AN EXPLICIT FORMULA OF THE BIFURCATION CURVE FOR A BOUNDARY BLOW-UP PROBLEM

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Abstract. We study the bifurcation curve of (sign-changing and nonnegative) solutions of the boundary blow-up problem

\[- \left( \varphi_p (u'(x)) \right)' = \lambda f_{a,b}(u(x)), \quad 0 < x < 1,\]

\[\lim_{x \to 0^+} u(x) = \infty = \lim_{x \to 1^-} u(x),\]

where \(p > 1, \varphi_p (y) = |y|^{p-2} y \) and \( (\varphi_p (u'))' \) is the one-dimensional \( p \)-Laplacian, \( \lambda > 0 \), and

\[f_{a,b}(u) := \begin{cases} -u^a & \text{with } a > p - 1 \quad \text{if } u \geq 0, \\ -k(-u)^b & \text{with } b > 0, k > 0 \quad \text{if } u < 0. \end{cases}\]

We give a simple explicit formula of the bifurcation curve of solutions. Thus we are able to determine the shape of the bifurcation curve and hence the exact multiplicity of solutions for any \( \lambda > 0 \). Moreover, when \( b = a > p - 1 > 0, k = 1 \), and \( f_{a,b}(u) = -|u|^a \), we investigate the variation of the bifurcation curves with respect to the exponent \( a \).

Keywords. boundary blow-up problem, nonnegative solution, sign-changing solution, exact multiplicity, bifurcation curve, gamma function.

AMS (MOS) subject classification: 34B18, 34C23.

1 Introduction

In this paper we study the bifurcation curve of (sign-changing and nonnegative) solutions of the boundary blow-up problem

\[- \left( \varphi_p (u'(x)) \right)' = \lambda f_{a,b}(u(x)), \quad 0 < x < 1,\]

\[\lim_{x \to 0^+} u(x) = \infty = \lim_{x \to 1^-} u(x),\]

where \(p > 1, \varphi_p (y) = |y|^{p-2} y \) and \( (\varphi_p (u'))' \) is the one-dimensional \( p \)-Laplacian, \( \lambda > 0 \) and

\[f_{a,b}(u) := \begin{cases} -u^a & \text{with } a > p - 1 \quad \text{if } u \geq 0, \\ -k(-u)^b & \text{with } b > 0, k > 0 \quad \text{if } u < 0. \end{cases}\]

For \( p = 2 \), \( (\varphi_p (u'(x)))' = u'' \), and (1) reduces to

\[-u''(x) = \lambda f_{a,b}(u(x)), \quad 0 < x < 1,\]

\[\lim_{x \to 0^+} u(x) = \infty = \lim_{x \to 1^-} u(x).\]
Also, when \( b = a > p - 1 > 0 \) and \( k = 1 \), then (2) reduces to
\[
f_{a,b} = f_a(u) := -|u|^a, \quad a > p - 1.
\]

Blow-up solutions of the boundary value problem
\[
-\Delta u(x) = f(u(x)), \quad x \in \Omega,
\]
\[
u \mid_{\partial \Omega} = \infty,
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) (\( N \geq 1 \)) have been extensively studied, see [1-15, 17-18]. A problem of this type was first considered by Bieberbach [6] in 1916, where \( f(u) = -e^u \) and \( N = 2 \). Bieberbach showed that if \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) such that \( \partial \Omega \) is a \( C^2 \) submanifold of \( \mathbb{R}^2 \), then there exists a unique \( u \in C^2(\Omega) \) such that \( \Delta u(x) = e^u \) in \( \Omega \) and \( |u(x) - \ln(d(x))^{-2}| \) is bounded on \( \Omega \). Here \( d(x) \) denotes the distance from a point \( x \) to \( \partial \Omega \).

Rademacher [15], using the idea of Bieberbach, extended to smooth bounded domain in \( \mathbb{R}^3 \). In this case the problem plays an important role, when \( N = 2 \), in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when \( N = 3 \), according to [15], in the study of the electric potential in a glowing hollow metal body.

For \( f(u) = -u^a \) with \( a > 1 \), problem (3) is of interest in the study of the subsonic motion of a gas when \( a = 2 \), see [14]. For the special case where \( f(u) = -u^{(N+2)/(N-2)} \) (\( N > 2 \)), Loewener and Nirenberg [11] studied uniqueness of positive solution and asymptotic behavior of (3). Then, Bandle and Marcus [3-5] and Marcus and Véron [12] extended the results of [11] to a much large class of nonlinearities including \( f(u) = -u^a, \quad a > 1 \). Díaz and Letelier [8] proved existence and uniqueness of positive solution when \( \Delta \) is replaced by the \( p \)-Laplacian \( \text{div}(|\nabla u|^{p-2}\nabla u), \quad f(u) = -u^a \) with \( a > p - 1 \) and \( p \neq 2 \). The first result of nonuniqueness of solutions for (3) was obtained by McKenna et al. [13], in the special case when the domain \( \Omega \) is a ball and \( f(u) = -|u|^a \). More precisely, they proved that for \( 1 < a < N^* \) (note that \( N^* = (N+2)/(N-2) \) for \( N \geq 3 \) and \( N^* = \infty \) for \( N = 1, 2 \), there are exactly two blow-up solutions: one is positive and the other one is sign-changing. For \( a \geq N^* \), there is a unique blow-up solution and it is positive.

(They have similar exact multiplicity results for radial blow-up solutions of (3) when \( \Delta \) is replaced by the \( p \)-Laplacian.) Subsequently, Aftalion and Reichel [1] extended the existence of at least two blow-up solutions for general nonlinearities \( f \) including
\[
f(u) = \begin{cases} 
-u^a \text{ with } a > 1 & \text{if } u > 0, \\
-(-u)^b \text{ with } 1 < b < N^* & \text{if } u < 0.
\end{cases}
\]

Anuradha et al. [2] and Wang [17] studied necessary and sufficient conditions for the existence of nonnegative solutions of (3) when \( N = 1 \). Their method is based on building a quadrature method which also applies for sign-changing solutions and can be extended for \( p \)-Laplacian problem.
Suppose that $u$ is a (nonnegative or sign-changing) solution of
\[- (\varphi_p(u'(x)))' = \lambda f(u(x)), \quad 0 < x < 1,\]
\[
\lim_{x \to 0^+} u(x) = \infty = \lim_{x \to 1^-} u(x). \quad \text{(4)}
\]

Define
\[F(s) = \int_0^s f(t)dt,\]
\[I = \{s \in \mathbb{R} : F(s) > F(u) \text{ for all } u > s \text{ and } \int_s^\infty \frac{du}{(F(s) - F(u))^{1/p}} < \infty\}.
\]

The next lemma is mainly due to Anuradha et al. [2, Lemma 2.1] after very slight generalization to $p$-Laplacian problem (4). We omit the proof.

**Lemma 1** Assume that $p > 1$. Let $f$ be a Lipschitz continuous function in $\mathbb{R}$ except possibly at some point $s = 0$ where $f(s) = 0$ and $f$ is continuous there. Then, given $\lambda > 0$, there exists a unique solution $u$ to problem (4) with
\[
\min_{x \in (0,1)} u(x) = \rho \in \mathbb{R}
\]
if and only if
\[G_f(\rho) := 2 \left(\frac{p-1}{p}\right)^{1/p} \int_\rho^\infty \frac{du}{(F(\rho) - F(u))^{1/p}} = \lambda^{1/p} \text{ for } \rho \in I. \quad \text{(5)}
\]

The next example is mainly due to Anuradha et al. [2, Example 5.1].

**Example 1** Assume that $p > 1$. Consider problem (4) with $f(u) = -e^u$. Then
\[
G_f(\rho) = 2 \left(\frac{p-1}{p}\right)^{1/p} \int_\rho^\infty \frac{du}{(F(\rho) - F(u))^{1/p}}
\]
\[
= 2 \left(\frac{p-1}{p}\right)^{1/p} \int_\rho^\infty \frac{du}{(e^u - e^\rho)^{1/p}}
\]
\[
= 2 \left(\frac{p-1}{p}\right)^{1/p} \int_{e^{\rho/p}}^{\infty} \frac{dw}{w(w^p - e^\rho)^{1/p}} \quad \text{(let } w = e^{u/p})
\]
\[
= 2 \left(\frac{p-1}{p}\right)^{1/p} e^{-\rho/p} \int_0^{\pi/2} \frac{p \tan \theta d\theta}{(\sec^p \theta - 1)^{1/p}} \quad \text{(let } w = e^{p/\sec \theta})
\]
\[
= 2 \left(\frac{p-1}{p}\right)^{1/p} e^{-\rho/p} \int_1^{\infty} \frac{pdv}{v(v^p - 1)^{1/p}} \quad \text{(let } v = \sec \theta)
\]
\[
= 2\pi \left(\frac{p-1}{p}\right)^{1/p} \csc \frac{\pi}{p} e^{-\rho/p} \text{ for } \rho \in I = \mathbb{R}
\]

by applying Lemma 5(i)-(ii) stated below.
2 Main Results

First, for (1) and for any fixed \( p > 1 \), in Theorem 2, we give a simple explicit formula of the bifurcation curve \( G_{f,a,b}(\rho) = \lambda^{1/p} \), \( \rho = \min_{x \in (0,1)} u(x) \in I = (-\infty, 0) \cup (0, \infty) \). Thus, in Corollary 3, we are able to determine the shape of \( G_{f,a,b}(\rho) \) and hence the exact multiplicity of solutions for any \( \lambda > 0 \). Also, in Theorem 4, for symmetric nonlinearity \( f = f_a(u) = -|u|^a \), \( a > p - 1 \), we investigate the variation of \( G_{f,a}(\rho) \), \( \rho \in I = (-\infty, 0) \cup (0, \infty) \) with respect to the exponent \( a \). In particular, we show that

\[
\frac{G_{f,a}(-\rho)}{G_{f,a}(\rho)} = \sin \frac{\pi(2a + 2 - p)}{2p(a + 1)} \csc \frac{\pi}{2(a + 1)} > 1 \text{ for } \rho > 0,
\]

which is simply a function of \( p \) and \( a \). This implies that, for any fixed \( p > 1 \),

\[
\lim_{a \to (p - 1)^+} \frac{G_{f,a}(-\rho)}{G_{f,a}(\rho)} = \sin \frac{\pi}{2p} \csc \frac{\pi}{2p} = 1,
\]

and hence the bifurcation curve \( G_{f,a}(\rho) \) is “almost symmetric” as the exponent \( a \) tends to \((p - 1)^+\), see Fig. 2 below.

Recall the Gamma function as follows (see e.g. [16, p. 6]):

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \ (z > 0).
\]

For convenience of notations used behind, for any fixed \( p > 1 \), \( a > p - 1 \) and \( b > 0 \), we first define the following positive constants

\[
M_{p,a} = 2 \left( \frac{p - 1}{p(a + 1)^{p-1}} \right)^{1/p} \frac{\Gamma \left( \frac{p-1}{p} \right) \Gamma \left( \frac{a+1-p}{p(a+1)} \right)}{\Gamma \left( \frac{a}{a+1} \right)}, \quad (6)
\]

\[
\hat{N}_{p,a} = 2 \left( \frac{p - 1}{p(a + 1)^{p-1}} \right)^{1/p} \frac{\Gamma \left( \frac{1}{a+1} \right) \Gamma \left( \frac{a+1-p}{p(a+1)} \right)}{\Gamma \left( \frac{1}{p} \right)}, \quad (7)
\]

\[
\hat{N}_{p,b} = 2 \left( \frac{p - 1}{p(b + 1)^{p-1}} \right)^{1/p} \frac{\Gamma \left( \frac{1}{b+1} \right) \Gamma \left( \frac{p-1}{p} \right)}{\Gamma \left( \frac{2p+pb-b-1}{p(b+1)} \right)}, \quad (8)
\]

\[
N_{p,a} = \hat{N}_{p,a} + \hat{N}_{p,a}
\]

\[
= 2 \left( \frac{p - 1}{p(a + 1)^{p-1}} \right)^{1/p} \frac{\Gamma \left( \frac{1}{a+1} \right) \Gamma \left( \frac{a+1-p}{p(a+1)} \right)}{\Gamma \left( \frac{1}{p} \right)} + \frac{\Gamma \left( \frac{p-1}{p} \right)}{\Gamma \left( \frac{2p+pa-a-1}{p(a+1)} \right)}. \quad (9)
\]
Assume that 

The explicit formulas of order(s) as 0 or (10)-(11) suggest possible asymptotic behaviors of critical point, a minimum, at $G_a > p$. It is also interesting to note that if

(ii) For $\rho < 0$,

$$G_{f_{a,b}}(\rho) = \left\{ \begin{array}{ll}
\frac{1}{k^{1/p}} \frac{\hat{N}_{p,b}}{(-\rho)^{1+1/p}} + \frac{\hat{N}_{p,a}}{(-\rho)^{1+1/p}} \left( \frac{b+1}{k(a+1)} \right) \frac{a+1-p}{a+1} & \text{if } b \neq p - 1, \\
\frac{2\pi}{p} \left( \frac{p-1}{k} \right)^{1/p} \csc \frac{\pi}{p} + \frac{\hat{N}_{p,a}}{(-\rho)^{1+1/p}} \left( \frac{p}{k(a+1)} \right) \frac{a+1-p}{a+1} & \text{if } b = p - 1.
\end{array} \right. \quad (11)$$

In particular, if $a = b > p - 1$, $k = 1$, and $f_{a,b} = f_a(u) = -|u|^a$, then

$$G_{f_a}(\rho) = \frac{\hat{N}_{p,a}}{(-\rho)^{1+1/p}} + \frac{\hat{N}_{p,a}}{(-\rho)^{1+1/p}} = \frac{N_{p,a}}{(-\rho)^{1+1/p}} \quad \text{for } \rho < 0. \quad (12)$$

Remark 1 The explicit formulas of $G_f(\rho) (= \lambda^{1/p})$ for $f = f_{a,b}(u)$ given in (10)-(11) suggest possible asymptotic behaviors of $G_f(\rho)$ as $\rho \to \infty$ or $-\infty$ or $0^+$ or $0^-$ for general nonlinearities $f$ being asymptotically of polynomial order(s) as $u \to \infty$ or $-\infty$ or $0^+$ or $0^-$. For example, it can be shown that

(i) If $f(u) \sim -u^a$ with $a > p - 1$ as $u \to \infty$, then

$$G_f(\rho) \sim \frac{M_{p,a}}{\rho^{1+1/p}} \quad \text{as } \rho \to \infty.$$ 

(ii) In addition to (i), if $f(u) \sim -u^\hat{a}$ with $\hat{a} \geq p - 1$ as $u \to 0^+$, then

$$G_f(\rho) \sim \left\{ \begin{array}{ll}
\frac{M_{p,a}}{\rho^{1+1/p}} & \text{if } \hat{a} > p - 1 \quad \text{as } \rho \to 0^+, \\
-2(p-1)^{1/p} \ln \rho & \text{if } \hat{a} = p - 1
\end{array} \right. \quad (13)$$

It is also interesting to note that if $f(u) \sim -u^\hat{a}$ with $0 < \hat{a} < p - 1$ as $u \to 0^+$, then $\lim_{\rho \to 0^+} G_f(\rho)$ exists, $0 < G_f(0) = \lim_{\rho \to 0^+} G_f(\rho) \leq \infty$, and $0 \in I$. Many possible cases are studied in the forthcoming paper [18].

The next corollary follows immediately by Theorem 2, and it generalizes [13, Theorem 1]. In particular, for $0 < b < p - 1$, by solving the equation $G_{f_{a,b}}'(\rho) = 0$ for $\rho < 0$, we show that $G_{f_{a,b}}(\rho)$ for $\rho < 0$ has exactly one critical point, a minimum, at

$$\rho = \rho_c := - \left( \frac{k(a+1)}{b+1} \right)^{\frac{a+1-p}{a+1}} \left( \frac{k^{1/p}(b+1)(a+1-p)\hat{N}_{p,b}}{(p-b-1)(a+1)N_{p,a}} \right)^{\frac{1}{a+1}}.$$

Theorem 2 Assume that $p > 1$. Let $f = f_{a,b}(u)$ be defined in (2) with $a > p - 1$, $b > 0$ and $k > 0$. Then

(i) For $\rho > 0$,

$$G_{f_{a,b}}(\rho) = \frac{M_{p,a}}{\rho^{1+1/p}}. \quad (10)$$

(ii) For $\rho < 0$,

$$G_{f_{a,b}}(\rho) = \left\{ \begin{array}{ll}
\frac{1}{k^{1/p}} \frac{\hat{N}_{p,b}}{(-\rho)^{1+1/p}} + \frac{\hat{N}_{p,a}}{(-\rho)^{1+1/p}} \left( \frac{b+1}{k(a+1)} \right) \frac{a+1-p}{a+1} & \text{if } b \neq p - 1, \\
\frac{2\pi}{p} \left( \frac{p-1}{k} \right)^{1/p} \csc \frac{\pi}{p} + \frac{\hat{N}_{p,a}}{(-\rho)^{1+1/p}} \left( \frac{p}{k(a+1)} \right) \frac{a+1-p}{a+1} & \text{if } b = p - 1.
\end{array} \right. \quad (11)$$

In particular, if $a = b > p - 1$, $k = 1$, and $f_{a,b} = f_a(u) = -|u|^a$, then

$$G_{f_a}(\rho) = \frac{\hat{N}_{p,a}}{(-\rho)^{1+1/p}} + \frac{\hat{N}_{p,a}}{(-\rho)^{1+1/p}} = \frac{N_{p,a}}{(-\rho)^{1+1/p}} \quad \text{for } \rho < 0. \quad (12)$$
Corollary 3 Assume that \( p > 1 \). Let \( f = f_{a,b}(u) \) be defined in (2) with \( a > p - 1 \), \( b > 0 \) and \( k > 0 \). Then \( G_{f_{a,b}}(\rho) \) is given explicitly as in (10)-(11). \( \lim_{\rho \to 0^+} G_{f_{a,b}}(\rho) = \infty \), \( \lim_{\rho \to -\infty} G_{f_{a,b}}(\rho) = 0 \), and \( G_{f_{a,b}}(\rho) \) is a strictly decreasing function on \((0, \infty)\). Moreover,

(i) If \( 0 < b < p - 1 \), then \( \lim_{\rho \to -\infty} G_{f_{a,b}}(\rho) = \infty \), \( \lim_{\rho \to -\infty} G_{f_{a,b}}(\rho) = \infty \), and \( G_{f_{a,b}}(\rho) \) has exactly one critical point, a minimum, at \( \rho = \rho_c \) on \((-\infty, 0)\).

Hence,

(A) for \( \lambda > \lambda_c := (G_{f_{a,b}}(\rho_c))^p \), problem (1) has exactly two sign-changing solutions and exactly one positive solution,

(B) for \( \lambda = \lambda_c \), problem (1) has exactly one sign-changing solutions and exactly one positive solution,

(C) for \( 0 < \lambda < \lambda_c \), problem (1) has exactly one positive solution and no sign-changing solutions.

(ii) If \( b = p - 1 \), then \( \lim_{\rho \to -\infty} G_{f_{a,b}}(\rho) = \infty \), \( \lim_{\rho \to -\infty} G_{f_{a,b}}(\rho) = \infty \), and \( G_{f_{a,b}}(\rho) \) is a strictly increasing function on \((-\infty, 0)\).

Hence,

(D) for \( \lambda > \left(\frac{2\pi}{p} \left(\frac{p-1}{k}\right)^{1/p} \csc \frac{\pi}{p}\right)^p \), problem (1) has exactly one sign-changing solutions and exactly one positive solution,

(E) for \( 0 < \lambda \leq \left(\frac{2\pi}{p} \left(\frac{p-1}{k}\right)^{1/p} \csc \frac{\pi}{p}\right)^p \), problem (1) has exactly one positive solution and no sign-changing solutions.

(iii) If \( b > p - 1 \), \( \lim_{\rho \to -\infty} G_{f_{a,b}}(\rho) = \infty \), \( \lim_{\rho \to -\infty} G_{f_{a,b}}(\rho) = 0 \), and \( G_{f_{a,b}}(\rho) \) is a strictly increasing function on \((-\infty, 0)\). Hence,

(F) for each \( \lambda > 0 \), problem (1) has exactly one sign-changing solutions and exactly one positive solution.
An example to Corollary 3(i). (See Fig. 1) Let \( p = 2 \) and

\[
f = f_{2,\frac{1}{3}}(u) = \begin{cases} -u^2 & \text{if } u \geq 0, \\ -(u)^{1/3} & \text{if } u < 0. \end{cases}
\]

Then

\[
\sqrt{\lambda} = G_{f_{2,\frac{1}{3}}} = \begin{cases} M_{2,2} \rho^{-1/2} & \text{for } \rho > 0, \\ \tilde{N}_{2,1/3}(-\rho)^{1/3} + (\frac{2}{3})^{1/3} \tilde{N}_{2,2}(-\rho)^{-2/9} & \text{for } \rho < 0. \end{cases}
\]

Therefore, \( G_{f_{2,\frac{1}{3}}} \) satisfies \( \lim_{\rho \to 0^-} G_{f_{2,\frac{1}{3}}}(\rho) = \infty \) and \( \lim_{\rho \to -\infty} G_{f_{2,\frac{1}{3}}}(\rho) = \infty \), and \( G_{f_{2,\frac{1}{3}}} \) has exactly one critical point, a minimum, at \( \rho = \rho_c = \sqrt[12]{\pi^2} \left( \frac{\tilde{N}_{2,2}}{N_{2,1/3}} \right)^{9/5} \approx -1.316 \) on \( (-\infty, 0) \). Note that \( \lambda_c = (G(\rho_c))^2 \approx 88.538 \).

**Remark 2** Assume that \( p > 1 \). For \( f = f_{a,b}(u) \) with \( a > p-1 > 0 \), \( 0 < b < 1 \) and \( k > 0 \), our exact multiplicity results in Corollary 3(i) prove a conjecture of [7] for problem (4), see [7, Fig. 1] for details.

**Theorem 4** (See Figs. 2 and 3) Assume that \( p > 1 \). Let \( f = f_a(u) = -|u|^a \), \( a > p-1 > 0 \). Then

(i) \( G_{f_a}(\rho) = \frac{M_{p,a}}{\rho^{\frac{a+1-p}{p}}} \) and \( G_{f_a}(-\rho) = \frac{M_{p,a}}{\rho^{\frac{a+1-p}{p}}} \) for \( \rho > 0 \).

(ii) \( \frac{G_{f_a}(-\rho)}{G_{f_a}(\rho)} = \frac{2^{(2a+2-p)}}{2p(a+1)\csc\frac{\pi}{2(a+1)}} > 1 \) for \( \rho > 0 \).

Furthermore, for fixed \( p > 1 \),

(A) \( \lim_{a \to (p-1)^+} \frac{G_{f_a}(-\rho)}{G_{f_a}(\rho)} = 1 \),

(B) \( \lim_{a \to \infty} \frac{G_{f_a}(-\rho)}{G_{f_a}(\rho)} = \infty \),

(C) \( G_{f_a}(-\rho) / G_{f_a}(\rho) \) is strictly increasing with respect to \( a \).

(iii) For any fixed \( \lambda > 0 \), suppose that

\[
\lambda^{1/p} = G_{f_a}(\hat{\rho}) = \frac{N_{p,a}}{|\hat{\rho}|^{\frac{a+1-p}{p}}} = \frac{M_{p,a}}{\rho^{\frac{a+1-p}{p}}} = G_{f_a}(\rho) \text{ for some } \hat{\rho} < 0 < \rho.
\]

Then

\[
\left| \frac{\hat{\rho}}{\rho} \right| = \left( \frac{N_{p,a}}{M_{p,a}} \right)^{\frac{p}{a+1-p}} = \left( \sin \frac{\pi(2a+2-p)}{2p(a+1)} \csc \frac{\pi}{2(a+1)} \right)^{\frac{p}{a+1-p}} > 1.
\]

Furthermore, for fixed \( p > 1 \),
\[ (D) \lim_{a \to (p-1)^+} \left| \frac{\hat{\rho}}{\rho} \right| = \exp \left( \frac{\pi}{p} \cot \frac{\pi}{2p} \right) \in (1, e^2), \]
\[ (E) \lim_{a \to \infty} \left| \frac{\hat{\rho}}{\rho} \right| = 1, \]
\[ (F) \left| \frac{\hat{\rho}}{\rho} \right| \text{ is strictly decreasing with respect to } a. \]

**Figure 2:** Graphs of \( G_{f_a}(-\rho) \) and \( G_{f_a}(\rho) \), \( \rho > 0 \). The vertical ratio \( \left| \frac{\hat{\rho}}{\rho} \right| \) is \( \sin \frac{\pi(2a+2-p)}{2p(a+1)} \csc \frac{\pi}{2(a+1)} > 1 \).

**Figure 3:** Graphs of \( G_{f_a}(-\rho) \) and \( G_{f_a}(\rho) \), \( \rho > 0 \). The horizontal ratio \( \left| \frac{\hat{\rho}}{\rho} \right| = \left( \sin \frac{\pi(2a+2-p)}{2p(a+1)} \csc \frac{\pi}{2(a+1)} \right)^{p \rho \rho^{-1}} > 1 \).

### 3 Proofs of Main Results

**Lemma 5** ([16, pp. 6-8 and 157]) The Gamma function \( \Gamma(x) \) satisfies
(i) \( \int_1^0 t^{x-1}(1-t)^{y-1}dt = \int_0^\infty p^{x-1}(1+p)^{-y}dp \)
\[ = k \int_1^\infty s^{k(1-x-y)-1}(s^k - 1)^{y-1}ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \text{ for all } x, y, k > 0. \]

(ii) \( \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} (z \neq 0, -1, -2, ...) \).

**Proof of Theorem 2.**

For function \( f = f_{a,b}(u) \) defined in (2),
\[ F = F_{a,b}(u) := \int_0^u f_{a,b}(t)dt = \begin{cases} -\frac{u^{a+1}}{a+1} & \text{for } u \geq 0, \\ \frac{k(-u)^{b+1}}{b+1} & \text{for } u < 0. \end{cases} \] (13)

(i) For \( \rho > 0 \), by (5) and (13), letting \( u = \rho v (1 < v < \infty) \), we compute that
\[ G_{f_{a,b}}(\rho) = 2 \left( \frac{p-1}{p} \right)^{1/p} \int_0^\infty \frac{du}{(F_{a,b}(\rho) - F_{a,b}(u))^{1/p}} \]
\[ = 2 \left( \frac{p-1}{p} \right)^{1/p} \int_0^\infty \frac{du}{(u^{a+1} - \rho^{a+1})^{1/p}} \]
\[ = 2 \left( \frac{p-1}{p} \frac{(b+1)}{p-k} \right)^{1/p} \int_0^\infty \frac{du}{\left(\frac{a+1}{b+1}\right)^{1/p} (\rho)_{\frac{a+1}{b+1}}} \]
\[ = \frac{M_{p,a,b}}{\rho^{a/b}} \]

by applying Lemma 5(i) and by (6).

(ii) For \( \rho < 0 \), by (5) and (13), we write
\[ G_{f_{a,b}}(\rho) = 2 \left( \frac{p-1}{p} \right)^{1/p} \int_0^\infty \frac{du}{(F_{a,b}(\rho) - F_{a,b}(u))^{1/p}} \]
\[ = 2 \left( \frac{p-1}{p} \right)^{1/p} \int_0^{\infty} \frac{du}{(F_{a,b}(\rho) - F_{a,b}(u))^{1/p}} \]
\[ + 2 \left( \frac{p-1}{p} \right)^{1/p} \int_0^{\infty} \frac{du}{(F_{a,b}(\rho) - F_{a,b}(u))^{1/p}} \]
\[ = 2 \left( \frac{(p-1)(b+1)}{kp} \right)^{1/p} \int_0^{\infty} \frac{du}{((-\rho)^{b+1} - (-u)^{b+1})^{1/p}} \]
\[ + 2 \left( \frac{(p-1)(a+1)}{p} \right)^{1/p} \int_0^{\infty} \frac{du}{(k(a+1)-u)^{b+1}(-\rho)^{b+1} + u^{a+1})^{1/p}}. \] (14)
Then we compute that
\[
\int_0^\infty \frac{du}{(\frac{k(a+1)}{b+1} \rho^{b+1} + \rho^{a+1})^{1/p}} \\
= \int_0^\infty \frac{du}{(\rho^{b+1} + u^{a+1})^{1/p}} \quad (\text{let } t = \left[ \frac{k(a+1)}{b+1} \rho^{b+1} \right]^{1/p} > 0)
\]
\[
= \frac{1}{l^{\frac{a+1-p}{p}}} \int_0^\infty \frac{dv}{(1 + v^{a+1})^{1/p}} \quad (\text{let } u = tv)
\]
\[
= \frac{1}{l^{\frac{a+1-p}{p}}} \int_0^\infty \frac{w^{n-p} dw}{(1 + w^{a+1})^{1/p}} \quad (\text{let } w = v^{a+1})
\]
\[
= \frac{1}{l^{\frac{a+1-p}{p}}} \left( \frac{b+1}{k(a+1)} \right)^{\frac{a+1-p}{p(n+1)}} \frac{1}{(a+1)} \Gamma \left( \frac{1}{n+1} \right) \Gamma \left( \frac{a+1-p}{p(n+1)} \right) \right) \quad (15)
\]
by applying Lemma 5(i). In addition,
(I) If \( b \neq p - 1 \), we compute that
\[
\int_0^\infty \frac{du}{((-\rho)^{b+1} - (-u)^{b+1})^{1/p}} \\
= \frac{1}{l^{\frac{a+1-p}{p}}} \int_0^1 \frac{dv}{(1 - v^{b+1})^{1/p}} \quad (\text{let } u = \rho v, \ 0 < v < 1)
\]
\[
= \frac{1}{l^{\frac{a+1-p}{p}}} \frac{1}{b+1} \int_0^1 \frac{t^{\frac{b}{b+1}} dt}{(1 - t)^{1/p}} \quad (\text{let } t = v^{b+1})
\]
\[
= \frac{1}{l^{\frac{a+1-p}{p}}} \frac{1}{b+1} \Gamma \left( \frac{1}{b+1} \right) \Gamma \left( \frac{p-1}{p} \right) \left( \frac{2p + pb - b - 1}{p(b+1)} \right) \quad (16)
\]
by applying Lemma 5(i). Thus, by (14)-(16) and (7)-(8), for \( \rho < 0 \), we obtain
\[
G_{f,a,b}(\rho)(\rho) = \frac{1}{k^{1/p}} \frac{\tilde{N}_{p,b}}{(-\rho)^{b+1+p}} + \frac{\tilde{N}_{p,a}}{(-\rho)^{\frac{k(a+1-p)}{p(n+1)}}} \left( \frac{b+1}{k(a+1)} \right)^{\frac{a+1-p}{p(n+1)}}.
\]
In particular, if \( a = b > p - 1, k = 1, \) and \( f = f_a(u) = -|u|^a \), then, by (9),

it follows immediately that (12) holds.

(II) If \( b = p - 1 \), letting \( u = \rho v \ (0 < v < 1) \), we compute that
\[
\int_\rho^0 \frac{du}{((-\rho)^{b+1} - (-u)^{b+1})^{1/p}} = \int_0^1 \frac{dv}{(1 - v\rho)^{1/p}} = \frac{\pi}{p} \csc \frac{\pi}{p} \quad (17)
\]
An Explicit Formula for the Bifurcation Curve

by Lemma 5(i)-(ii). Thus, by (14)-(15), (17) and (7), for \( \rho < 0 \), we obtain

\[
G_{f_{\alpha,p}}(\rho) - 1 = \frac{2\pi}{p} \left( \frac{p-1}{k} \right)^{1/p} \csc \frac{\pi}{p} \left( \frac{p}{k(a+1)} \right)^{\frac{1-p}{p(a+1)}}.
\]

The proof of Theorem 2 is complete.

**Proof of Theorem 4.**

(i) For fixed \( p > 1 \) and \( f = f_{\alpha}(u) = -|u|^a \), \( a > p-1 > 0 \), in (10) and (12), we obtain

\[
G_{f_{\alpha}}(\rho) = \frac{M_{p,a}}{\rho^{\frac{p-1}{p}}} \quad \text{and} \quad G_{f_{\alpha}}(-\rho) = \frac{N_{p,a}}{\rho^{\frac{p-1}{p}}} \quad \text{for} \quad \rho > 0.
\]

(ii) For \( \rho > 0 \), we then simplify that

\[
\frac{G_{f_{\alpha}}(-\rho)}{G_{f_{\alpha}}(\rho)} = \frac{N_{p,a}}{M_{p,a}} \quad \left( \frac{1}{\alpha+1} \right) \left( \frac{\Gamma\left( \frac{a+1-p}{p(a+1)} \right)}{\Gamma\left( \frac{1}{p} \right)} + \frac{\Gamma\left( \frac{p-1}{p} \right)}{\Gamma\left( \frac{2p+1-b-1}{p(b+1)} \right)} \right)
\]

\[
= \frac{\Gamma\left( \frac{1}{\alpha+1} \right) \Gamma\left( \frac{a}{\alpha+1} \right)}{\Gamma\left( \frac{1}{p} \right) \Gamma\left( \frac{p-1}{p} \right)} \left( 1 + \frac{\Gamma\left( \frac{1}{p} \right) \Gamma\left( \frac{p-1}{p} \right)}{\Gamma\left( \frac{a+1-p}{p(a+1)} \right) \Gamma\left( \frac{2p+1-b-1}{p(b+1)} \right)} \right)
\]

\[
= \frac{\pi}{\sin\frac{\pi}{\alpha+1}} \left( 1 + \frac{\pi}{\sin\frac{\pi}{p}} \right) \left( 1 + \frac{\pi}{\sin\frac{a+1-p}{p(a+1)}} \right) \quad \text{(by applying Lemma 5(ii))}
\]

\[
= \frac{\sin\frac{\pi}{\alpha+1}}{\sin\frac{\pi}{p}} \left( \sin\frac{\pi}{p} + \sin\frac{(a+1-p)\pi}{p(a+1)} \right)
\]

\[
= \frac{1}{\sin\frac{\pi}{\alpha+1}} \left( \frac{\pi}{2p(a+1)} \cos\frac{\pi}{2(a+1)} \right)
\]

\[
= \frac{\pi(2a+2-p)}{2p(a+1)} \csc\frac{\pi}{2(a+1)} > 1
\]
since \( \sin \frac{\pi}{a+1} = 2 \sin \frac{\pi}{2(a+1)} \cos \frac{\pi}{2(a+1)} \) and

\[
\sin \frac{\pi(2a + 2 - p)}{2p(a + 1)} \csc \frac{\pi}{2(a+1)} - 1 = \frac{\sin \frac{\pi(2a + 2 - p)}{2p(a + 1)} - \sin \frac{\pi}{2(a+1)}}{\sin \frac{\pi}{2(a+1)}} \sin \frac{\pi}{2} \frac{2p(a + 1)}{2p(a + 1)} \frac{\cos \frac{\pi}{2}}{\cos \frac{\pi}{2}} > 0
\]

for \( a > p - 1 \) and \( p > 1 \). By above, for fixed \( p > 1 \),

(A)

\[
\lim_{a \to (p-1)^+} \frac{G_{f_a}(-\rho)}{G_{f_a}(\rho)} = \sin \frac{\pi}{2p} \csc \frac{\pi}{2p} = 1.
\]

(B)

\[
\lim_{a \to -\infty} \frac{G_{f_a}(-\rho)}{G_{f_a}(\rho)} = \lim_{a \to -\infty} \left( \sin \frac{\pi(2a + 2 - p)}{2p(a + 1)} \csc \frac{\pi}{2(a+1)} \right) = \infty.
\]

(C) We compute that

\[
\frac{\partial}{\partial a} \left( \frac{G_{f_a}(-\rho)}{G_{f_a}(\rho)} \right) = \frac{\partial}{\partial a} \left( \frac{N_{p,a}}{M_{p,a}} \right) = \frac{\partial}{\partial a} \left( \sin \frac{\pi(2a + 2 - p)}{2p(a + 1)} \csc \frac{\pi}{2(a+1)} \right) = \frac{\pi}{2(a+1)^2} \csc^2 \frac{\pi}{2(a+1)} \sin \frac{\pi}{p} > 0
\]

after simplification. Therefore, \( \frac{G_{f_a}(-\rho)}{G_{f_a}(\rho)} \) (\( = \frac{N_{p,a}}{M_{p,a}} \)) is strictly increasing with respect to \( a \).

(iii) For any fixed \( \lambda > 0 \), suppose that

\[
\lambda^{1/p} = G_{f_a}(\hat{\rho}) = \frac{N_{p,a}}{|\hat{\rho}|^{\frac{p}{p+1}}} = \frac{M_{p,a}}{\rho^{\frac{p}{p+1}}} = G_{f_a}(\rho)
\]

for some \( \hat{\rho} < 0 < \rho \).

see Fig. 3. Then by Theorem 4(ii), for \( a > p - 1 \),

\[
\frac{\hat{\rho}}{\rho} = \left( \frac{N_{p,a}}{M_{p,a}} \right)^{\frac{p}{p+1}} = \left( \sin \frac{\pi(2a + 2 - p)}{2p(a + 1)} \csc \frac{\pi}{2(a+1)} \right)^{\frac{p}{p+1}} > 1.
\]

By using l'Hôpital's rule, for fixed \( p > 1 \), it can be computed that
(D) \[
\lim_{a \to (p-1)^+} \left| \frac{\hat{\rho}}{\rho} \right| = \exp \left( \lim_{a \to (p-1)^+} \frac{p}{a + 1 - p} \ln \left( \sin \frac{\pi (2a + 2 - p)}{2p(a + 1)} \csc \frac{\pi}{2(a + 1)} \right) \right)
\]
\[
= \exp \left( \frac{\pi}{p} \cot \frac{\pi}{2p} \right) \in (1, e^2)
\]
since \(0 < \frac{\pi}{p} \cot \frac{\pi}{2p} < 2\) for \(p > 1\).

(E) \[
\lim_{a \to \infty} \left| \frac{\hat{\rho}}{\rho} \right| = \exp \left( \lim_{a \to \infty} \frac{p}{a + 1 - p} \ln \left( \sin \frac{\pi (2a + 2 - p)}{2p(a + 1)} \csc \frac{\pi}{2(a + 1)} \right) \right) = 1.
\]

(F) The result that \(\left| \frac{\hat{\rho}}{\rho} \right|\) is strictly decreasing with respect to \(a\) can be proved analytically. However the proof is tedious; we omit it.

The proof of Theorem 4 is complete. ■

4 Acknowledgements

Much of the computation in this paper has been checked using the symbolic manipulator Mathematica 4.0.

5 References


Received September 2001; revised March 2002.