows that
\[ \lim_{u \to 0^+} F(u)/u^3 = f''(0)/6 \leq 0. \]

(a) Assume that \( f''(0) < 0 \). Note that
\[ \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{1+1/p}} \approx \frac{\theta_p(\beta)}{(-f''(0)/6)^{1+1/p}} \frac{1}{u^{3(1+1/p)}} \quad \text{near } 0^+. \]
Since \( 3(1 + 1/p) > 1 \) and \( \theta_p(\beta)(-f''(0)/6)^{-1-1/p} < 0 \), easy computation shows that \( I_0 = -\infty \).
(b) Assume that \( f''(0) = 0 \). Note that \( \lim_{u \to 0^+} (-F(u)/u^3)^{-1-1/p} = \infty \), then there exist \( \hat{K} > 0 \) and \( 0 < \hat{\varepsilon} < \delta_p \) such that
\[ \left( -\frac{F(u)}{u^3} \right)^{1-1/p} \frac{1}{u^{3(1+1/p)}} \geq \hat{K} \frac{1}{u^{3(1+1/p)}} \quad \text{for } 0 < u < \hat{\varepsilon}. \]
On the other hand, \( \theta_p(\beta) - \theta_p(u) < 0 \) for \( 0 < u < \hat{\varepsilon} < \delta_p \), then
\[ \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{1+1/p}} = \left( -\frac{F(u)}{u^3} \right)^{-1-1/p} \frac{\theta_p(\beta) - \theta_p(u)}{u^{3(1+1/p)}} \leq \hat{K} \frac{\theta_p(\beta) - \theta_p(u)}{u^{3(1+1/p)}} \quad \text{for } 0 < u < \hat{\varepsilon}. \]
and
\[ \frac{\theta_p(\beta) - \theta_p(u)}{u^{3(1+1/p)}} \approx \frac{\theta_p(\beta)}{u^{3(1+1/p)}} \quad \text{near } 0^+. \]

Since \(3(1 + 1/p) > 1\) and \(\theta_p(\beta) < 0\), it follows that
\[ \int_0^\varepsilon \frac{\theta_p(\beta) - \theta_p(u)}{u^{3(1+1/p)}} \, du = -\infty, \]

which implies that \(I_0 = -\infty\).

The proof of Lemma 4.5 is complete. \(\square\)

**Lemma 4.6.** For \(p > 1\), \(S''_p(\alpha) + (p/\alpha)S'_p(\alpha) > 0\) for all \(\alpha \in (\max\{\sigma_p, \beta\}, c)\).

**Proof.** Easy computations show that for all \(\alpha \in (\beta, c)\),
\[ S''_p(\alpha) = \frac{p + 1}{p \alpha^2} \int_0^\alpha \frac{(\theta_p(\alpha) - \theta_p(u))^2}{(F(\alpha) - F(u))^{2+1/p}} \, du + \frac{1}{p \alpha^2} \int_0^\alpha \frac{\Phi_p(\alpha) - \Phi_p(u)}{(F(\alpha) - F(u))^{1+1/p}} \, du, \]
where
\[ \Phi_p(u) := -p(p + 1)F(u) + 2puF(u) - u^2f'(u) \]
for \(u \in (0, c)\). Then
\[ S''_p(\alpha) + \frac{p}{\alpha} S'_p(\alpha) = \frac{1}{p \alpha^2} \int_0^\alpha \frac{\Psi_p(\alpha) - \Psi_p(u)}{(F(\alpha) - F(u))^{1+1/p}} \, du \]
\[ + \frac{p + 1}{p \alpha^2} \int_0^\alpha \frac{(\theta_p(\alpha) - \theta_p(u))^2}{(F(\alpha) - F(u))^{2+1/p}} \, du \]
for all \(\alpha \in (\beta, c)\), where
\[ \Psi_p(u) := \Phi_p(u) + p\theta_p(u) = u\theta'_p(u) - \theta_p(u) \]
for all \(u \in (0, c)\). Hence
\[ S''_p(\alpha) + \frac{p}{\alpha} S'_p(\alpha) > \frac{1}{p \alpha^2} \int_0^\alpha \frac{\Psi_p(\alpha) - \Psi_p(u)}{(F(\alpha) - F(u))^{1+1/p}} \, du. \quad (4.11) \]

Note that \(\Psi'_p(u) = u\theta''_p(u)\). Therefore, by (f4), it follows that \(\Psi_p\) is strictly increasing on \([0, r_p]\), is strictly decreasing on \((r_p, s_p)\) and is strictly increasing on \((s_p, c)\). On the other hand,
\[ \Psi_p(0) = 0 \cdot \theta'_p(0) - \theta_p(0) = 0, \]
\[ \Psi'_p(0) = 0 \cdot \theta''_p(0) = 0, \]
\[ \Psi_p(\sigma_p) = \sigma_p \theta'_p(\sigma_p) - \theta_p(\sigma_p) = -\theta_p(\sigma_p) > 0. \]
Therefore, $\Psi_p(r_p) > 0$ and by (f5) it follows that $\Psi_p(z) - \Psi_p(u) > 0$ for all $u \in (0, z)$ and all $z \in (\max\{\sigma_p, \beta\}, c)$. The typical graph of $\Psi_p(u)$ on $[0, c]$ is depicted in Fig. 5. Therefore, the integral in (4.11) is strictly positive for all $z \in (\max\{\sigma_p, \beta\}, c)$, which proves Lemma 4.6.

**Lemma 4.7.** For $p > 1$, $S_p(z)$ has exactly one critical point, a minimum, on $(\beta, c)$. More precisely, there exists a unique $m_p \in (\beta, c)$ such that $S_p(z)$ is strictly decreasing on $(\beta, m_p)$ and is strictly increasing on $(m_p, c)$.

**Proof.** We prove existence and uniqueness of $m_p$ separately.

**Uniqueness of $m_p$:** We distinguish two cases:

(i) $\beta < \sigma_p$. By Lemma 4.6, the uniqueness of $m_p$ in $(\sigma_p, c)$ follows, and by Lemma 4.5(ii), the non-existence of $m_p$ in $(\beta, \sigma_p]$ follows. So uniqueness of $m_p$ in $(\beta, c)$ follows.

(ii) $\beta \geq \sigma_p$. The uniqueness of $m_p$ in $(\beta, c)$ follows by Lemma 4.6 immediately.

**Existence of $m_p$:** We distinguish two cases:

(i) $1 < p \leq 2$. By Lemmas 4.3 and 4.4, it follows that $S_p(\beta) = S_p(c) = \infty$. So the existence of $m_p$ follows.

(ii) $p > 2$. By Lemmas 4.3 and 4.5(i), it follows that $S_p(c) < \infty$ and $S_p' (c) = \infty$. On the other hand, there are many cases to be considered for $S_p(\beta)$. However, by Lemmas 4.4 and 4.5, it follows that in all these cases, either $S_p(\beta) = \infty$ or $(S_p(\beta) < \infty$ and $S_p(z)$ is strictly decreasing on a right neighborhood of $\beta)$. So the existence of $m_p$ follows in all cases.

The proof of Lemma 4.7 is complete. 

Let $u$ be a positive solution of problem (1.1), by Lemma 2.1,

(i) if $\beta < \|u\|_{\infty} \leq c$ then $u'(-1) = (p'\lambda F(\|u\|_{\infty}))^{1/p} > 0$, and hence $u \in A_0^+ \cup A_1^+$,
(ii) if \( \|u\|_\infty = \beta \) then \( u'(-1) = (p' \lambda F(\|u\|_\infty))^{1/p} = (p' \lambda F(\beta))^{1/p} = 0 \), and hence \( u \in B_0^+ \).

By Lemma 4.3, \( S_p(c) < \infty \) for \( p > 2 \). In this case, suppose for \( \lambda = \nu_p = (S_p(c))^{1/p} \), \( u_{\gamma_p} \) is the corresponding solution of problem (1.1) satisfying \( \|u_{\nu_p}\|_\infty = u_{\nu_p}(0) = c \), \( u_{\gamma_p}'(-1) = (p' \nu_p F(c))^{1/p} > 0 \), and \( u_{\gamma_p} \in A_0^+ \). Then for each \( \gamma > \nu_p \),

\[
u_p((\gamma_p)^{1/p}(|x| - 1 + (\frac{\gamma_p}{2})^{1/p})), \quad \text{if } 1 - (\frac{\gamma_p}{2})^{1/p} < |x| \leq 1,
\]

\[
u_p((\frac{\gamma_p}{2})^{1/p}(|x| - 1 + (\frac{\gamma_p}{2})^{1/p})), \quad \text{if } |x| \leq 1 - (\frac{\gamma_p}{2})^{1/p}
\]

is a \( C^1 \) dead-core solution of problem (1.1) satisfying \( \|u_{\gamma}\|_\infty = c \),

\[
u_p((\gamma_p)^{1/p}(|x| - 1 + (\frac{\gamma_p}{2})^{1/p})), \quad \text{if } 1 - (\frac{\gamma_p}{2})^{1/p} < |x| \leq 1,
\]

\[
u_p((\frac{\gamma_p}{2})^{1/p}(|x| - 1 + (\frac{\gamma_p}{2})^{1/p})), \quad \text{if } |x| \leq 1 - (\frac{\gamma_p}{2})^{1/p}
\]

and \( u_{\gamma} \in A_1^+ \).

Hence Theorems 3.1–3.4 follow immediately by Theorem 2.2 and Lemmas 4.1–4.4 and 4.5–4.7.

### 4.1. Proofs of the examples

The proofs of Theorems 3.5 and 3.6 are quite similar.

**Proof of Theorem 3.5.** It is easy to see that \( f(u) = -u^2(u - b)(u - c) \) for \( c > \frac{5}{3}b > 0 \) satisfies (f1), (f2) and (f4) with its two inflection points at

\[
r = \frac{1}{4}(b + c - \sqrt{(b + c)^2 - \frac{8}{3}bc}) > 0, \quad s = \frac{1}{4}(b + c + \sqrt{(b + c)^2 - \frac{8}{3}bc}) < c,
\]

since \( f''(0) = -2bc < 0 \). Also \( f \) satisfies (f3) since

\[
\int_0^c f(u) \, du = \frac{1}{20}c^4(c - \frac{5}{3}b) > 0.
\]

Finally, for (f5), it can be shown that there exists a unique number

\[
\sigma = \frac{1}{3}(b + c + \sqrt{b^2 + c^2 - bc})
\]

satisfying \( f(u) - uf'(u) = u^2(3u^2 - 2(b + c)u + bc) = 0 \) on \((b, c)\). We then compute that

\[
207360(\Psi(\sigma) - \Psi(r)) = 256(b + c + \sqrt{b^2 + c^2 - bc})^3(b^2 + c^2 - 2bc + (b + c)\sqrt{b^2 + c^2 - bc})
\]

\[
+ 243(b + c - \sqrt{b^2 + c^2 - \frac{2}{5}bc})^3(b^2 + c^2 - \frac{14}{9}bc)
\]

\[
- (b + c)\sqrt{b^2 + c^2 - \frac{2}{5}bc}).
\]
To estimate the above expression we need the following lemma which follows easily by (3.8). We omit the proof.

**Lemma 4.8.** Assume (3.8). Then

(i) \( b^2 + c^2 - \frac{14}{9} bc > b^2 + c^2 - 2bc > \frac{4}{25} c^2, \)

(ii) \( \frac{2}{5} c < \sqrt{b^2 + c^2 - bc} < \frac{\sqrt{34}}{5} c, \)

(iii) \( \frac{2}{5} c < \sqrt{b^2 + c^2 - \frac{2}{3} bc} < \frac{\sqrt{34}}{5} c. \)

Now by Lemma 4.8(i) and (ii),

\[
\begin{align*}
&b^2 + c^2 - 2bc + (b + c)\sqrt{b + c - bc} > \frac{4}{25} c^2 + (b + c)\left(\frac{2}{5} c\right) > 0, \\
&256(b + c + \sqrt{b^2 + c^2 - bc})^3 > 256(b + c + \frac{2}{5} c)^3 > 0.
\end{align*}
\]

Also, by Lemma 4.8(iii),

\[
0 < 243(b + c - \sqrt{b^2 + c^2 - \frac{2}{3} bc})^3 < 243(b + c - \frac{2}{3} c)^3.
\]

Since \( b, c > 0, \) we obtain

\[
(0 <) \frac{b^2 + c^2 - \frac{14}{9} bc}{\sqrt{b^2 + c^2 - \frac{2}{3} bc}} < \sqrt{b^2 + c^2 - \frac{2}{3} bc} < b + c
\]

and hence

\[
(b^2 + c^2 - \frac{14}{9} bc) - (b + c)\sqrt{b^2 + c^2 - \frac{2}{3} bc} < 0.
\]

Then by Lemma 4.8(i) and (iii), it follows that

\[
\left( \frac{4}{25} c^2 - (b + c)\frac{\sqrt{34}}{5} c \right) < (b^2 + c^2 - \frac{14}{9} bc)
\]

\[
- (b + c)\sqrt{b^2 + c^2 - \frac{2}{3} bc} < 0.
\]

Let

\[
1 < x := \frac{b + c}{c} < \frac{3}{5} c + c = \frac{8}{5}.
\]

Then by (4.12)–(4.16), we compute that

\[
207360(\Psi(\sigma) - \Psi(r)) > 256(b + c + \frac{2}{5} c)^3\left( \frac{4}{25} c^2 + (b + c)\left(\frac{2}{5} c\right) \right)
\]

\[
+ 243(b + c - \frac{2}{3} c)^3\left( \frac{4}{25} c^2 - (b + c)\frac{\sqrt{34}}{5} c \right)
\]
\[
\begin{align*}
&= c^5 \{ 256(x + 2 \cdot 2)^3 (\frac{4}{25} + 2 \cdot 2x) + 243(x - 2 \cdot 2)^3 (\frac{4}{25} - \sqrt{34}x) \} \\
&= c^5 \{ \frac{312}{375} (5x + 2)^4 + \frac{243}{25} (x - 2 \cdot 2)^3 (4 - 5\sqrt{34}x) \}
\end{align*}
\]
for \(1 < x < \frac{5}{2}\); we omit the proof of the last inequality. Therefore \((f5)\) follows. This completes the proof of Theorem 3.5. \(\square\)

**Proof of Theorem 3.6.** To prove Theorem 3.6, we need the following lemma.

**Lemma 4.9.** If, in addition to (3.9) and (3.10), \(f(u) = -u(u - \hat{a})(u - b)(u - c)\) satisfies
\[
\hat{a}b + bc + c\hat{a} > 0.
\]
then:

(i) \(0 < -\hat{a} < \frac{1}{\sqrt{2}} c\).

(ii) \(0 < b < \frac{5}{2} c\).

(iii) \(\hat{a} + b + c > 0\).

(iv) \(\hat{a}^2 + b^2 + c^2 - 2(\hat{a}b + bc + c\hat{a}) > \frac{4}{25} c^2\).

(v) \(\hat{a} + b + c - \sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{5}(\hat{a}b + bc + c\hat{a})} > 0\).

(vi) \([\hat{a}^2 + b^2 + c^2 - \frac{14}{9}(\hat{a}b + bc + c\hat{a})] - [(\hat{a} + b + c)\sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{5}(\hat{a}b + bc + c\hat{a})}] < 0\).

(vii) \(\hat{a}^2 + b^2 + c^2 - (\hat{a}b + bc + c\hat{a}) > \hat{a}^2 + b^2 + c^2 - \frac{14}{9}(\hat{a}b + bc + c\hat{a}) > \frac{4}{25} c^2\).

(viii) \(\frac{3}{5} c < \sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{5}(\hat{a}b + bc + c\hat{a})} < \sqrt{\frac{33}{5}} c\).

**Proof.** (i) By (3.10), we obtain \(\frac{3}{2} c^2 - c(\hat{a} + b) + 2\hat{a}b > 0\), which implies \(\frac{3}{2} c^2 + 3\hat{a}b > \hat{a}b + bc + c\hat{a}\), and by (4.18), it follows that \(\frac{3}{2} c^2 > -\hat{a}b\). Also, by (4.18), it follows that \(\hat{a} + b > -\hat{a}b/c > 0\), and hence \(b > -\hat{a} > 0\). Then \(-\hat{a}b > \hat{a}^2\) and therefore \(\frac{3}{2} c^2 > \hat{a}^2 > 0\). This implies immediately that \(0 < -\hat{a} < \frac{1}{\sqrt{2}} c\).

(ii) By (3.10) and the fact that \(\hat{a} < 0 < b\), we obtain
\[
\frac{3}{2} c > \frac{1}{10} (5\hat{a} + 5b + \sqrt{(5\hat{a} + 5b)^2 - 120\hat{a}b})
= \frac{1}{10} (5\hat{a} + 5b + \sqrt{(5b - 5\hat{a})^2 - 20\hat{a}b})
> \frac{1}{10} (5\hat{a} + 5b + \sqrt{(5b - 5\hat{a})^2})
= b > 0.
\]

(iii) Condition (4.18) implies \(\hat{a} + b > -\hat{a}b/c > 0\), and hence \(\hat{a} + b + c > 0\).

(iv) Since \(\hat{a}^2 > 0\), \(-2(\hat{a}b + \hat{a}c) > 0\), and by assertion (ii), we obtain
\[
\hat{a}^2 + b^2 + c^2 - 2(\hat{a}b + bc + c\hat{a}) > b^2 + c^2 - 2bc = (c - b)^2 > \frac{4}{25} c^2.
\]
(v) It is easy to prove that $\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a}) > 0$ by making use of assertion (iv) and (4.18). In fact,

$$\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a}) = \hat{a}^2 + b^2 + c^2 - 2(\hat{a}b + bc + c\hat{a}) + \frac{4}{3}(\hat{a}b + bc + c\hat{a}) > \frac{4}{25} c^2 + \frac{4}{3}(\hat{a}b + bc + c\hat{a}) > 0.$$ 

Next, the inequality in assertion (v) follows easily by (4.18) and assertion (iii).

(vi) Since $b, c > 0$, we obtain

$$0 < \frac{\hat{a}^2 + b^2 + c^2 - \frac{14}{9}(\hat{a}b + bc + c\hat{a})}{\sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})}} < \sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})} < \hat{a} + b + c$$

and hence

$$[\hat{a}^2 + b^2 + c^2 - \frac{14}{9}(\hat{a}b + bc + c\hat{a})] - [(\hat{a} + b + c)\sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})}] < 0.$$

(vii) By (4.18),

$$\hat{a}^2 + b^2 + c^2 - (\hat{a}b + bc + c\hat{a}) > \hat{a}^2 + b^2 + c^2 - \frac{14}{9}(\hat{a}b + bc + c\hat{a}).$$

In addition, since $\hat{a}^2 > 0$, $-(\hat{a}b + c\hat{a}) > 0$ and $bc > 0$, by assertion (ii),

$$\hat{a}^2 + b^2 + c^2 - \frac{14}{9}(\hat{a}b + bc + c\hat{a}) > b^2 + c^2 - 2bc = (c - b)^2 > \frac{4}{25} c^2.$$ 

(viii) By (4.18) and assertions (i) and (ii),

$$\sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})} < \sqrt{\hat{a}^2 + b^2 + c^2} < \sqrt{\frac{1}{3} c^2 + \frac{9}{25} c^2} = \frac{\sqrt{39}}{5} c.$$

On the other hand, since $\hat{a}^2 > 0$, $-\hat{a}(b + c) > 0$, and $bc > 0$, by assertion (ii),

$$\sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})} > \sqrt{b^2 + c^2 - 2bc} = c - b > \frac{2}{5} c.$$

The proof of Lemma 4.9 is complete. □

We are now ready to prove Theorem 3.6.

Case 1: $\hat{a}b + bc + c\hat{a} \leq 0$. In this case, it is easy to see that $f(u) = -u(u - \hat{a})(u - b)(u - c)$ satisfying (3.9) satisfies conditions (1.3) and (1.4) in Theorem 1.1. Also, $f$ satisfies (1.5) since

$$\int_0^c f(u) du = \frac{1}{60} c^3(3c^2 - 5c(\hat{a} + b) + 10\hat{a}b) > 0$$
by (3.10). Finally, since \( f''(0) = -2(\hat{a}b + bc + c\hat{a}) \geq 0 \) and by Lemma 4.9(v), the quadratic polynomial \( f''(u) = -12u^2 + 6(\hat{a} + b + c)u - 2(\hat{a}b + \hat{a}c + bc) \) has exactly one positive zero \( r \) given by

\[
r = \frac{1}{4}(\hat{a} + b + c + \sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})}) < c.
\]

So (1.6) holds. We summarize that \( f \) satisfies (1.3)–(1.6) in Theorem 1.1.

**Case 2:** \( \hat{a}b + bc + c\hat{a} > 0 \). In this case, for \( f(u) = -u(u - \hat{a})(u - b)(u - c) \) satisfying (3.9), it is easy to see that \( f' \) satisfies conditions (f1) and (f2) in Theorem 3.1. Also, \( f \) satisfies (f3) since

\[
\int_0^c f(u) du = \frac{1}{60}c^3(3c^2 - 5c(\hat{a} + b) + 10\hat{a}b) > 0
\]

by (3.10). In addition, since \( f''(0) = -2(\hat{a}b + bc + c\hat{a}) < 0 \), the quadratic polynomial

\[
f''(u) = -12u^2 + 6(\hat{a} + b + c)u - 2(\hat{a}b + \hat{a}c + bc)
\]

has exactly two positive zeros \( r < s \) given by

\[
r = \frac{1}{4}(\hat{a} + b + c - \sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})}) > 0,
\]

\[
s = \frac{1}{4}(\hat{a} + b + c + \sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})}) < c.
\]

So \( f' \) satisfies (f4). Finally, for (f5), it can be shown that there exists a unique number

\[
\sigma = \frac{1}{2}(\hat{a} + b + c + \sqrt{\hat{a}^2 + b^2 + c^2 - (\hat{a}b + bc + c\hat{a})})
\]

satisfying \( f(u) - uf'(u) = u^2(3u^2 - 2(\hat{a} + b + c)u + \hat{a}b + bc + c\hat{a}) = 0 \) on \((b, c)\). Similarly as before, let

\[
1 < x := \frac{\hat{a} + b + c}{c} < \frac{b + c}{c} < \frac{\frac{3}{2}c + c}{c} = \frac{8}{5}.
\]

Then by Lemma 4.9, we compute that

\[
207360(\Psi(\sigma) - \Psi(r))
\]

\[
= 256(\hat{a} + b + c + \sqrt{\hat{a}^2 + b^2 + c^2 - (\hat{a}b + bc + c\hat{a})})^3
\]

\[
\times (\hat{a}^2 + b^2 + c^2 - 2(\hat{a}b + bc + c\hat{a}) + (\hat{a} + b + c)
\]

\[
\sqrt{\hat{a}^2 + b^2 + c^2 - (\hat{a}b + bc + c\hat{a})}
\]

\[
+ 243(\hat{a} + b + c - \sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})})^3
\]

\[
\times (\hat{a}^2 + b^2 + c^2 - \frac{14}{9}(\hat{a}b + bc + c\hat{a}) - (\hat{a} + b + c)
\]

\[
\sqrt{\hat{a}^2 + b^2 + c^2 - \frac{2}{3}(\hat{a}b + bc + c\hat{a})}
\]
for $1 < x < \frac{8}{5}$ (cf. (4.17)); we omit the proof of the last inequality. So (f5) holds. We summarize that $f$ satisfies (f1)–(f5) in Theorem 3.1.

The proof of Theorem 3.6 is complete. □

Acknowledgements

The authors thank Professors P. Korman and J. Shi for sending many of their recent papers. Much of the computation in this paper has been checked using the symbolic manipulator 

Mathcad 7 Professional for I. Addou and 

Mathematica 4.0 for S.-H. Wang.

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