An exact multiplicity theorem involving concave–convex nonlinearities and its application to stationary solutions of a singular diffusion problem

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1. Introduction

In this paper we study the exact multiplicity of positive solutions of the two-point Dirichlet boundary value problem

\[ u''(x) + f(u(x)) = 0, \quad -L < x < L, \]  
\[ u(-L) = u(L) = 0, \]  

where \( L > 0 \) is a bifurcation parameter and the nonlinearity \( f \in C^2(0, \infty) \cap C[0, \infty) \) satisfies

(H1) \( f(u) > 0 \) for \( u \geq 0 \),
(H2) \( \lim_{u \to \infty} f(u)/u = m_\infty \leq \infty \), and
(H3) \( f''(u) < 0 \) for \( 0 < u < C \) and \( f''(u) > 0 \) for \( u > C \) for some number \( C \geq 0 \).

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(When \( C > 0 \), \( f \) is a concave–convex function; when \( C = 0 \), we assume that \( f''(u) > 0 \) for \( u > 0 \); i.e., \( f \) is a convex function.)

The motivating example is the following stationary plasma problem:

\[
\begin{align*}
u'' + (1 - u^{1/m})^{-\beta} &= 0, \quad -L < x < L, \\
u(-L) &= u(L) = 0,
\end{align*}
\]  

(1.3) (1.4)

where \( m > 1 \) and \( \beta \) are positive constants. For this problem, \( f(u) = (1 - u^{1/m})^{-\beta} \) defined on \((0, 1)\) satisfies (H1) and (H3) with \( C > 0 \). Problem (1.3), (1.4) is discussed and exact multiplicity of (classical) positive solutions \( u \) with \( 0 < u < 1 \) is obtained in Section 2.

Exact multiplicity results are usually difficult to establish; see, e.g., [11]. This paper is also motivated by the work of Castro and Shivaji [2], Korman and Ouyang [7], Ouyang [12], Korman et al. [6], and Laetsch [8]. When \( f \in C^2(0, \infty) \cap C[0, \infty) \) satisfies

(i) \( f(0) < 0 \),

(ii) \( 0 \leq \lim_{u \to \infty} f(u) \), and

(iii) \( f''(u) > 0 \) on \((0, \infty)\),

Castro and Shivaji [2] studied (1.1), (1.2) for two classes of nonlinearities \( f(u) \) satisfying (iv) \( f''(u) > 0 \) for \( u > 0 \), or (v) (H3) with \( C > 0 \). If \( f''(u) > 0 \) for \( u > 0 \), Castro and Shivaji [2, Theorem 1.1, Case A] showed that there exists \( L^* > 0 \) such that (1.1), (1.2) has a unique positive solution for \( 0 < L < L^* \) and has no positive solutions for \( L > L^* \). However, if \( f \) satisfies (H3) with \( C > 0 \), under certain additional hypotheses on \( f \), Castro and Shivaji [2, Theorem 1.1, Case C] showed that (1.1), (1.2) has at least three positive solutions for a range of \( L > 0 \), while the exact multiplicity of positive solutions for (1.1), (1.2) is not known yet.

Korman and Ouyang [7] studied the exact multiplicity of positive solutions of the problem involving nonconvex nonlinearity (in \( u \)):

\[
\begin{align*}
u''(x) + \lambda u + h(x)u^p &= 0, \quad -1 < x < 1, \\
u(-1) &= u(1) = 0
\end{align*}
\]  

(1.5) (1.6)

with \( p > 1 \), and \( \lambda \) is a real parameter. Under the assumptions that

\[
\begin{align*}h(x) &\in C^1(-1, 1) \cap C^0[-1, 1] \quad \text{and} \quad h(-x) = h(x) \quad \text{for all } x, \\
h(0) &> 0 \quad \text{and} \quad h'(x) < 0 \quad \text{for all } x \in (-L, L), \\
\int_{-1}^{1} h(x)\varphi_1^{p+1}(x) \, dx &< 0,
\end{align*}
\]

where \( \varphi_1(x) \) is the principal eigenfunction of the operator \(-d^2/dx^2\) on \((-1, 1)\) corresponding to the principal eigenvalue \( \lambda_1 = \pi^2/4 \), Korman and Ouyang [7, Theorem 2] showed that there exist numbers \( \lambda_0 > \lambda_1 \) such that problem (1.5), (1.6) has a unique positive solution for \(-\infty < \lambda < \lambda_1 \) and \( \lambda = \lambda_0 \), exactly two positive solutions for \( \lambda_1 < \lambda < \lambda_0 \), and no positive solutions for \( \lambda > \lambda_0 \), see [7, Fig. 2].
Ouyang [12] obtained multiplicity result (but not exact) of positive solutions of the semilinear equation 
\[ \Delta u(x) + \lambda u + h(x)u^p = 0 \]
on the compact Riemannian manifolds \((M, g)\), where \(\lambda > 0\), \(p > 1\) and \(h(x)\) is a smooth function with both positive and negative parts, and \(\int_M h(x)\,dx < 0\).

Korman et al. [6] obtained exact multiplicity results for boundary value problems of the type
\[ u''(x) + f(x, u(x)) = 0, \quad -L < x < L, \]
\[ u(-L) = u(L) = 0. \] (1.7) (1.8)

Their first result concerns the case when the nonlinearity \(f\) is independent of \(x\) and behaves like a cubic 
\[-(u - a)(u - b)(u - c)\] with \(0 < a < b < c\) and \(a + c > 2b\). To get the exact multiplicity of nonnegative solutions, they assume that \(f''(u)\) changes sign exactly once when \(u > 0\) and \(f''(u)\) has exactly one positive root. They showed that there exists \(L_0\) such that problem (1.7), (1.8) with \(f = f(u)\) has exactly one nonnegative solution for \(0 < L < L_0\), exactly two nonnegative solutions for \(L = L_0\), and exactly three nonnegative solutions for \(L > L_0\). Similar results were obtained by Wang [14] under the same conditions when \(f(0) = 0\) and under one additionally different condition when \(f(0) > 0\). Their second result deals with a class of nonlinearities with explicit \(x\) dependence. The main tool used in [6,7,12] is a bifurcation theorem of Crandall and Rabinowitz [3]. Their approach may apply to problem (1.1), (1.2) with \(f(u)\) satisfying (H1)–(H3). However, we remark that our exact multiplicity result do not strictly require that \(f''(u)\) changes sign exactly once when \(u > 0\), see Theorem 1.2 and Remark 1.3 stated later.

Laetsch [8] studied (1.1), (1.2) for \(f\) satisfies (H1)–(H3) with \(C = 0\). A quadrature formula, called time map, is applied by Laetsch [8] and also in this paper to study (1.1), (1.2). It can be derived easily as follows:

Multiplying (1.1) by \(u'(x)\) and integrating, we obtain
\[ \frac{(u'(x))^2}{2} + F(u(x)) = \text{constant}, \] (1.9)

where \(F(u) = \int_0^u f(s)\,ds\). Since we are dealing only with positive solutions, \(u(x)\) has to be symmetric with respect to \(x = 0\), and \(u'(x) > 0\) for \(-L < x < 0\). Thus \(\|u\|_\infty = u(0)\).

Let \(z = u(0)\) and substitute \(x = 0\) into (1.9), then
\[ u'(x) = 2^{1/2}(F(z) - F(u))^{1/2}, \quad -L < x < 0. \] (1.10)

Now integrating (1.10) on \([-L, 0]\),
\[ L = 2^{-1/2} \int_0^x (F(z) - F(u))^{-1/2} \, du := T(z) \quad \text{for } 0 < z < \infty. \] (1.11)

Solutions \(u\) of (1.1), (1.2) correspond to \(\|u\|_\infty = z\) and \(T(z) = L\). Thus to study the number of positive solutions of (1.1), (1.2) is equivalent to study the shape of the time map \(T(z)\) on \((0, \infty)\).
Theorem 1.1 (Laetsch [8, Theorem 3.2], see Fig. 1). Suppose that \( f \in C^2(0, \infty) \cap C[0, \infty) \) and satisfies (H1)-(H3) with \( C = 0 \). Then
\[
\lim_{x \to 0} T(x) = 0, \quad \lim_{x \to \infty} T(x) = \frac{\pi}{2} (m_\infty)^{-1/2} := L_\infty \geq 0
\]
and the time map \( T(x) \) has exactly one critical point, a maximum, on (0, \( \infty \)). Moreover, let
\[
\hat{L} = \max_{x \in (0, \infty)} T(x).
\]

(i) If \( L_\infty = 0 \), then (1.1), (1.2) has exactly two positive solutions for \( 0 < L < \hat{L} \), exactly one positive solution for \( L = \hat{L} \), and no positive solutions for \( L > \hat{L} \).

(ii) If \( L_\infty > 0 \), then (1.1), (1.2) has exactly two positive solutions for \( L_\infty < L < \hat{L} \), exactly one positive solution for \( 0 < L \leq L_\infty \) and \( L = \hat{L} \), and no positive solutions for \( L > \hat{L} \).

We note that when \( f \) satisfies (H3) with \( C > 0 \); i.e., \( f \) is a concave–convex function on (0, \( \infty \)), only partial information on the exact number of positive solutions for (1.1), (1.2) is known even though \( f \) is assumed to be increasing on (0, \( \infty \)).

\( T'(x) \) can be easily computed from (1.11), see e.g. [13, p. 273]. We have
\[
T'(x) = 2^{-3/2} \int_0^x \frac{\theta(x) - \theta(u)}{(\Delta F)^{3/2}} \frac{du}{\bar{x}},
\]
where \( \Delta F = F(x) - F(u) \) and
\[
\theta(x) = 2F(x) - xf(x).
\]

We compute that
\[
\theta'(x) = f(x) - xf'(x),
\]
\[
\theta''(x) = -xf''(x)
\]
which are useful in our analysis later.

Assuming that \( f \) satisfies (H1)-(H3), we easily see that
\[
\theta(0) = 0, \quad \theta'(0) = f(0) > 0, \quad \lim_{x \to \infty} \theta(x) = -\infty.
\]

In addition, by (1.15) and (1.16), there exist numbers \( A \) and \( B \) with \( 0 \leq C < A < B \) such that
\[
\theta'(x) = f(x) - xf'(x) > 0 \quad \text{on} \ (0, A),
\]
\[
\theta'(A) = f(A) - Af'(A) = 0,
\]
\[
\theta'(x) = f(x) - xf'(x) < 0 \quad \text{on} \ (A, \infty)
\]
Fig. 1. Bifurcation curve $T(x)$. (a) $L_\infty = 0$. (b) $L_\infty > 0$.

and

$$\theta(x) = 2F(x) - xf(x) > 0 \text{ on } (0,B),$$

$$\theta(B) = 2F(B) - Bf(B) = 0,$$

$$\theta(x) = 2F(x) - xf(x) < 0 \text{ on } (B,\infty).$$

Thus, it is easy to see as in (1.13) that $T'(x) > 0$ on $(0,A)$, $T'(x) < 0$ on $(B,\infty)$, and $T(x)$ has at least one critical point, a maximum, on $(0,\infty)$. Under one additional reasonable hypothesis on the function $uf'(u)/f(u)$ on $(0,B)$, we are able to show
that the corresponding time map $T(x)$ for (1.1), (1.2) has exactly one critical point, a maximum, on $(0,\infty)$ and thus are able to determine the exact number of positive solutions of (1.1), (1.2).

**Theorem 1.2** (See Fig. 1). In addition to (H1)-(H3), suppose that $f \in C^2(0,\infty) \cap C[0,\infty)$ and satisfies

(H4) $\frac{af''(u)}{f(u)} \geq \frac{-1}{3}$ on $(0,A)$ and $\frac{af''(u)}{f(u)}$ is increasing on $(A,B)$.

Then

$$
\lim_{x \to 0} T(x) = 0, \quad \lim_{x \to \infty} T(x) = \frac{\pi}{2}(m_\infty)^{-1/2} := L_\infty \geq 0
$$

(1.20)

and the time map $T(x)$ has exactly one critical point, a maximum, on $(A,B)$ and hence on $(0,\infty)$ too. Moreover, let

$$
\hat{L} = \max_{x \in (0,\infty)} T(x).
$$

(i) If $L_\infty = 0$, then (1.1), (1.2) has exactly two positive solutions for $0 < L < \hat{L}$, exactly one positive solution for $L = \hat{L}$, and no positive solutions for $L > \hat{L}$.

(ii) If $L_\infty > 0$, then (1.1), (1.2) has exactly two positive solutions for $L_\infty < L < \hat{L}$, exactly one positive solution for $0 < L \leq L_\infty$ and $L = \hat{L}$, and no positive solutions for $L > \hat{L}$.

**Remark 1.3.** Actually, in Theorem 1.2, hypothesis (H3) can be weakened as (1.18) and (1.19). That is, we allow that $f''$ (the convexity of $f$) changes sign many times when $u > 0$. For example, the nonlinearity $f(u) = u^4 - 2u^3 + u^2 + 12u + 12$ satisfies (H1), (H2), (H4), (1.18) and (1.19) with $A \approx 1.81684$ and $B \approx 2.59593$. For which nonlinearity, $f''(u) = 12u^2 - 12u + 2$ changes sign twice when $u > 0$.

**Remark 1.4.** Theorem 1.2 allows $f'(0) < 0$, e.g., $f(u) = 2u^3 - u^2 - 4u + 10$ which satisfies $f''(0) < 0$ on $(0,\infty)$. Theorem 1.2 also applies for many classes of convex nonlinearities. In particular, let (i) $p(u)$ be a real polynomial of degree $\geq 2$ with nonnegative coefficients and with positive constant term, (ii) $\tilde{p}(u)$ be a real polynomial of degree $\geq 0$ with nonnegative coefficients and with positive constant term, and (iii) $q(u)$ be a real polynomial of degree $\geq 1$ with nonnegative coefficients except the constant term. Then it can be easily checked that all the nonlinearities $f(u) = p(u)$, $\exp(q(u))$, and $\tilde{p}(u)\exp(q(u))$ satisfy (H1)-(H4) with $C = 0$ in Theorem 1.2.

This paper is organized as follows: In the next section, we give an important application of Theorem 1.2 on exact multiplicity of positive solutions of problem (1.3), (1.4). In Section 3, we give two other applications of Theorem 1.2. In Section 4, we complete the paper by giving the proof of Theorem 1.2.
2. Stationary solutions of a singular diffusion problem

In this section we consider exact multiplicity of stationary solutions of a singular diffusion equation with Dirichlet boundary conditions. The steady states satisfy

\[(v^m)_{xx} + (1 - v)^{1-\beta} = 0, \quad -L < x < L, \quad (2.1)\]
\[v(-L) = v(L) = 0, \quad (2.2)\]

where \(m, L\) and \(\beta\) are positive constants. The associated diffusion problem to (2.1), (2.2) takes the form

\[v_t = (v^m)_{xx} + (1 - v)^{1-\beta}, \quad -L < x < L, \quad t > 0, \quad (2.3)\]
\[v(-L,t) = v(L,t) = 0, \quad t > 0, \quad (2.4)\]
\[v(x,0) = v_0(x), \quad -L < x < L \quad (2.5)\]

with different values of \(m\) and \(\beta\). This problem arises in many applications, for example, in the area of plasmic physics, where \(v \geq 0\) denote plasma density, see [4,5,9,10]. For \(0 < m < 1\), Eq. (2.3) is called the plasma (fast diffusion) equation, since the diffusion coefficient \(mv^{m-1} \to \infty\) as \(v \to 0\). For \(m = 1\), (2.3) reduces to the standard heat equation while for \(m > 1\), Eq. (2.3) is called the porous medium (slow diffusion) equation, since the diffusion coefficient \(mv^{m-1} \to 0\) as \(v \to 0\).

Problem (2.1), (2.2) can be written in the form of (1.1), (1.2) with

\[u = v^m, \quad f(u) = (1 - u^{1/m})^{-\beta}. \quad (2.6)\]

That is, problem (2.1), (2.2) is equivalent to

\[u'' + (1 - u^{1/m})^{1-\beta} = 0, \quad -L < x < L, \quad (2.7)\]
\[u(-L) = u(L) = 0. \quad (2.8)\]

In (2.7), the nonlinearity \(f(u) = (1 - u^{1/m})^{-\beta}\) has a singularity at \(u = 1\). So we look for classical \(C^2\) solutions \(u\) of (2.7), (2.8) satisfying \(0 \leq u < 1\). When \(m = 1\), problem (2.7), (2.8) was completely studied by Levine [9] by studying the time map \(T(x)\) defined by (1.11) for problem (2.7), (2.8) since then \(T(x)\) is solvable, see [9, Section 2] for details. Whereas, when \(m \neq 1\), \(T(x)\) can only be represented implicitly. A similar problem has been studied by Guo [5] with Dirichlet boundary conditions replaced by Robin boundary conditions. When \(0 < m < 1\), problem (2.7), (2.8) was studied by Deng [4]. It was shown that, if \(0 < m < 1\), there exists a positive number \(\beta^* = \beta^*(m)\) such that the next theorem holds.

**Theorem 2.1** (Deng [4, Theorem 3.3, Figs. 1 and 2], Levine [9, Theorem 2.1 A, Figs. 1 and 2]) (See Figs. 2 and 3). For \(0 < m \leq 1\), there exists a number \(\beta^* > 0\) (\(\beta^* = \infty\) if \(m = 1\)). Let \(\beta \leq \beta^*\). Then
(i) If $0 < \beta < 1$, there exist two numbers $L_1(\beta)$ and $L_2(\beta)$ with $0 < L_1(\beta) < L_2(\beta)$ such that (2.7), (2.8) has exactly one positive solution for $0 < L \leq L_1$ or $L = L_2$, exactly two positive solutions for $L_1 < L < L_2$, and no positive solutions for $L > L_2$ ($L_1 = 2((1 - \beta)^{1/2})/(1 + \beta)$ for $m = 1$);

(ii) If $\beta \geq 1$, there exists a number $L_3(\beta) > 0$ such that (2.7), (2.8) has exactly two positive solutions for $0 < L < L_3$, exactly one positive solution for $L = L_3$, and no positive solutions for $L > L_3$.

In (2.6), $f(u) = (1 - u^{1/m})^{-\beta} \in C^2(0,1)$, $f(u) > 0$ on $[0,1)$ and $f$ satisfies

$$\lim_{u \to 1^-} f(u) = \infty,$$

(2.9)

$$f'(u) = \beta m^{-1}(1 - u^{1/m})^{-\beta - 1}u^{(1/m)-1} > 0 \text{ on } (0,1),$$

(2.10)

$$f''(u) = \beta m^{-2}(1 - u^{1/m})^{-\beta - 2}u^{(1/m)-2}((\beta + m)u^{1/m} + 1 - m).$$

(2.11)

Eq. (2.11) implies

$$f''(u) > 0 \text{ on } (0,1) \text{ if } 0 < m \leq 1$$

(2.12)

and

$$f''(u) \begin{cases} < 0 & \text{on } (0, C) \text{ if } m > 1, \\ > 0 & \text{on } (C, 1) \text{ if } m > 1, \end{cases}$$

(2.13)

where

$$C = \left(\frac{m - 1}{\beta + m}\right)^m < 1.$$ 

(2.14)

(We may assume $C = 0$ if $0 < m \leq 1$ so that $f$ satisfies (H3) with $C \geq 0$ for any $m > 0$.)

Fig. 2. Bifurcation curve $T(\alpha)$ of (2.7), (2.8) on $(0,1)$ for $0 < \beta < 1$. $L_1 = 2((1 - \beta)^{1/2})/(1 + \beta)$ for $m = 1$, $0 < \beta < 1$. 

- $L_1$ and $L_2$ are bifurcation points.
- $L_3$ is a singular solution.
- Classical solutions are below $L_1$ and above $L_2$. 

Diagram shows $T(\alpha)$ axis with $L_1$ and $L_2$ and singular solutions line.
We remark that, for $0 < m \leq 1$, Theorem 2.1 actually also follows from Theorem 1.1 since then $f(u)$ is a convex function on $(0, 1)$ by (2.12). But for $m > 1$, as mentioned before, Theorem 1.1 does not apply to (2.7), (2.8) since then $f(u)$ is a nonconvex function on $(0, 1)$.

The method of Deng [4] used to show Theorem 2.1 for $0 < m \leq 1$, similar to that of Theorem 1.2 given in Section 4, is based upon showing the following results:

(i) $T'(z) \in C^2(0, 1)$ and there exist $z_1, z_2$ in $(0, 1)$ with $z_1 < z_2$ such that $T''(z) > 0$ on $(0, z_1)$ and $T'(z) < 0$ on $(z_2, 1)$.
(ii) $T(z_1) \in C^2(0, 1)$, and $T'(z^*) = 0$ implies $T''(z^*) < 0$.

Deng [4, Remark 3.2] conjectured that Theorem 2.1 still holds for any $\beta > 0$ when $0 < m < 1$ (fast diffusion case) and $m > 1$ (slow diffusion case). We prove this conjecture in the next theorem which is an easy corollary of Theorem 1.2.

**Theorem 2.2** (See Figs. 2 and 3). For any $m > 0$,

(i) if $0 < \beta < 1$, there exist two numbers $L_1(\beta)$ and $L_2(\beta)$ with $0 < L_1(\beta) < L_2(\beta)$ such that (2.7), (2.8) has exactly one positive solution for $0 < L \leq L_1$ or $L = L_2$, exactly two positive solutions for $L_1 < L < L_2$, and no positive solutions for $L > L_2$;

(ii) if $\beta \geq 1$, there exists a number $L_3(\beta) > 0$ such that (2.7), (2.8) has exactly two positive solutions for $0 < L < L_3$, exactly one positive solution for $L = L_3$, and no positive solutions for $L > L_3$.

**Proof.** Similarly as before, we show Theorem 2.2 by studying the time map $T(x)$ defined by (1.11) on $(0, 1)$ for $f(u)$ in (2.6). First, for any $m > 0$, exactly the same arguments used to prove [4, Theorem 3.3] can apply to show that

$$T(0) = 0,$$
Thus to complete the proof of Theorem 2.2, it suffices to show that, for any $m > 0$, the time map $T(x)$ has exactly one critical point, a maximum, on $(0, 1)$.

In (1.14), $\theta(u) = 2F(u) - uf(u)$ satisfies

\[
\lim_{u \to 1^-} T(x) \begin{cases} > 0 & \text{if } 0 < \beta < 1, \\
= 0 & \text{if } \beta \geq 1.
\end{cases}
\]

by (1.15)–(1.16) and (2.9)–(2.14). So there exist two positive numbers $A$ and $B$ with $C < A < B < 1$ such that

\[
\theta'(u) > 0 \quad \text{on } (0, A),
\]

\[
\theta'(u) < 0 \quad \text{on } (A, 1),
\]

\[
\theta(B) = 0.
\]

Hence by (1.13),

\[
T'(x) > 0 \quad \text{on } (0, A), \tag{2.15}
\]

\[
T'(x) < 0 \quad \text{on } (B, 1). \tag{2.16}
\]

Moreover, for any $m > 0$, $\beta > 0$, $f(u) = (1 - u^{1/m})^{-\beta} > 0$ on $(0, 1)$. We compute that

\[
f'(u) = \beta m^{-1}(1 - u^{1/m})^{-\beta - 1}u^{(1/m) - 1} > 0 \quad \text{on } (0, 1),
\]

\[
\left[ \frac{uf'(u)}{f(u)} \right]' = \beta m^{-2}(1 - u^{1/m})^{-2}u^{(1/m) - 1} > 0 \quad \text{on } (0, 1).
\]

By above, it is easy to see that $f(u)$ satisfies (H1), (H3) with $C > 0$ and (H4). If we confine $T(x)$ on $(A, B) \subset (0, 1)$, by Theorem 1.2, $T(x)$ has exactly one critical point, a maximum, on $(A, B)$. This completes the proof of Theorem 2.2. ∎

We finally remark that a similar problem to (2.1), (2.2) with $f(0) = 0$ has been studied by Aronson et al. [1]. They studied the bifurcation of positive solutions $v$ with $0 < v < 1$ of the following Dirichlet problem involving cubic nonlinearities:

\[
(v^m)_{xx} + v(v - a)(1 - v) = 0, \quad 0 < a < \frac{m + 1}{m + 3}, \quad -L < x < L,
\]
\[ v(-L) = v(L) = 0, \]

where \( m > 1 \), and \( L \) are positive constants. They showed that, which is opposite to ours, the bifurcation curve has exactly one critical point, a local minimum, on the \((\|v\|_{\infty}, L)\)-plane, see [1, Fig. 1].

3. Two other applications

(I) Consider the problem

\[ u'' + u^3 - au^2 + bu + c = 0, \quad -L < x < L, \quad (3.1) \]

\[ u(-L) = u(L) = 0. \quad (3.2) \]

We are able to show that \( f(u) := u^3 - au^2 + bu + c \) satisfies (H1)–(H4) with \( m_\infty = \infty \) and \( C > 0 \) if

\[ a > 0, \quad b \geq \frac{49}{160}a^2 \quad (3.3) \]

and \( c > 0 \) is small enough.

**Theorem 3.1** (See Fig. 1(a)). Suppose that \( f(u) = u^3 - au^2 + bu + c \) satisfy (3.3) and \( c > 0 \) is small enough. Then there exists a number \( \tilde{L} > 0 \) such that problem (3.1), (3.2) has exactly two positive solutions for \( 0 < L < \tilde{L} \), exactly one positive solution for \( L = \tilde{L} \), and no positive solutions for \( L > \tilde{L} \).

**Proof.** Suppose that \( f(u) = u^3 - au^2 + bu + c \) satisfies (3.3) and \( c > 0 \). By (1.14)–(1.16),

\[ \theta(u) = 2F(u) - uf(u) = u(-3u^3 + 2au^2 + 6c)/6, \]

\[ \theta'(u) = f(u) - uf'(u) = -2u^3 + au^2 + c, \]

\[ \theta''(u) = -uf''(u) = 2u(-3u + a). \]

So \( \theta(0) = \theta'(0) = \theta''(a/3) = 0, \lim_{u \to -\infty} \theta(u) = -\infty \), and \( \theta(u) \) has exactly one positive critical point at \( u = A \) and exactly one positive zero at \( u = B \) for some numbers \( A, B \) with \( a/3 < A < B \). Moreover,

\[ A \to a/2, \quad B \to 2a/3 \quad \text{as} \quad c \to 0. \quad (3.4) \]

By above, it is easy to see that

(i) \( f \) satisfies (H1) since \( b \geq (49/160)a^2 > a^2/4 \) and \( c > 0 \).
(ii) \( f \) satisfies (H2) with \( m_\infty = \lim_{u \to -\infty} f(u)/u = \infty \).
(iii) \( f \) satisfies (H3) for any constant \( c > 0 \).
(iv) To show that \( f(u) \) satisfies (H4), we first let

\[
3uf'(u) + f(u) = u(10u^2 - 7au + 4b) + c = uR(u) + c.
\]

By (3.4), it is easy to see that the quadratic polynomial \( R(u) = 10u^2 - 7au + 4b \) has its unique critical point, a minimum, at \( u = 7a/20 < A \) if \( c > 0 \) is small enough. Thus if \( f \) satisfies (3.3) and \( c > 0 \) is small enough, we obtain

\[
b \geq (49/160)a^2 \iff R(7a/20) \geq 0 \iff R(u) \geq 0 \text{ on } (0, \infty)
\]

\[
\Rightarrow \frac{uf'(u)}{f(u)} \geq -\frac{1}{3} \text{ on } (0, A).
\]

Hence,

\[
\frac{uf'(u)}{f(u)} \geq -\frac{1}{3} \text{ on } (0, A).
\]

Secondly, we compute that

\[
\left[ \frac{uf'(u)}{f(u)} \right]' = \frac{c(9u^2 - 4au + b) + u^2(-au^2 + 4bu - ab)}{(u^3 - au^2 + bu + c)^2}.
\]

Define

\[
N(u) = -au^2 + 4bu - ab
\]

which is a quadratic polynomial with negative leading coefficient. We compute that

\[
N(a/2) = a(4b - a^2)/4 > 0 \iff b > a^2/4,
\]

\[
N(2a/3) = a(15b - 4a^2)/9 > 0 \iff b > (4/15)a^2.
\]

For (3.8) and (3.9), \( \max(a^2/4, (4/15)a^2) = (4/15)a^2 < (49/160)a^2 \). So, if \( b \geq (49/160)a^2 \),

\[
N(u) > 0 \text{ on } \left[ a/2, 2a/3 \right].
\]

Moreover, by the continuity of \( N(u) \) on \( u \), we can find a small positive number \( \delta \) independent of \( c \) such that

\[
N(u) > 0 \text{ on } \left[ a/2 - \delta, 2a/3 + \delta \right].
\]

By (3.4)–(3.10), choosing \( c > 0 \) small enough, we obtain that

\[
\frac{a}{2} - \delta < A < B < \frac{2a}{3} + \delta,
\]

\[
N(u) > 0 \text{ on } [A, B],
\]

\[
\left[ \frac{uf'(u)}{f(u)} \right]' > 0 \text{ on } (A, B).
\]
Hence the function \( uf'(u)/f(u) \) is strictly increasing on \((A, B)\). This and (3.5) implies that \( f \) satisfies (H4) if \( f \) satisfies (3.3) and \( c > 0 \) is small enough.

Hence Theorem 3.1 follows immediately from Theorem 1.2. \( \square \)

(II) Similarly, consider the problem

\[
(v^m)_{xx} + (1 + v)^p = 0, \quad -L < x < L, \tag{3.11}
\]

\[
v(-L) = v(L) = 0, \tag{3.12}
\]

which arises in porous medium combustion. The same as before, problem (3.11), (3.12) can be written in the form of (1.1), (1.2) with

\[
u = v^m, \quad f(u) = (1 + u^{1/m})^p. \tag{3.13}
\]

That is, problem (3.11), (3.12) is equivalent to

\[
u'' + (1 + u^{1/m})^p = 0, \quad -L < x < L, \tag{3.14}
\]

\[
u(-L) = \nu(L) = 0. \tag{3.15}
\]

It is easy to show that, if \( p > m > 0 \), \( f(u) = (1 + u^{1/m})^p \) satisfies (H1)–(H4) with \( m_\infty = \infty \) and \( C > 0 \). Thus, the next theorem follows immediately from Theorem 2.2.

**Theorem 3.2** (See Fig. 1(a)). *For any constants \( p \) and \( m \) with \( p > m > 0 \), there exists a number \( \tilde{L} > 0 \) such that (3.11), (3.12) has exactly two positive solutions for \( 0 < L < \tilde{L} \), exactly one positive solution for \( L = \tilde{L} \), and no positive solutions for \( L > \tilde{L} \).

4. **Proof of Theorem 1.2**

To prove Theorem 1.2, our method is based on a modification of a time map technique due to Smoller and Wasserman [13] in which, among other things, they studied problem (1.1), (1.2) for cubic polynomials \( f(u) = -(u - a)(u - b)(u - c) \) satisfying

\[
a < b < 0 < c, \quad ab + bc + ca > 0, \quad (a + b + c) > \left( \frac{27}{4} \right)^2 \pi.
\]

First, \( T''(x) \) can be computed from (1.13), see [13, p. 273].

\[
T''(x) = \frac{2^{-3/2}}{x^2} \int_0^x \frac{-\frac{3}{2} (\Delta \theta)(\Delta \tilde{\theta}) + \Delta F(\Delta \tilde{\theta})'}{(\Delta F)^{5/2}} \, du, \tag{4.1}
\]
where $\Delta \theta = \theta(x) - \theta(u)$,

$$\Delta \tilde{f} = xf(x) - uf(u),$$ \hfill (4.2)

$$\Delta \tilde{\theta} = x\theta(x) - u\theta(u).$$ \hfill (4.3)

To prove Theorem 1.2, we first show the following Lemmas 4.1–4.3 which will be needed subsequently.

**Lemma 4.1.** Suppose that $f$ satisfies (H1)–(H4), then it follows that

$$\max_{0 \leq u \leq z} \frac{\Delta \tilde{f}}{\Delta F} = \min_{0 \leq u \leq z} \frac{\Delta \tilde{\theta}'}{\Delta \tilde{f}} = 1 \text{ for } z \in (A, B),$$

where $\Delta \tilde{f} = xf(x) - uf(u)$ and $\Delta \tilde{\theta}' = x\theta'(x) - u\theta'(u)$.

Lemma 4.1 follows easily from the following Lemmas 4.2 and 4.3.

**Lemma 4.2.** Suppose that $f$ satisfies (H1)–(H4), then it follows that the maximum of $\Delta \tilde{f}/\Delta F$ on $[0, z]$ occurs at $u = z$ for $z \in (A, B)$, and $\max_{0 \leq u \leq z}(\Delta \tilde{f}/\Delta F) = (f(z) + xf'(z))/f(z)$ for $z \in (A, B)$.

**Proof.** For fixed $z \in (A, B)$, by (1.18) and (1.19), $\theta(z) = 2F(z) - zf(z) > 0$ and $\theta'(z) = f(z) - zf'(z) < 0$. So

$$\left. \frac{\Delta \tilde{f}}{\Delta F} \right|_{u=0} = \frac{xf(z) - uf(u)}{F(z) - F(u)} \Big|_{u=0} = \frac{x f(z)}{F(z)} < 2,$$

$$\left. \frac{\Delta \tilde{f}}{\Delta F} \right|_{u=z} = \frac{f(z) + zf'(z)}{f(z)} > \frac{f(z)}{f(z)} = 2$$

by L'Hôpital’s rule. So for fixed $z \in (A, B)$, the maximum of $\Delta \tilde{f}/\Delta F$ occurs at $u = z$ or at some internal point on $(0, z)$. Set $G(u) = (f(u)+uf'(u))/f(u) = 1+(uf'(u)/f(u))$. Then by (1.18) and (H4), we have

$$G(u) < 2 \quad \text{for } 0 < u < A,$$

$$G(A) = 2,$$

$$G(u) \text{ is increasing on } (A, B).$$ \hfill (4.4)

This implies that

$$G(z) - G(u) \geq 0 \quad \text{for } 0 < u < z \text{ and } z \in (A, B).$$ \hfill (4.5)

For $z \in (A, B)$, suppose that the maximum of $\Delta \tilde{f}/\Delta F$ does not occur at $u = z$, then the maximum of $\Delta \tilde{f}/\Delta F$ occurs at an internal point $u_0$ on $(0, z)$ for some $u_0$. Hence $(\Delta \tilde{f}/\Delta F)'|_{u=u_0} = 0$ which implies that

$$f(u_0)[zf(z) - uf(u_0)] - [F(z) - F(u_0)][u_0f'(u_0) + f(u_0)] = 0.$$
So
\[ \frac{\alpha f(\alpha) - u_0 f(u_0)}{F(\alpha) - F(u_0)} = \frac{f(u_0) + u_0 f'(u_0)}{f(u_0)}. \]

That is,
\[ \frac{\Delta \tilde{f}}{\Delta F} \bigg|_{u=u_0} = G(u_0). \]

Thus for \( \alpha \in (A, B) \),
\[ \frac{\Delta \tilde{f}}{\Delta F} \bigg|_{u=u_0} = G(u_0) \leq G(\alpha) = \frac{\Delta \tilde{f}}{\Delta F} \bigg|_{u=\alpha} \]
by (4.5) and L’Hôpital’s rule. This contradicts our assumption that the maximum of \( \Delta \tilde{f}/\Delta F \) does not occur at \( u=\alpha \) for \( \alpha \in (A, B) \). Hence, for \( \alpha \in (A, B) \), the maximum of \( \Delta \tilde{f}/\Delta F \) occurs at \( u=\alpha \), and \( \max_{0 \leq u \leq \alpha} \Delta \tilde{f}/\Delta F = (f(\alpha) + \alpha f'(\alpha))/f(\alpha) \) for \( \alpha \in (A, B) \). This completes the proof of Lemma 4.2.

Lemma 4.3. Suppose that \( f \) satisfies (H1)–(H4), then it follows that the minimum of \( \Delta \tilde{f}/\Delta \tilde{f} \) on \( [0, \alpha] \) occurs at \( u = 0 \) for \( \alpha \in (A, B) \), and \( \min_{0 \leq u \leq \alpha} (\Delta \tilde{f}/\Delta \tilde{f}) = \alpha f'(\alpha)/f(\alpha) \) for \( \alpha \in (A, B) \).

Proof. By (H4), \( [xf(x)]' = f(x) + xf'(x) > (1/3)[f(x) + 3xf'(x)] \geq 0 \) on \((0, A)\). In addition, by (1.18), we obtain \( xf'(x) > f(x) > 0 \) on \((A, \infty)\). So \( [xf(x)]' = f(x) + xf'(x) > 0 \) on \((A, \infty)\). Hence \( [xf(x)]' > 0 \) on \((0, \infty)\). So \( \alpha f(x) - uf(u) > 0 \) for \( 0 < u < \alpha < B \), that is,
\[ \Delta \tilde{f} > 0 \quad \text{for} \quad 0 < u < \alpha < B. \quad (4.6) \]

For \( \alpha \in (A, B) \) and \( 0 < u < \alpha \), by (4.5) and (4.6), it can be computed that
\[ \frac{\Delta \tilde{f}'}{\Delta \tilde{f}} - \left( \frac{\Delta \tilde{f}'}{\Delta \tilde{f}} \bigg|_{u=0} \right) = \frac{u f(u) [\alpha f'(\alpha)/f(\alpha) - uf'(u)/f(u)]}{xf(x) - uf(u)} \]
\[ = \frac{u f(u) (G(x) - G(u))}{xf(x) - uf(u)} \geq 0. \]

In addition, by L’Hôpital’s rule again,
\[ \frac{\Delta \tilde{f}'}{\Delta \tilde{f}} \bigg|_{u=\alpha} = \lim_{u \to \alpha} \frac{\alpha^2 f'(\alpha) - u^2 f'(u)}{\alpha f'(\alpha) - uf'(u)} = \frac{2\alpha f'(\alpha) + \alpha^2 f''(\alpha)}{f(x) + \alpha f'(x)} \]
\[ = \frac{2\alpha f'(\alpha)}{f(x) + \alpha f'(x)}. \]

Since \( f \in C^2(0, \infty) \), the result in (4.4) that \( G(u) \) is increasing on \((A, B)\) is equivalent to saying that \( G'(u) \geq 0 \) on \((A, B)\). So it can be computed that
\[ \left( \frac{\Delta \tilde{f}'}{\Delta \tilde{f}} \bigg|_{u=\alpha} \right) - \left( \frac{\Delta \tilde{f}'}{\Delta \tilde{f}} \bigg|_{u=0} \right) = \frac{u G'(u) f(u)}{f(u) + uf'(u)} \bigg|_{u=\alpha} \geq 0 \]
for $x \in (A,B)$. Hence the minimum of $\Delta \hat{f}'/\Delta \hat{f}$ on $[0,z]$ occurs at $u=0$ for $x \in (A,B)$, and $\min_{0 \leq u \leq z}(\Delta \hat{f}'/\Delta \hat{f}) = xf'(x)/f(x)$ for $x \in (A,B)$. This completes the proof of Lemma 4.3. □

We are now in a position to show Theorem 1.2.

Proof of Theorem 1.2. First, (1.20) is well known, see e.g. [8, Theorem 3.1]. Secondly, for (1.13), by (1.18) and (1.19), it is easy to see that $T'(x) > 0$ for $x \in (0,A]$ and $T'(x) < 0$ for $x \in [B,\infty)$. So $T(x)$ has at least one critical point, a local maximum, on $(A,B)$. We then show that $T(x)$ has exactly one critical point, a maximum, on $(0,\infty)$.

Suppose that

$$
M = \max_{0 \leq u \leq x} \frac{\Delta \hat{f}}{\Delta F}, \quad m = \min_{0 \leq u \leq x} \frac{\Delta \hat{f}'}{\Delta \hat{f}} \tag{4.7}
$$

then $M - m = 1$ by Lemma 4.1. Recall a result of Smoller and Wasserman [13, p. 282],

$$
T''(x) + \frac{M}{2x} T'(x) = \int_0^x \frac{M/2[2(\Delta F)^2 - (\Delta \hat{f}) (\Delta F)] + \frac{3}{2}(\Delta \hat{f})^2 - 2(\Delta \hat{f}) (\Delta F) - (\Delta \hat{f}') (\Delta F)}{2x^2(\Delta F)^{3/2}} \, du.
$$

Let the numerator of the integrand of the above integral be $Q$ and $\lambda = \Delta \hat{f}/\Delta F$. Define $\Gamma(x) = xf(x) - (2/3)F(x)$. By hypothesis (H4), $\Gamma'(x) = (1/3)[f(x) + 3xf'(x)] \geq 0$ on $(0,B)$. So $[xf(x) - (2/3)F(x)] - [uf(u) - (2/3)F(u)] \geq 0$ for $0 < u < x < B$. Hence,

$$\frac{xf(x) - uf(u)}{F(x) - F(u)} \geq \frac{2}{3} \quad \text{for } 0 < u < x < B,
$$

i.e.,

$$\lambda = \frac{\Delta \hat{f}}{\Delta F} \geq \frac{2}{3} \tag{4.8}
$$

Now by (4.6) and (4.8), we have

$$Q \leq \frac{3}{2}(\Delta \hat{f})^2 - \left(2 + m + \frac{M}{2}\right)(\Delta \hat{f}) (\Delta F) + M(\Delta F)^2
$$

$$= (\Delta F)^2 \left[\frac{3}{2} \lambda^2 - \lambda(2 + m + \frac{M}{2}) + M\right]
$$

for $x \in (A,B)$, $0 < u < x$. Denoting the quadratic polynomial in $\lambda$ by $p(\lambda)$, and noting that $m = M - 1$ by Lemma 4.1, we have

$$p(\lambda) = \frac{3}{2} \lambda^2 - \lambda \left(2 + m + \frac{M}{2}\right) + M = \frac{3}{2} \lambda^2 - \lambda \left(\frac{3}{2}M + 1\right) + M
$$

$$= \frac{3}{2}(\lambda - M) \left(\lambda - \frac{2}{3}\right).$$
Since \(2/3 \leq \lambda \leq M,\) \(p(\lambda) \leq 0.\) It follows that \(Q \leq 0.\) By a more careful analysis, it can be shown that \(Q\) is not identically zero for fixed \(x \in (A,B),\) \(0 < u < x.\) Hence

\[
T''(x) + \frac{M}{2x} T'(x) < 0 \quad \text{for} \quad x \in (A,B).
\]

Thus \(T(x)\) has exactly one critical point, a local maximum, on \((A,B).\) By above, \(T(x)\) has exactly one critical point, a maximum, on \((0, \infty).\) Thus the results in (i)–(ii) of Theorem 1.2 follow immediately by (1.20). The proof of Theorem 1.2 is now complete.

We finally remark that when \(f(u)\) satisfies (H2)–(H4) and \(f(0)=0\) instead of (H1) \(f(0) > 0,\) almost the same arguments can be used to show that the bifurcation curve \(T(x)\) for (1.1), (1.2) satisfies

(A) \(\lim_{x \to 0} T(x) = \pi/2(m_0)^{-1/2}\) and \(\lim_{x \to \infty} T(x) = \pi/2(m_\infty)^{-1/2},\) where \(m_0 = \lim_{u \to 0} f(u)/u\) and \(m_\infty = \lim_{u \to \infty} f(u)/u,\)

(B) \(T(x)\) has exactly one critical point, a maximum, on \((0, \infty).\)

This applies for the concave–convex polynomial nonlinearities including

(i) \(f(u) = u^p + \sum_{k=1}^{m} a_k u^{nk}\) with constants \(0 < p < 1 \leq n_1 < n_2 < \cdots < n_m, n_m > 1,\) and \(a_k > 0\) for \(k = 1, 2, \ldots, m,\)

(ii) \(f(u) = u^3 - au^2 + bu\) with constants \(a > 0, b \geq \frac{49}{160} a^2,\)

(iii) \(f(u) = u^p - au^2 + bu\) with constants \(a > 0, p > 2\) and

\[
b \geq \frac{1}{4}(p-2)(3 + p + 1) \left(\frac{7}{(p-1)(3 + p + 1)}\right)^{(p-1)/(p-2)} a^2,
\]

(iv) \(f(u) = \sum_{k=1}^{m} a_k u^{nk} + u^3 - au^2 + bu\) with constants \(a > 0, a_k > 0\) for \(k = 1, 2, \ldots, m,\) and \(3 < n_1 < n_2 < \cdots < n_m,\) and

\[
b \geq \max \left(\frac{(n_1 - 2)^3}{4n_1(n_1 - 3)^2 a^2}, \frac{49}{160} a^2\right).
\]

These results will appear elsewhere, see [15].

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References


